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STRONG CONVERGENCE THEOREMS BY HYBRID METHODS FOR FAMILIES OF DEMIMETRIC MAPPINGS IN BANACH SPACES AND APPLICATIONS

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Dedicated to Professor Makoto Tsukada on the occasion of his 65th birthday

Abstract. In this paper, using a family of new nonlinear mappings called demimetric and the C-Q method, we first prove a strong convergence theorem for finding a common fixed point for the family in a Banach space which generalizes simultaneously the result for one-paramter nonexpansive semigroups by Nakajo and Takahashi [15] and the result for proximal point iterations by Ohsawa and Takahashi [16]. Furthermore, using the family and the shrinking projection method, we prove another strong convergence theorem in a Banach space. We apply these results to obtain well-known and new strong convergence theorems for families of demimetric mappings in a Hilbert space and a Banach space, respectively.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. For a mapping $U : C \to C$, we denote by $F(U)$ the set of fixed points of U. Let k be a real number with $0 \leq k < 1$. A mapping $U : C \to C$ is called a k-strict pseudo-contraction [5] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. A mapping $U : C \to C$ is called generalized hybrid [8] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping U is called (α, β)-generalized hybrid. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [9, 10] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

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It is also hybrid [22] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [7].

Recently, Takahashi [23] introduced a new nonlinear mapping as follows: Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to C$ with $F(U) \neq \emptyset$ is called η-demimetric if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2.$$

We know from [23] that a k-strict pseudo-contraction U with $F(U) \neq \emptyset$ is k-demimetric and an $(\alpha,\ \beta)$-generalized hybrid mapping U with $F(U) \neq \emptyset$ is 0-demimetric. We also know from [23] that there exists such a mapping in a Banach space. Let E be a smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then, for the metric resolvent J_λ of B for $\lambda > 0$, we have from [20] that, for any $x \in E$ and $q \in B^{-1}0$,

$$\langle J_\lambda x - q, J(x - J_\lambda x)\rangle \geq 0.$$

Then we get

$$\langle J_\lambda x - x + x - q, J(x - J_\lambda x)\rangle \geq 0$$

and hence

$$\langle x - q, J(x - J_\lambda x)\rangle \geq \|x - J_\lambda x\|^2 = \frac{1 - (-1)}{2}\|x - J_\lambda x\|^2.$$

So, the metric resolvent J_λ with $B^{-1}0 \neq \emptyset$ is (-1)-demimetric.

On the other hand, we know the C-Q method introduced by Solodov and Svaiter [17] for finding a solution of an optimization problem; see also [15,16]. Furthermore, we know the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [24] for finding a fixed point of a nonexpansive mapping.

In this paper, using a family of new nonlinear mappings called demimetric and the C-Q method, we first prove a strong convergence theorem for finding a common fixed point for the family in a Banach space which generalizes simultaneously the result for one-paramter nonexpansive semigroups by Nakajo and Takahashi [15] and the result for proximal point iterations by Ohsawa and Takahashi [16]. Furthermore, using the family and the shrinking projection method, we prove another strong convergence theorem in a Banach space. We apply these results to obtain well-known and new strong convergence theorems for families of demimetric mappings in a Hilbert space and a Banach space, respectively.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

for every ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. It is known that a Banach space E is uniformly convex if and only if, for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1 \text{ and } \lim_{n\to\infty} \|x_n + y_n\| = 2,$$

$\lim_{n\to\infty} \|x_n - y_n\| = 0$ holds. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [6].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [19] and [20]. We know the following result:

Lemma 2.1 ([19]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \ge 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \le \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C.

Lemma 2.2 ([19]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x$;
(2) $\langle z - y, J(x - z) \rangle \ge 0, \quad \forall y \in C$.

Let E be a Banach space and let A be a mapping of E into 2^{E^*}. The effective domain of A is denoted by $\mathrm{dom}(A)$, that is, $\mathrm{dom}(A) = \{x \in E : Ax \ne \emptyset\}$. A multi-valued mapping A on E is said to be monotone if $\langle x - y, u^* - v^* \rangle \ge 0$ for all $x, y \in \mathrm{dom}(A)$, $u^* \in Ax$, and $v^* \in Ay$. A monotone operator A on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [4]; see also [20, Theorem 3.5.4].

Theorem 2.3 ([4]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let A be a monotone operator of E into 2^{E^*}. Then A is maximal if and only if for any $r > 0$,*

$$R(J + rA) = E^*,$$

where $R(J + rA)$ is the range of $J + rA$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let A be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$0 \in J(x_r - x) + rAx_r.$$

This equation has a unique solution x_r. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the metric resolvents of A. The set of null points of A is defined by $A^{-1}0 = \{z \in E : 0 \in Az\}$. We know that $A^{-1}0$ is closed and convex; see [20].

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \operatorname*{s-Li}_n C_n = \operatorname*{w-Ls}_n C_n, \tag{2.2}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [12] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [12]. The following lemma was proved by Tsukada [25].

