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NEW GEOMETRIC NOTIONS OF BANACH SPACES USING ψ-DIRECT SUMS

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Abstract. This paper is a survey on our two papers concerning new geometric properties of Banach spaces that generalize p-uniform smoothness and q-uniform convexity. Some basic properties including the duality property are given.

1. Introduction and preliminaries

Banach space geometry has been provided many fundamental notions and useful tools for various fields of functional analysis. It is widely applied in Banach space theory, nonlinear analysis, probability theory and so on. We have many important geometric properties of Banach spaces such as strict convexity, smoothness and uniform non-squareness. A lot of generalizations and refinements of such geometric properties were also introduced and studied by many mathematicians.

Uniform smoothness and uniform convexity have refinements called p-uniform smoothness and q-uniform convexity, respectively. In this paper, we introduce generalizations of these notions using ψ-direct sums, and present some basic properties of them.

For a Banach space X, let B_X and S_X denote its unit ball and unit sphere, respectively. Moreover, let

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in S_X,\ \|x-y\| = \varepsilon\right\}$$

for each $\varepsilon \in (0,2]$, and let

$$\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\| + \|x-\tau y\|}{2} - 1 : x, y \in S_X\right\}$$

for each $\tau \geq 0$. These constants are, respectively, the moduli of convexity and smoothness of X. Let $1 < p \leq 2 \leq q < \infty$. Then a Banach space X is said to be

(i) *uniformly convex* if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0,2]$,

(ii) *q-uniformly convex* if there exists $C > 0$ such that $\delta_X(\varepsilon) \geq C\varepsilon^q$ for each $\varepsilon \in (0,2]$,

(iii) *uniformly smooth* if $\lim_{\tau\to 0^+} \rho_X(\tau)/\tau = 0$, and

(iv) *p-uniformly smooth* if there exists $K > 0$ such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau \geq 0$.

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Obviously the implications (ii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (iii) hold. Some basic properties of p-uniform smoothness and q-uniform convexity are collected in the following.

(a) Suppose that $1 < p_1 \leq p_2 \leq 2$. If X is p_2-uniformly smooth, then it is p_1-uniformly smooth.

(b) Suppose that $2 \leq q_1 \leq q_2 < \infty$. If X is q_1-uniformly convex, then it is q_2-uniformly convex.

(c) Suppose that $1 < p \leq 2$, and that $1/p + 1/q = 1$. Then X (resp. X^*) is p-uniformly smooth if and only if X^* (resp. X) is q-uniformly convex.

For more details about p-uniform smoothness and q-uniform convexity, the readers are referred to [1, 18].

A norm $\|\cdot\|$ on $\mathbb{R}^2$ is said to be absolute if $\|(x,y)\| = \|(|x|,|y|)\|$ for all $(x,y) \in \mathbb{R}^2$, normalized if $\|(1,0)\| = \|(0,1)\| = 1$, and symmetric if $\|(x,y)\| = \|(y,x)\|$. The set of all absolute normalized norms on $\mathbb{R}^2$ is denoted by AN_2. Bonsall and Duncan [4] showed the following characterization of absolute normalized norms on $\mathbb{R}^2$. Namely, the set AN_2 of all absolute normalized norms on $\mathbb{R}^2$ is in a one-to-one correspondence with the set Ψ_2 of all convex functions ψ on $[0,1]$ satisfying $\max\{1-t, t\} \leq \psi(t) \leq 1$ for each $t \in [0,1]$ (cf. [14]). The correspondence is given by the equation $\psi(t) = \|(1-t,t)\|$ for each $t \in [0,1]$. Remark that the norm $\|\cdot\|_\psi$ associated with the function $\psi \in \Psi_2$ is given by

$$\|(x,y)\|_\psi = \begin{cases} (|x|+|y|)\psi\left(\dfrac{|y|}{|x|+|y|}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Now let AN_2^S be the collection of all symmetric absolute normalized norms on $\mathbb{R}^2$, and let $\Psi_2^S = \{\psi \in \Psi_2 : \psi(1-t) = \psi(t) \text{ for each } t \in [0,1]\}$. Then it follows that $\|\cdot\|_\psi \in AN_2^S$ if and only if $\psi \in \Psi_2^S$. In other words, the symmetric absolute normalized norms on $\mathbb{R}^2$ and the convex functions in Ψ_2 that are symmetric with respect to $1/2$ are in a one-to-one correspondence under the same equation. For example, the function ψ_p corresponding to the ℓ_p-norm $\|\cdot\|_p$ is given by

$$\psi_p(t) = \begin{cases} ((1-t)^p + t^p)^{1/p} & \text{if } 1 \leq p < \infty, \\ \max\{1-t, t\} & \text{if } p = \infty, \end{cases}$$

and satisfies $\psi_p(1-t) = \psi_p(t)$ for each $t \in [0,1]$.