Lemma 2.4 ([25]). *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = M\text{-lim}_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of E onto C_n and C_0, respectively.*

3. Main results

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let $\{U_n\}$ be a sequence of mappings of C into itself such that $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$. The sequence $\{U_n\}$ is said to satisfy the condition (I) [2] if for any bounded sequence $\{z_n\}$ of C such that $\lim_{n\to\infty} \|z_n - U_n z_n\| = 0$, every weak cluster point of $\{z_n\}$ belongs to $\cap_{n=1}^{\infty} F(U_n)$. In this section, using the C-Q method, we first prove a strong convergence theorem for finding a common fixed point of a family of demimetric mappings in a Banach space. Before proving the result, we need the following lemma for demimetric mappings by Takahashi [23].

Lemma 3.1 ([23]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of C into itself. Then $F(U)$ is closed and convex.*

Theorem 3.2. *Let E be a uniformly convex and smooth Banach space and let J_E be the duality mapping on E. Let C be a nonempty, closed and convex subset of E. Let $\{\eta_n\}$ be a sequence of real numbers with $\eta_n \in (-\infty, 1)$ and let $\{U_n\}$ be a family of η_n-demimetric mappings of C into itself with $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$ satisfying the condition (I). Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = \alpha_n x_n + (1-\alpha_n) U_n x_n, \\ C_n = \{z \in C : 2\langle x_n - z, J_E(x_n - z_n)\rangle \geq (1-\eta_n)\|x_n - z_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where for some $a, b \in \mathbb{R}$,

$$0 \leq \alpha_n \leq a < 1 \text{ and } 0 < b \leq 1 - \eta_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{n=1}^{\infty} F(U_n)$, where $z_0 = P_{\cap_{n=1}^{\infty} F(U_n)} x_1$.

Proof. It is obvious that $C_n \cap Q_n$ is closed and convex for all $n \in \mathbb{N}$. To show that $\cap_{n=1}^{\infty} F(U_n) \subset C_n$ for all $n \in \mathbb{N}$, let us show that

$$2\langle x_n - z, J_E(x_n - z_n)\rangle \geq (1-\eta_n)\|x_n - z_n\|^2$$

for all $z \in \cap_{n=1}^{\infty} F(U_n)$ and $n \in \mathbb{N}$. In fact, we have that for all $z \in \cap_{n=1}^{\infty} F(U_n)$ and $n \in \mathbb{N}$,

$$\begin{aligned} 2\langle x_n - z, J_E(x_n - z_n)\rangle &= 2\langle x_n - z, J_E((1-\alpha_n)(x_n - U_n x_n))\rangle \\ &= 2(1-\alpha_n)\langle x_n - z, J_E(x_n - U_n x_n)\rangle \\ &\geq (1-\alpha_n)(1-\eta_n)\|x_n - U_n x_n\|^2 \\ &= \frac{(1-\alpha_n)^2}{1-\alpha_n}(1-\eta_n)\|x_n - U_n x_n\|^2 \\ &= \frac{1-\eta_n}{1-\alpha_n}\|x_n - z_n\|^2 \\ &\geq (1-\eta_n)\|x_n - z_n\|^2. \end{aligned} \tag{3.1}$$

Then we have that $\cap_{n=1}^{\infty} F(U_n) \subset C_n$ for all $n \in \mathbb{N}$. We show that $\cap_{n=1}^{\infty} F(U_n) \subset Q_n$ for all $n \in \mathbb{N}$. Since $Q_1 = \{z \in C : \langle x_1 - z, J_E(x_1 - x_1)\rangle \geq 0\} = C$, it is obvious that $\cap_{n=1}^{\infty} F(U_n) \subset Q_1$. Suppose that $\cap_{n=1}^{\infty} F(U_n) \subset Q_k$ for some $k \in \mathbb{N}$. Then $\cap_{n=1}^{\infty} F(U_n) \subset C_k \cap Q_k$. By $x_{k+1} = P_{C_k \cap Q_k} x_1$, we have that

$$\langle x_{k+1} - z, J_E(x_1 - x_{k+1})\rangle \geq 0, \quad \forall z \in C_k \cap Q_k$$

and hence

$$\langle x_{k+1} - z, J_E(x_1 - x_{k+1})\rangle \geq 0, \quad \forall z \in \cap_{n=1}^{\infty} F(U_n).$$

Then we get $\cap_{n=1}^{\infty} F(U_n) \subset Q_{k+1}$. We have by mathematical induction that $\cap_{n=1}^{\infty} F(U_n) \subset Q_n$ for all $n \in \mathbb{N}$. Thus, we have that $\cap_{n=1}^{\infty} F(U_n) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $\cap_{n=1}^{\infty} F(U_n)$ is nonempty, closed and convex from Lemma 3.1, there exists $z_0 \in F(U)$ such that $z_0 = P_{\cap_{n=1}^{\infty} F(U_n)} x_1$. By $x_{n+1} = P_{C_n \cap Q_n} x_1$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - y\|$$

for all $y \in C_n \cap Q_n$. Since $z_0 \in \cap_{n=1}^{\infty} F(U_n) \subset C_n \cap Q_n$, we have that

$$\|x_1 - x_{n+1}\| \leq \|x_1 - z_0\|. \tag{3.2}$$

This means that $\{x_n\}$ is bounded.

Next, we show that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From the definition of Q_n, we have that $x_n = P_{Q_n} x_1$. From $x_{n+1} = P_{C_n \cap Q_n} x_1$ we have that $x_{n+1} \in Q_n$. Thus

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|$$

for all $n \in \mathbb{N}$. This implies that $\{\|x_1 - x_n\|\}$ is bounded and nondecreasing. Then there exists the limit of $\{\|x_1 - x_n\|\}$. Put $\lim_{n\to\infty} \|x_n - x_1\| = c$. If $c = 0$, then $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Assume that $c > 0$. Since $x_n = P_{Q_n} x_1$, $x_{n+1} \in Q_n$ and $\frac{x_n + x_{n+1}}{2} \in Q_n$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - \frac{x_n + x_{n+1}}{2}\| \leq \frac{1}{2}(\|x_1 - x_n\| + \|x_1 - x_{n+1}\|)$$

and hence

$$\lim_{n\to\infty} \|x_1 - \frac{x_n + x_{n+1}}{2}\| = c.$$

Since E is uniformly convex, we get that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

We have from $x_{n+1} \in C_n$ that

$$(1 - \eta_n)\|x_n - z_n\|^2 \leq 2\langle x_n - x_{n+1}, J_E(x_n - z_n)\rangle. \tag{3.3}$$

We also have that for $z \in \cap_{n=1}^{\infty} F(U_n)$ and $n \in \mathbb{N}$,

$$(1 - \eta_n)\|x_n - z_n\|^2 \leq 2\langle x_n - z, J_E(x_n - z_n)\rangle \leq 2\|x_n - z\|\|x_n - z_n\|$$

and hence

$$(1 - \eta_n)\|x_n - z_n\| \leq 2\|x_n - z\|.$$

Since $0 < b \leq 1 - \eta_n$, $\{z_n\}$ is bounded. Since $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ and $\{z_n\}$ is bounded, we have from (3.3) that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Since

$$\|x_n - z_n\| = (1 - \alpha_n)\|x_n - U_n x_n\| \geq (1 - a)\|x_n - U_n x_n\|,$$

we get that

$$\lim_{n\to\infty} \|x_n - U_n x_n\| = 0. \tag{3.4}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converging weakly to w. Since $\{U_n\}$ with $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$ satisfies the condition (I), we have that $w \in \cap_{n=1}^{\infty} F(U_n)$.

From $z_0 = P_{\cap_{n=1}^{\infty} F(U_n)} x_1$ and $w \in \cap_{n=1}^{\infty} F(U_n)$, we have from (3.2) that

$$\begin{aligned} \|x_1 - z_0\| \leq \|x_1 - w\| &\leq \liminf_{i\to\infty} \|x_1 - x_{n_i}\| \\ &\leq \limsup_{i\to\infty} \|x_1 - x_{n_i}\| \leq \|x_1 - z_0\|. \end{aligned}$$

Then we get that

$$\lim_{i\to\infty} \|x_1 - x_{n_i}\| = \|x_1 - w\| = \|x_1 - z_0\|.$$

From the Kadec-Klee property of E, we have that $x_1 - x_{n_i} \to x_1 - w$ and hence

$$x_{n_i} \to w = z_0.$$

Therefore, we have $x_n \to z_0$. This completes the proof. $\square$

Next, using the shrinking projection method, we prove a strong convergence theorem for finding a common fixed point of a family of demimetric mappings in a Banach space.

Theorem 3.3. *Let E be a uniformly convex and smooth Banach space and let J_E be the duality mapping on E. Let C be a nonempty, closed and convex subset of E. Let $\{\eta_n\}$ be a sequence of real numbers with $\eta_n \in (-\infty, 1)$ and let $\{U_n\}$ be a family of η_n-demimetric mappings of C into itself with $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$ satisfying the condition (I). Let $x_1 \in C$ and let $C_1 = C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = \alpha_n + (1-\alpha_n)U_n x_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, J_E(x_n - z_n)\rangle \geq (1-\eta_n)\|x_n - z_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where for some $a, b \in \mathbb{R}$,

$$0 \leq \alpha_n \leq a < 1 \text{ and } 0 < b \leq 1 - \eta_n, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{n=1}^{\infty} F(U_n)$, where $z_0 = P_{\cap_{n=1}^{\infty} F(U_n)} x_1$.