2. The ψ-direct sums

In this section, we collect some results on the ψ-direct sums. The notion of ψ-direct sums of Banach spaces was first introduced by Takahashi, Kato and Saito [19]. Let X and Y be Banach spaces, and let $\psi \in \Psi_2$. Then the ψ-direct sum $X \oplus_\psi Y$ of X and Y is defined to be the space $X \times Y$ endowed with the norm

$$\|(x,y)\|_\psi = \|(\|x\|, \|y\|)\|_\psi$$

for each $(x,y) \in X \times Y$. Naturally, the ψ_p-direct sum coincides with the usual p-direct sum.

As in the case of the p-direct sums, the ψ-direct sum $X \oplus_\psi Y$ is often inherited the geometric structure from the summands X and Y. Recall that a Banach space X is said to be *strictly convex* if $x, y \in S_X$ and $x \neq y$ imply that $\|x+y\| < 2$.

Theorem 2.1 (Takahashi, Kato and Saito [19]). *Let X and Y be Banach spaces, and let $\psi \in \Psi_2$. Then $X \oplus_\psi Y$ is strictly convex if and only if X and Y are strictly convex and ψ is strictly convex.*

Remark 2.2. It is known that if $X \oplus_p Y$ is strictly convex if and only X and Y are. The preceding theorem says that this result is essentially caused by the strict convexity of the function ψ_p.

In fact, we have the following result.

Proposition 2.3 (Takahashi, Kato and Saito [19]). *Let $\psi \in \Psi_2$. Then $(\mathbb{R}^2, \|\cdot\|_\psi)$ is strictly convex if and only if ψ is strictly convex.*

Thus Theorem 2.1 is interpreted as follows: The direct sum $X \oplus_\psi Y$ is strictly convex if and only if X, Y and $(\mathbb{R}^2, \|\cdot\|_\psi)$ are strictly convex.

In the case of uniform convexity, we have the following result. Remark that strict convexity and uniform convexity are equivalent for finite-dimensional spaces.

Theorem 2.4 (Saito and Kato [13]). *Let X and Y be Banach spaces, and let $\psi \in \Psi_2$. Then $X \oplus_\psi Y$ is uniformly convex if and only if X, Y and $(\mathbb{R}^2, \|\cdot\|_\psi)$ are uniformly convex.*

For smoothness, one has the following characterization, where a Banach space X is said to be *smooth* if each $x \in S_X$ has exactly one support functional.

Proposition 2.5 (Mitani, Saito and Suzuki [12]). *Let $\psi \in \Psi_2$. Then $(\mathbb{R}^2, \|\cdot\|_\psi)$ is smooth if and only if ψ is differentiable on $(0,1)$, $\psi'_L(0) = -1$ and $\psi'_R(1) = 1$, where ψ'_L and ψ'_R are, respectively, the left and right derivative of ψ.*

As in the case of convexity properties, we have the following theorem.

Theorem 2.6 (Mitani, Oshiro and Saito [9]). *Let X and Y be Banach spaces, and let $\psi \in \Psi_2$. Then $X \oplus_\psi Y$ is smooth if and only if X, Y and $(\mathbb{R}^2, \|\cdot\|_\psi)$ are smooth.*

It is well-known that a Banach space X is uniformly convex (resp. uniformly smooth) if and only if its dual space X^* is uniformly smooth (resp. uniformly convex). Moreover, it was shown in [9] that

$$(X \oplus_\psi Y)^* = X^* \oplus_{\psi^*} Y^*,$$

where ψ^* is an element of Ψ_2 that satisfies $(\mathbb{R}^2, \|\cdot\|_\psi)^* = (\mathbb{R}^2, \|\cdot\|_{\psi^*})$, and is given by

$$\psi^*(t) = \max_{0 \le s \le 1} \frac{(1-s)(1-t) + st}{\psi(s)}$$

for each $t \in [0,1]$. From these facts and Theorem 2.4, we immediately have the following result. Needless to say, smoothness and uniform smoothness are equivalent for finite-dimensional spaces.

Corollary 2.7 (Mitani, Oshiro and Saito [9]). *Let X and Y be Banach spaces, and let $\psi \in \Psi_2$. Then $X \oplus_\psi Y$ is uniformly smooth if and only if X, Y and $(\mathbb{R}^2, \|\cdot\|_\psi)$ are uniformly smooth.*

We also have the same result for uniform non-squareness. A Banach space X is said to be *uniformly non-square* if there exists $\delta > 0$ such that $\min\{\|x+y\|, \|x-y\|\} < 2(1-\delta)$ whenever $x, y \in S_X$. It should be mentioned that $(\mathbb{R}^2, \|\cdot\|_\psi)$ is uniformly non-square if and only if $\psi \notin \{\psi_\infty, \psi_1\}$; see [14].

Theorem 2.8 (Kato, Saito and Tamura [7]). *Let X and Y be Banach spaces, and let $\psi \in \Psi_2$. Then $X \oplus_\psi Y$ is uniformly non-square if and only if X, Y and $(\mathbb{R}^2, \|\cdot\|_\psi)$ are uniformly non-square.*

The same results remain true for the ψ-direct sums of n Banach spaces; see [6, 9], and also [15] for details of absolute norms on $\mathbb{R}^n$. For infinite dimensional analog of these results, the readers are referred to [11, 21].

The ψ-direct sum $X \oplus_\psi X$ of a Banach space X can be used to characterize the geometric properties of X itself. For example, it is known that a Banach space X is strictly convex if and only if $\|x+y\|^p < 2^{p-1}(\|x\|^p + \|y\|^p)$ whenever $x \neq y$, or equivalently,

$$\|x+y\| < \frac{1}{\psi_p(1/2)} \|(x,y)\|_p$$

for each $x, y \in X$ with $x \neq y$. In [10], Mitani and Saito gave the following characterization of strict convexity.

Theorem 2.9 (Mitani and Saito [10]). *Let X be a Banach spaces, and let $\psi \in \Psi_2$. Suppose that ψ takes the minimum only at t_0. Then X is strictly convex if and only if*

$$\|(1-t_0)x + t_0 y\| < \frac{1}{\psi(t_0)} \|((1-t_0)x, t_0 y)\|_\psi$$

for each $x, y \in X$ with $x \neq y$.

Remark that the function ψ_p is takes the minimum only at $1/2$.

We conclude this section with the following characterizations of uniform convexity and uniform non-squareness analogous to that in the preceding theorem.

Theorem 2.10 (Mitani and Saito [10]). *Let X be a Banach spaces, and let $\psi \in \Psi_2$. Suppose that ψ takes the minimum only at t_0. Then X is uniformly convex if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\|(1-t_0)x + t_0 y\| \leq \frac{1-\delta}{\psi(t_0)} \|((1-t_0)x, t_0 y)\|_\psi$$

whenever $x, y \in B_X$ with $\|x-y\| \geq \varepsilon$.

Theorem 2.11 (Mitani and Saito [10]). *Let X be a Banach spaces, and let $\psi \in \Psi_2$. Suppose that ψ takes the minimum at $t_0 \in (0,1)$. Then X is uniformly non-square if and only if there exists $\delta > 0$ such that*

$$\|(1-t_0)x + t_0 y\| \leq \frac{1-\delta}{\psi(t_0)} \|((1-t_0)x, t_0 y)\|_\psi$$

whenever $x, y \in B_X$ with $\|(1-t_0)x - t_0 y\| \geq 1-\delta$.

3. Generalized Beckner inequalities

To study new geometric properties, generalized Beckner's inequalities play a fundamental role. The original Becker inequality is the following: Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then the inequality

$$\left(\frac{|u+\gamma_{p,q}v|^q + |u-\gamma_{p,q}v|^q}{2}\right)^{1/q} \leq \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{1/p}$$

holds for each $u, v \in \mathbb{R}$. This was shown in 1975 by Beckner [2]. It is also known that $\gamma_{p,q}$ in the above inequality is the best constant, that is, if $\gamma \in [0,1]$ and the inequality

$$\left(\frac{|u+\gamma v|^q + |u-\gamma v|^q}{2}\right)^{1/q} \leq \left(\frac{|u+v|^p + |u-v|^p}{2}\right)^{1/p}$$

holds for each $u, v \in \mathbb{R}$, then we have $\gamma \leq \gamma_{p,q}$. In [20], we constructed an elementary proof of these facts.