Proof. It is obvious that C_n are closed and convex for all $n \in \mathbb{N}$. We show that $\cap_{n=1}^{\infty} F(U_n) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\cap_{n=1}^{\infty} F(U_n) \subset C = C_1$. Suppose that $\cap_{n=1}^{\infty} F(U_n) \subset C_k$ for some $k \in \mathbb{N}$. To show that $\cap_{n=1}^{\infty} F(U_n) \subset C_n$ for all $n \in \mathbb{N}$, let us show that

$$2\langle x_k - z, J_E(x_k - z_k)\rangle \geq (1-\eta_k)\|x_k - z_k\|^2$$

for all $z \in \cap_{n=1}^{\infty} F(U_n)$. In fact, we have that for all $z \in \cap_{n=1}^{\infty} F(U_n)$,

$$\begin{aligned} 2\langle x_k - z, J_E(x_k - z_k)\rangle &= 2\langle x_k - z, (1-\alpha_k)J_E(x_k - U_k x_k)\rangle \\ &\geq (1-\alpha_k)(1-\eta_k)\|x_k - U_k x_k\|^2 \\ &= \frac{(1-\alpha_k)^2}{1-\alpha_k}(1-\eta_k)\|x_k - U_k x_k\|^2 \\ &= \frac{1-\eta_k}{1-\alpha_k}\|x_k - z_k\|^2 \\ &\geq (1-\eta_k)\|x_k - z_k\|^2. \end{aligned} \tag{3.5}$$

Then, $\cap_{n=1}^{\infty} F(U_n) \subset C_{k+1}$. We have by mathematical induction that $\cap_{n=1}^{\infty} F(U_n) \subset C_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $\cap_{n=1}^{\infty} F(U_n)$ is nonempty, closed and convex from Lemma 3.1, there exists $z_0 \in \cap_{n=1}^{\infty} F(U_n)$ such that $z_0 = P_{\cap_{n=1}^{\infty} F(U_n)} x_1$. From $x_n = P_{C_n} x_1$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - y\|$$

for all $y \in C_n$. Since $z_0 \in \cap_{n=1}^{\infty} F(U_n) \subset C_n$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - z_0\|. \tag{3.6}$$

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Since $C_0 \supset \cap_{n=1}^{\infty} F(U_n) \neq \emptyset$, we have that C_0 is nonempty. Since $C_0 = \text{M-}\lim_{n\to\infty} C_n$ and $x_n = P_{C_n} x_1$ for every $n \in \mathbb{N}$, by Lemma 2.4 we have

$$x_n \to x_0 = P_{C_0} x_1.$$

This also implies that

$$\|x_n - x_{n+1}\| \to 0. \tag{3.7}$$

We have from $x_{n+1} \in C_{n+1}$ that

$$(1-\eta_n)\|x_n - z_n\|^2 \leq 2\langle x_n - x_{n+1}, J_E(x_n - z_n)\rangle. \tag{3.8}$$

We also have that for $z \in \cap_{n=1}^{\infty} F(U_n)$ and $n \in \mathbb{N}$,

$$(1-\eta_n)\|x_n - z_n\|^2 \leq 2\langle x_n - z, J_E(x_n - z_n)\rangle \leq 2\|x_n - z\|\|x_n - z_n\|$$

and hence

$$(1-\eta_n)\|x_n - z_n\| \leq 2\|x_n - z\|.$$

Since $0 < b \leq 1 - \eta_n$, $\{z_n\}$ is bounded. Since $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ from (3.7) and $\{z_n\}$ is bounded, we have from (3.8) that $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Since

$$\|x_n - z_n\| = (1-\alpha_n)\|x_n - U_n x_n\| \geq (1-a))\|x_n - U_n x_n\|,$$

we get that

$$\lim_{n\to\infty} \|x_n - U_n x_n\| = 0. \tag{3.9}$$

Since $\{x_n\}$ converges strongly to x_0, $\{x_n\}$ converges weakly to x_0. Since $\{U_n\}$ with $\cap_{n=1}^{\infty} F(U_n) \neq \emptyset$ satisfies the condition (I), we have that $x_0 \in \cap_{n=1}^{\infty} F(U_n)$.

From $z_0 = P_{\cap_{n=1}^{\infty} F(U_n)} x_1$, $x_0 \in \cap_{n=1}^{\infty} F(U_n)$, $x_n \to x_0$ and (3.6), we have that

$$\begin{aligned}\|x_1 - z_0\| \leq \|x_1 - x_0\| &= \lim_{n\to\infty} \|x_1 - x_n\| \\ &\leq \|x_1 - z_0\|.\end{aligned}$$

Then we get that $x_0 = z_0$. Therefore, we have $x_n \to x_0 = z_0$. This completes the proof. □

4. Applications

In this section, using Theorems 3.2 and 3.3, we get well-known and new strong convergence theorems in Hilbert spaces and Banach spaces, respectively. We know the following lemma obtained by Marino and Xu [11].

Lemma 4.1 ([11])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to C$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following lemma from Kocourek, Takahashi and Yao [8].