Beckner's inequality is easily extended to Banach spaces; see [8, Corollary 1.e.15] for the proof.

Theorem 3.1. *Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then the inequality*

$$\left(\frac{\|x+\gamma_{p,q}y\|^q + \|x-\gamma_{p,q}y\|^q}{2}\right)^{1/q} \leq \left(\frac{\|x+y\|^p + \|x-y\|^p}{2}\right)^{1/p}$$

holds for each $x, y \in X$.

Using the functions ψ_p and ψ_q, Beckner's inequality can be viewed as follows: Let $1 < p \leq q < \infty$, and let $\gamma_{p,q} = \sqrt{(p-1)/(q-1)}$. Then the inequality

$$\frac{\|(u+\gamma_{p,q}v, u-\gamma_{p,q}v)\|_q}{2\psi_q(\frac{1}{2})} \leq \frac{\|(u+v, u-v)\|_p}{2\psi_p(\frac{1}{2})}$$

holds for each $u, v \in \mathbb{R}$. From this observation, we considered in [16] generalized Beckner's inequality. Namely, for each $\varphi, \psi \in \Psi_2$, let

$$\Gamma(\varphi,\psi) = \left\{\gamma \in [0,1] : \frac{\varphi(\frac{1-\gamma u}{2})}{\psi(\frac{1-u}{2})} \leq \frac{\varphi(\frac{1}{2})}{\psi(\frac{1}{2})} \text{ for all } u \in [0,1]\right\},$$

and let $\gamma_{\varphi,\psi} = \max\Gamma(\varphi,\psi)$. Then we have the following result.

Theorem 3.2 (Generalized Beckner's inequality [16])**.** *Let X be a Banach space. Suppose that $\varphi, \psi \in \Psi_2^S$, and that $\gamma \in \Gamma(\varphi,\psi)$. Then the inequality*

$$\frac{\|(x+\gamma y, x-\gamma y)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \frac{\|(x+y, x-y)\|_\psi}{2\psi(\frac{1}{2})}$$

holds for each $x, y \in X$.

We remark that the value $\gamma_{\varphi,\psi}$ is the best constant for the inequality in the preceding theorem. Some conditions that $\gamma_{\varphi,\psi} > 0$ can be found in [16].

4. ψ-UNIFORM SMOOTHNESS

We start this section with the definition of ψ-uniform smoothness.

Definition 4.1 ([17])**.** *Let $\psi \in \Psi_2$. Then a Banach space X is said to be* ψ-uniformly smooth *if there exists $M > 0$ such that $\rho_X(\tau) \leq \|(1, M\tau)\|_\psi - 1$ for each $\tau \in [0, 1]$.*

As the following lemma shows, p-uniform smoothness is equivalent to ψ_p-uniform smoothness, and so the notion of ψ-uniform smoothness is a natural generalization of that of p-uniform smoothness.

Proposition 4.2 ([17])**.** *Let X be a Banach space, and let $1 < p \leq 2$. Then X is p-uniformly smooth if and only if it is ψ_p-uniformly smooth.*

For each $\psi \in \Psi_2$, let ψ'_R denote the right derivative of ψ, that is, let

$$\psi'_R(s) = \lim_{t \to 0^+} \frac{\psi(s+t) - \psi(s)}{t}$$

for each $s \in [0, 1)$.

The relationship between uniform smoothness and ψ-uniform smoothness is as follows.

Proposition 4.3 ([17])**.** *Suppose that $\psi \in \Psi_2$ and that $\psi'_R(0) = -1$. Then every ψ-uniformly smooth Banach space is uniformly smooth.*

A function $\psi \in \Psi_2$ is said to have the property $(*)$ if there exists $M > 0$ satisfying

$$\|(1, \tau)\|_\psi + \tau^2 \leq \|(1, M\tau)\|_\psi$$

for each $\tau \in [0, 1]$. We can prove that the function ψ_p has the property $(*)$ for each $1 \leq p \leq 2$.

The property $(*)$ will be frequently used in the rest of this paper.