Lemma 4.2 ([8])**.** *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to C$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $k \in [0, 1)$. Let $U : C \to C$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Define $U_n = \beta_n I + (1-\beta_n)U$ for all $n \in \mathbb{N}$*

such that $0 \leq \beta_n < 1$ *and* $\sup_{n\in\mathbb{N}} \beta_n < 1$. *Let* $x_1 \in C$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = U_n x_n, \\ C_n = \{z \in C : 2\langle x_n - z, x_n - z_n\rangle \geq (1-k)\|x_n - z_n\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ *converges strongly to a point* $z_0 \in F(U)$, *where* $z_0 = P_{F(U)} x_1$.

Proof. Since U is a k-strict pseudo-contraction of C into itself such that $F(U) \neq \emptyset$, U is k-demimetric. We also have that for $x \in C$ and $p \in F(U_n) = F(U)$,

$$\begin{aligned} 2\langle x - p, x - U_n x\rangle &= 2\langle x - p, x - (\beta_n x + (1-\beta_n)Ux)\rangle \\ &= 2(1-\beta_n)\langle x - p, x - Ux\rangle \\ &\geq (1-\beta_n)(1-k)\|x - Ux\|^2 \\ &= (1-\beta_n)^2 \frac{1-k}{1-\beta_n}\|x - Ux\|^2 \\ &= \frac{1-k}{1-\beta_n}\|x - (\beta_n x + (1-\beta_n)Ux)\|^2 \\ &\geq (1-k)\|x - (\beta_n x + (1-\beta_n)Ux)\|^2 \\ &= (1-k)\|x - U_n x\|^2 \end{aligned}$$

and hence $\{U_n\}$ is a family of k-demimetric mappings of C into C such that $F(U) = \cap_{n=1}^{\infty} F(U_n)$. Furthermore, let $\{u_n\}$ be a bounded sequence of C such that $u_n - U_n u_n \to 0$. Then we have

$$(1-\beta_n)(u_n - Uu_n) = u_n - U_n u_n \to 0$$

and hence $u_n - Uu_n \to 0$. It follows from Lemma 4.1 that every weak cluster point of $\{u_n\}$ belongs to $F(U) = \cap_{n=1}^{\infty} F(U_n)$. This means that the family $\{U_n\}$ satisfies the condition (I). Therefore, putting $\alpha_n = 0$ in Theorem 3.2, we have the desired result from Theorem 3.2. □

Theorem 4.4. *Let* H *be a Hilbert space and let* C *be a nonempty, closed and convex subset of* H. *Let* $U : C \to C$ *be a generalized hybrid mapping with* $F(U) \neq \emptyset$. *Define* $U_n = \beta_n I + (1-\beta_n)U$ *for all* $n \in \mathbb{N}$ *such that* $0 \leq \beta_n < 1$ *and* $\sup_{n\in\mathbb{N}} \beta_n < 1$. *Let* $x_1 \in C$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = U_n x_n, \\ C_n = \{z \in C : \|z - z_n\| \leq \|z - x_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ *converges strongly to a point* $z_0 \in F(U)$, *where* $z_0 = P_{F(U)} x_1$.

Proof. Since U is a generalized hybrid mapping of C into itself such that $F(U) \neq \emptyset$, U is 0-demimetric. Thus, the inequality

$$2\langle x_n - z, J_E(x_n - z_n)\rangle \geq (1-\eta_n)\|x_n - z_n\|^2$$

in Theorem 3.2 is

$$2\langle x_n - z, x_n - z_n\rangle \geq \|x_n - z_n\|^2.$$

Using this inequality and $2\langle x_n - z, x_n - z_n\rangle = \|x_n - z_n\|^2 + \|z - x_n\|^2 - \|z - z_n\|^2$, we have $\|z - z_n\|^2 \leq \|z - x_n\|^2$, that is, $\|z - z_n\| \leq \|z - x_n\|$ in Theorem 4.4. As in the proof of Theorem 4.3, we also have that for $x \in C$ and $p \in F(U_n) = F(U)$,

$$\begin{aligned} 2\langle x - p, x - U_n x\rangle &= 2\langle x - p, x - (\beta_n x + (1 - \beta_n)Ux)\rangle \\ &= 2(1 - \beta_n)\langle x - p, x - Ux\rangle \\ &\geq (1 - \beta_n)\|x - Ux\|^2 \\ &\geq \|x - U_n x\|^2 \end{aligned}$$

and hence $\{U_n\}$ is a family of 0-demimetric mappings of C into C such that $F(U) = \cap_{n=1}^{\infty} F(U_n)$. Furthermore, let $\{u_n\}$ be a bounded sequence of C such that $u_n - U_n u_n \to 0$. Then we have

$$(1 - \alpha_n)(u_n - Uu_n) = u_n - U_n u_n \to 0$$

and hence $u_n - Uu_n \to 0$. It follows from Lemma 4.2 that every weak cluster point of $\{u_n\}$ belongs to $F(U) = \cap_{n=1}^{\infty} F(U_n)$. This means that the family $\{U_n\}$ satisfies the condition (I). Therefore, putting $\alpha_n = 0$ in Theorem 3.2, we have the desired result from Theorem 3.2. □

As a direct result of Theorem 4.4, we have the following theorem proved by Nakajo and Takahashi [15].