Proposition 4.4 ([17])**.** *Let $\psi \in \Psi_2$ with the property $(*)$. Then every 2-uniformly smooth Banach space is ψ-uniformly smooth.*

We shall characterize ψ-uniform smoothness using norm inequalities in the ψ-direct sum $X \oplus_\psi X$ (and the φ-direct sum $X \oplus_\varphi X$).

Theorem 4.5 ([17])**.** *Let X be a Banach space and $\psi \in \Psi_2^S$ with the property $(*)$. Suppose that $\gamma_{\psi,\psi_2} > 0$. Then the following are equivalent.*

(i) *The space X is ψ-uniformly smooth.*

(ii) *There exists $M > 0$ such that*

$$\frac{\|(x+y, x-y)\|_\psi}{2\psi(\frac{1}{2})} \leq \|(x, My)\|_\psi$$

for each $x, y \in X$.

(iii) *For any $\varphi \in \Psi_2^S$ with $\gamma_{\varphi,\psi} > 0$ there exists an $M_\varphi > 0$ such that*

$$\frac{\|(x+y, x-y)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \|(x, M_\varphi y)\|_\psi$$

for each $x, y \in X$.

(iv) *For some* $\varphi \in \Psi_2^S$ *with* $\gamma_{\varphi,\psi} > 0$ *there exists an* $M_\varphi > 0$ *such that*

$$\frac{\|(x+y, x-y)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \|(x, M_\varphi y)\|_\psi$$

for each $x, y \in X$.

As a consequence, we have the following well-known characterizations of p-uniform smoothness; see, for example, Beauzamy [1] and Takahashi-Hashimoto-Kato [18].

Corollary 4.6. *Let* X *be a Banach space and let* $1 < p \leq 2$. *Then the following are equivalent.*

(i) *The space* X *is* p*-uniformly smooth.*

(ii) *There exists* $M > 0$ *such that*

$$\frac{\|x+y\|^p + \|x-y\|^p}{2} \leq \|x\|^p + \|My\|^p$$

for each $x, y \in X$.

(iii) *For any* $s \in (1, \infty)$, *there exists* $M_s > 0$ *such that*

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \leq (\|x\|^p + \|M_s y\|^p)^{1/p}$$

for each $x, y \in X$.

(iv) *For some* $s \in (1, \infty)$, *there exists* $M_s > 0$ *such that*

$$\left(\frac{\|x+y\|^s + \|x-y\|^s}{2}\right)^{1/s} \leq (\|x\|^p + \|M_s y\|^p)^{1/p}$$

for each $x, y \in X$.

5. ψ^*-uniform convexity and duality

We next consider the notion of ψ^*-uniform convexity of Banach spaces. Recall that for each $\psi \in \Psi_2$ the function ψ^* defined by

$$\psi^*(t) = \max_{0 \leq s \leq 1} \frac{(1-s)(1-t) + st}{\psi(s)}$$

for each $t \in [0, 1]$ satisfies $\psi^* \in \Psi_2$ and $(\mathbb{R}^2, \|\cdot\|_\psi)^* = (\mathbb{R}, \|\cdot\|_{\psi^*})$, and so is called the *dual function* of ψ; see [9]. Clearly, $\psi \in \Psi_2^S$ if and only if $\psi^* \in \Psi_2^S$.

Definition 5.1 ([17])**.** *Let* $\psi \in \Psi_2$. *Then a Banach space* X *is said to be* ψ-uniformly convex *if there exists* $K > 0$ *such that* $\|(1 - \delta_X(\varepsilon), K\varepsilon)\|_\psi \leq 1$ *for each* $\varepsilon \in [0, 2]$.

The following result shows that this is also a natural generalization of the notion of q-uniform convexity. Remark that for $1 \leq p \leq q \leq \infty$ with $p^{-1} + q^{-1} = 1$ we have $\psi_q^* = \psi_p$.

Proposition 5.2 ([17])**.** *Let* $2 \leq q < \infty$. *Then a Banach space* X *is* q*-uniformly convex if and only if it is* ψ_q*-uniformly convex.*

The following proposition shows a basic duality between the functions ψ and ψ^*. We remark that $\psi^{**} = \psi$.

Proposition 5.3 ([17]). *Let $\psi \in \Psi_2$. Then $\psi'_R(0) = -1$ if and only if $\psi^*(t) > 1-t$ for each $t \in (0, 1/2]$.*

This duality provides the following natural implication.

Proposition 5.4 ([17]). *Suppose that $\psi \in \Psi_2$ and that $\psi'_R(0) = -1$. Then every ψ^*-uniformly convex Banach space is uniformly convex.*

The notion of ψ^*-uniform convexity also has characterizations using norm inequalities. To show the characterization, we make use of the following duality between two norm inequalities concerning with the pairs φ, ψ and φ^*, ψ^*. The fact that $(X \oplus_\psi X)^* = X^* \oplus_{\psi^*} X^*$ plays an important role in the proof.

Lemma 5.5 ([17]). *Let $\varphi, \psi \in \Psi_2^S$ and $K > 0$. Then the following are equivalent.*

(i) *The inequality*

$$\frac{\|(x+y, x-y)\|_\varphi}{2\varphi(\frac{1}{2})} \leq \|(x, Ky)\|_\psi.$$

holds for each $x, y \in X$.

(ii) *The inequality*

$$\frac{\|(f+g, f-g)\|_{\varphi^*}}{2\varphi^*(\frac{1}{2})} \geq \|(f, K^{-1}g)\|_{\psi^*}.$$

holds for each $f, g \in X^*$.

The equivalence remains true even if X is replaced with X^*.

We now present characterizations using norm inequalities.

Theorem 5.6 ([17]). *Let X be a Banach space and $\psi \in \Psi_2^S$ with the property $(*)$. Suppose that $\gamma_{\psi,\psi_2} > 0$. Then the following are equivalent*

(i) *The space X is ψ^*-uniformly convex.*

(ii) *There exists $M > 0$ such that*

$$\frac{\|(x+y, x-y)\|_{\psi^*}}{2\psi(\frac{1}{2})} \geq \|(x, My)\|_{\psi^*}$$

for each $x, y \in X$.

(iii) *For any $\varphi \in \Psi_2^S$ with $\gamma_{\varphi,\psi} > 0$ there exists an $M_\varphi > 0$ such that*

$$\frac{\|(x+y, x-y)\|_\varphi}{2\varphi(\frac{1}{2})} \geq \|(x, M_\varphi y)\|_{\psi^*}$$

for each $x, y \in X$.

(iv) *For some $\varphi \in \Psi_2^S$ with $\gamma_{\varphi,\psi} > 0$ there exists an $M_\varphi > 0$ such that*

$$\frac{\|(x+y, x-y)\|_\varphi}{2\varphi(\frac{1}{2})} \geq \|(x, M_\varphi y)\|_{\psi^*}$$

for each $x, y \in X$.

As well as the case of ψ-uniform smoothness, we have the following characterizations of q-uniform convexity as a corollary.

Corollary 5.7. *Let X be a Banach space, and let $2 \leq q < \infty$. Then, the following are equivalent:*

(i) *The space X is q-uniformly convex.*

(ii) *There exists $M > 0$ such that*

$$\frac{\|x+y\|^q + \|x-y\|^q}{2} \geq \|x\|^q + \|My\|^q$$

for each $x, y \in X$.

(iii) *For any $t \in (1, \infty)$, there exists $M_t > 0$ such that*

$$\left(\frac{\|x+y\|^t + \|x-y\|^t}{2}\right)^{1/t} \geq (\|x\|^q + \|M_t y\|^q)^{1/q}$$

for each $x, y \in X$.

(iv) *For some $t \in (1, \infty)$, there exists $M_t > 0$ such that*

$$\left(\frac{\|x+y\|^t + \|x-y\|^t}{2}\right)^{1/t} \geq (\|x\|^q + \|M_t y\|^q)^{1/q}$$

for each $x, y \in X$.

Theorem 5.6 together with Lemma 5.5 and Theorem 4.5 also show the duality between ψ-uniform smoothness and ψ^*-uniform convexity.

Corollary 5.8 ([17])**.** *Let X be a Banach space and $\psi \in \Psi_2^S$ with the property $(*)$. Suppose that $\gamma_{\psi,\psi_2} > 0$.*

(i) *The space X is ψ-uniformly smooth if and only if X^* is ψ^*-uniformly convex.*

(ii) *The space X^* is ψ-uniformly smooth if and only if X is ψ^*-uniformly convex.*

We conclude this paper with the following consequence of Corollary 5.8.

Corollary 5.9 ([17])**.** *Let $\psi \in \Psi_2$ with the property $(*)$. Suppose that $\gamma_{\psi,\psi_2} > 0$. Then every 2-uniformly convex Banach space is ψ^*-uniformly convex.*

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