Theorem 4.5 ([15]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $U : C \to C$ be a nonexpansive mapping with $F(U) \neq \emptyset$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = \beta_n x_n + (1 - \beta_n)Ux_n, \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\beta_n\}$ is chosen so that, for some $a \in \mathbb{R}$,

$$0 \leq \beta_n \leq a < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in F(U)$, where $z_0 = P_{F(U)} x_1$.

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. For $\alpha > 0$, a mapping $A : C \to H$ is called α-inverse strongly monotone if

$$\langle x - y, Ax - Ay\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

If A is α-inverse strongly monotone and $0 < \lambda \leq 2\alpha$, then $I - \lambda A : C \to H$ is nonexpansive. In fact, we have that for all $x, y \in C$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + \lambda^2\|Ax - Ay\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \|x-y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}
$$

Thus, $I - \lambda A$ is nonexpansive; see [1, 13, 21] for more results of inverse strongly monotone mappings. The variational inequalty problem for $A : C \to H$ is to find a point $u \in C$ such that

$$\langle Au, x - u\rangle \geq 0, \quad \forall x \in C. \tag{4.1}$$

The set of solutions of (4.1) is denoted by $VI(C, A)$. We also have that, for $\lambda > 0$, $u = P_C(I - \lambda A)u$ if and only if $u \in VI(C, A)$. In fact, let $\lambda > 0$. Then, for $u \in C$,

$$
\begin{aligned}
u = P_C(I - \lambda A)u &\iff \langle (I - \lambda A)u - u, u - y\rangle \geq 0, \quad \forall y \in C \\
&\iff \langle -\lambda Au, u - y\rangle \geq 0, \quad \forall y \in C \\
&\iff \langle Au, u - y\rangle \leq 0, \quad \forall y \in C \\
&\iff \langle Au, y - u\rangle \geq 0, \quad \forall y \in C \\
&\iff u \in VI(C, A).
\end{aligned}
$$

Using these results, we obtain the following theorem for inverse strongly monotone operators in a Hilbert space.

Theorem 4.6. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\gamma > 0$ and let $A : C \to H$ be a γ-inverse strongly monotone operator with $VI(C, A) \neq \emptyset$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
z_n = \alpha_n x_n + (1 - \alpha_n)P_C(I - \lambda_n A)x_n, \\
C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\
Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\
x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}.
\end{cases}
$$

where, for some $a, b \in \mathbb{R}$,

$$0 \leq \alpha_n \leq a < 1 \quad \text{and} \quad 0 < b \leq \lambda_n \leq 2\gamma, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in VI(C, A)$, where $z_0 = P_{VI(C,A)}x_1$.

Proof. Define $U_n = P_C(I - \lambda_n A)$ for all $n \in \mathbb{N}$. Since U_n is a nonexpansive mapping of C into itself, U_n is 0-demimetric. As in the proof of Theorem 4.4, the inequality

$$2\langle x_n - z, J_E(x_n - z_n)\rangle \geq (1 - \eta_n)\|x_n - z_n\|^2$$

in Theorem 3.2 is $\|z - z_n\| \leq \|z - x_n\|$ in Theorem 4.6. We also know that $\cap_{n=1}^{\infty} F(U_n) = VI(C, A)$. Furthermore, let $\{z_n\}$ be a bounded sequence of C such that $z_n - P_C(I - \lambda_n A)z_n \to 0$. Then, without loss of generality, we may assume that $z_n \rightharpoonup p$ for some $p \in C$ and $z_n - P_C(I - \lambda_n A)z_n \to 0$. Put $v_n = P_C(I - \lambda_n A)z_n$. Since P_C is the metric projection, we have that

$$\langle (I - \lambda_n A)z_n - v_n, v_n - u\rangle \geq 0, \quad \forall u \in C.$$

Thus, we have that for all $n \in \mathbb{N}$,

$$\langle \lambda_n Au - \lambda_n Av_n, u - v_n\rangle \geq \langle (I - \lambda_n A)z_n - v_n, u - v_n\rangle + \langle \lambda_n Au - \lambda_n Av_n, u - v_n\rangle.$$

Then, we have that

$$\langle \lambda_n Au, u - v_n\rangle \geq \langle (I - \lambda_n A)z_n - v_n + \lambda_n Av_n, u - v_n\rangle + \langle \lambda_n Au - \lambda_n Av_n, u - v_n\rangle.$$

Since A is monotone and $\lambda_n > 0$, we have that

$$\begin{aligned}
\langle Au, u - v_n\rangle &\geq \frac{1}{\lambda_n}\langle (I - \lambda_n A)z_n - v_n + \lambda_n Av_n, u - v_n\rangle + \langle Au - Av_n, u - v_n\rangle \\
&= \left\langle \frac{z_n - v_n}{\lambda_n} + Av_n - Az_n, u - v_n \right\rangle + \langle Au - Av_n, u - v_n\rangle \\
&= \left\langle \frac{z_n - v_n}{\lambda_n} + Av_n - Au + Au - Az_n, u - v_n \right\rangle + \langle Au - Av_n, u - v_n\rangle \\
&= \left\langle \frac{z_n - v_n}{\lambda_n} + Au - Az_n, u - v_n \right\rangle \\
&= \left\langle \frac{z_n - v_n}{\lambda_n}, u - v_n \right\rangle + \langle Au - Az_n, u - z_n + z_n - v_n\rangle \\
&\geq \left\langle \frac{z_n - v_n}{\lambda_n}, u - v_n \right\rangle + \langle Au - Az_n, z_n - v_n\rangle.
\end{aligned} \tag{4.2}$$

From $z_n - v_n \to 0$ and $\frac{z_n - v_n}{\lambda_n} \to 0$, we have

$$\langle Au, u - p\rangle \geq 0. \tag{4.3}$$

Take $u_t = (1-t)p + ty$ for all $t \in (0,1)$ and $y \in C$. From (4.3) and $t > 0$, we have that $\langle Au_t, u_t - p\rangle \geq 0$ and hence $\langle Au_t, y - p\rangle \geq 0$. Since $u_t = (1-t)p + ty \to p$ as $t \to 0$ and A is continuous, we have that $\langle Ap, y - p\rangle \geq 0$. This implies $p \in VI(C, A)$ from which it follows that the family $\{U_n\}$ satisfies the condition (I). Therefore, we have the desired result from Theorem 3.2. $\square$

Let C be a nonempty subset of a Hilbert space H. A family $\mathbf{S} = \{T(t) : t \in [0,\infty)\}$ of mappings of C into itself satisfying the following conditions is said to be a one-parameter nonexpansive semigroup on C:

(1) For each $t \in [0,\infty)$, $T(t)$ is nonexpansive;
(2) $T(0) = I$;
(3) $T(t+s) = T(t)T(s)$ for every $t, s \in [0,\infty)$;
(4) for each $x \in C$, $t \mapsto T(t)x$ is continuous.

Theorem 4.7. *Let H be a Hilbert space and let C be a nonempty, closed convex subset of H. Let $\mathbf{S} = \{T(t) : t \in [0,\infty)\}$ be a one-parameter nonexpansive semigroup on C with the common fixed point set $F(\mathbf{S}) = \cap_{t\in[0,\infty)} F(T(t)) \neq \emptyset$. Define $U_n x = \frac{1}{t_n}\int_0^{t_n} T(s)x ds$ for all $x \in C$ and $n \in \mathbb{N}$ with $t_n \to \infty$. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n = \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_1 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where, for some $a \in \mathbb{R}$,

$$0 \leq \alpha_n \leq a < 1, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $z_0 \in F(\mathbf{S})$*, where* $z_0 = P_{F(\mathbf{S})}x_1$.

Proof. Since U_n is a nonexpansive mapping of C into itself, U_n is 0-demimetric. Thus, the inequality

$$2\langle x_n - z, J_E(x_n - z_n)\rangle \geq (1-\eta_n)\|x_n - z_n\|^2$$

in Theorem 3.2 is

$$2\langle x_n - z, x_n - z_n\rangle \geq \|x_n - z_n\|^2.$$

Using this inequality and $2\langle x_n - z, x_n - z_n\rangle = \|x_n - z_n\|^2 + \|z - x_n\|^2 - \|z - z_n\|^2$, we have $\|z - z_n\|^2 \leq \|z - x_n\|^2$, that is, $\|z - z_n\| \leq \|z - x_n\|$ in Theorem 4.7. We also know from [19] that $\cap_{n=1}^{\infty} F(U_n) = F(\mathbf{S})$. Furthermore, let $\{u_n\}$ be a bounded sequence of C such that $u_n - U_n u_n \to 0$. Then we have from [14] that $u_n - T(s)u_n \to 0$ for all $s \in [0, \infty)$. Since $T(s)$ is nonexpansive, every weak cluster point u_0 of $\{u_n\}$ belongs to $F(T(s))$; see [21]. Then, $u_0 \in \cap_{n=1}^{\infty} F(U_n) = F(\mathbf{S})$. This means that the family $\{U_n\}$ satisfies the condition (I). Therefore, we have the desired result from Theorem 3.2. □

Theorem 4.8. *Let* E *be a uniformly convex and smooth Banach space. Let* J_E *be the duality mapping on* E*. Let* A *be a maximal monotone operator of* E *into* E^* *and let* J_λ *be the metric resolvent of* A *for* $\lambda > 0$*. Let* $x_1 \in E$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = \alpha_n x_n + (1-\alpha_n)J_{\lambda_n}x_n, \\ C_n = \{z \in E : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ Q_n = \{z \in E : \langle x_n - z, J_E(x_1 - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}x_1, \quad \forall n \in \mathbb{N}. \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$ *and* $\{\lambda_n\} \subset (0,\infty)$ *satisfy the conditions such that*

$$0 \leq \alpha_n \leq a < 1, \quad \textit{and} \quad 0 < b \leq \lambda_n$$

for some $a, b \in \mathbb{R}$*. Then* $\{x_n\}$ *converges strongly to a point* $z_0 \in A^{-1}0$*, where* $z_0 = P_{A^{-1}0}x_1$.

Proof. Since J_{λ_n} is the metric resolvent of A, J_{λ_n} is (-1)-demimetric. Thus, the inequality

$$2\langle x_n - z, J_E(x_n - z_n)\rangle \geq (1-\eta_n)\|x_n - z_n\|^2$$

in Theorem 3.2 is

$$\langle x_n - z, J_E(x_n - z_n)\rangle \geq \|x_n - z_n\|^2.$$

Using this inequality and $\|x_n - z_n\|^2 = \langle x_n - z_n, J_E(x_n - z_n)\rangle$, we have the inequality

$$\langle z_n - z, J_E(x_n - z_n)\rangle \geq 0$$

in Theorem 4.8. Assume that $\{x_n\}$ is a sequence in E such that $x_n \rightharpoonup p$ and $x_n - J_{\lambda_n}x_n \to 0$. It is clear that $J_{\lambda_n}x_n \rightharpoonup p$ and $\|J_E(x_n - J_{\lambda_n}x_n)\| = \|x_n - J_{\lambda_n}x_n\| \to 0$. Since J_{λ_n} is the metric resolvent of A, we have that

$$\frac{x_n - J_{\lambda_n}x_n}{\lambda_n} \in AJ_{\lambda_n}x_n.$$

Since A is monotone, we have

$$\left\langle J_{\lambda_n}x_n - u, \frac{x_n - J_{\lambda_n}x_n}{\lambda_n} - v^* \right\rangle \geq 0$$

for all $(u, v^*) \in A$. From $J_{\lambda_n}x_n \rightharpoonup p$, $J_E(x_n - J_{\lambda_n}x_n) \to 0$ and $0 < b \leq \lambda_n$, we have that

$$\langle p - u, -v^* \rangle \geq 0.$$

Since A is maximal, we get $0 \in Ap$ and hence $p = J_{\lambda_n}p$. Therefore, we have the desired result from Theorem 3.2. □

Theorem 4.8 is the result of Ohsawa and Takahashi [16]. Similarly, using Theorem 3.3, we have the following results.

Theorem 4.9. *Let H be a Hilbert space. Let k be a real number with $k \in [0, 1)$. Let C be a nonempty, closed and convex subset of H and let $U : C \to C$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Define $U_n = \beta_n I + (1 - \beta_n)U$ for all $n \in \mathbb{N}$ such that $0 \leq \beta_n < 1$ and $\sup_{n\in\mathbb{N}} \beta_n < 1$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = U_n x_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, x_n - z_n\rangle \geq (1 - k)\|x_n - z_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ converges strongly to $z_0 \in F(U)$, where $z_0 = P_{F(U)}x_1$.

Theorem 4.10. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $U : C \to C$ be a generalized hybrid mapping with $F(U) \neq \emptyset$. Define $U_n = \beta_n I + (1 - \beta_n)U$ for all $n \in \mathbb{N}$ such that $0 \leq \beta_n < 1$ and $\sup_{n\in\mathbb{N}} \beta_n < 1$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = U_n x_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|x_n - z_n\|\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}. \end{cases}$$

Then $\{x_n\}$ converges strongly to $z_0 \in F(U)$, where $z_0 = P_{F(U)}x_1$.

Using Theorem 3.3, we also have the following strong convergence theorem for finding a zero point of a maximal monotone operator in a Banach space.

Theorem 4.11. *Let E be a uniformly convex and smooth Banach space. Let J_E be the duality mapping on E. Let A be a maximal monotone operator of E. Let J_λ be the metric resolvent of A for $\lambda > 0$. Suppose that $A^{-1}0 \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = \alpha_n x_n + (1 - \alpha_n)J_{\lambda_n}x_n, \\ C_{n+1} = \{z \in C_n : \langle z_n - z, J_E(x_n - z_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy the conditions such that

$$0 \leq \alpha_n \leq a < 1, \quad \textit{and} \quad 0 < b \leq \lambda_n$$

for some $a, b \in \mathbb{R}$. *Then* $\{x_n\}$ *converges strongly to a point* $z_0 \in A^{-1}0$, *where* $z_0 = P_{A^{-1}0}x_1$.

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