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YOUNG'S INEQUALITY IS A HEAVEN'S BLESSING

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Dedicated to Professor Makoto Tsukada on the occasion of his retirement from Toho University

Abstract. We will show that Young's inequality is special in the sense that any best Young-type inequality with no redundancy is reduced to the original Young's inequality. Moreover, we will clarify the necessity of the exponent condition of Young's inequality from a general point of view.

1. Introduction

From ancient times, a sense of beauty would be inspired by the discovery of a certain relationship between different kinds of things. In the Genpei war, warlord Nasu no Yoichi of Genji was connected with the court lady of Heike by shooting an arrow into her fan, where we feel the beauty. We would feel further beauty in that Yoichi might be prepared for suicide if he failed to hit the fan. In mathematics, most mathematicians may feel the supreme beauty in Euler's formula that connects exponential functions and trigonometric functions or Galois theory that builds a bridge between groups and fields. We believe that the well-known Young's inequality also belongs to such a category. Let us explain the reason.

Young's inequality[1] is as follows:

Given real numbers p and q with $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy \text{ for } \forall x, y > 0.$$

This inequality immediately follows from the fact that the logarithmic function is concave:

$$\log(\frac{x^p}{p} + \frac{y^q}{q}) \geq \frac{1}{p}\log x^p + \frac{1}{q}\log y^p = \log(xy).$$

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[1]In this paper, we will focus on the discrete version of Young's inequality only.

In particular, when p and q are rational integers, it is easy to see that neither can be other than 2, and then Young's inequality has become

$$\frac{x^2+y^2}{2} \geq xy \text{ for } \forall x, y > 0,$$

or (by an appropriate variable transformation) equivalently

$$\frac{x+y}{2} \geq \sqrt{xy} \text{ for } \forall x, y > 0.$$

Namely, as is well-known, Young's inequality can be thought of as a generalization or a weighted version of the arithmetic-geometric mean inequality. However, we would like to claim that Young's inequality itself has some profound meanings such as

> "The product of real numbers can be estimated by a certain kind of sum of real numbers."

In other words, Young's inequality reveals a relationship hidden behind the two different kinds of operations: product and sum. This might be one reason why it is applied in various areas of mathematics. For example, Hölder-Rogers' inequality which leads to Minkowski's inequality is derived from Young's inequality.

Along these lines, we have recently found that Young's inequality is special in the sense that any best one in the family of Young-type inequalities is reduced to the original Young's inequality, and so might be a "Heaven's blessing". In this paper, we shall show what it means in detail. Moreover, we shall present an answer to the simple question "Why the exponent condition $\frac{1}{p}+\frac{1}{q}=1$ is necessary for Young's inequality?" from a general point of view.

2. Why Young's inequality is special?

Given real numbers p, q, α and β, let us consider the inequality

$$\mathrm{Y}_{(\alpha,\beta)}: \ \alpha x^p + \beta y^q \geq xy \text{ for } \forall x, y > 0. \tag{2.1}$$

We can see that each point (α, β) in the real plain $\mathbb{R}^2$ one-to-one corresponds to the Young-type inequality $\mathrm{Y}_{(\alpha,\beta)}$ in (2.1). This view also appears in [1, 2] and so on. Obviously, when

$$(\alpha, \beta, p, q) = \left(\frac{1}{t}, \frac{t-1}{t}, t, \frac{t}{t-1}\right) \text{ where } t > 1,$$

$\mathrm{Y}_{(\alpha,\beta)}$ is the original Young's inequality itself, but as will be seen, we can determine (α, β, p, q) for which $\mathrm{Y}_{(\alpha,\beta)}$ holds.

If $\mathrm{Y}_{(\alpha,\beta)}$ holds, it follows that $p, q > 1$ and $\alpha, \beta > 0$. In fact, obviously, it is impossible that $\alpha, \beta \leq 0$. Suppose that $\alpha > 0$ and $\beta = 0$. Take $x = a$ and $y = b$ where $a, b > 0$. Then we have $\alpha a^p \geq ab$, but as $b \to \infty$ we will arrive at a contradiction. Next, suppose that $\alpha > 0$ and $\beta < 0$. Take $x = a$ and $y = b^{\frac{1}{q}}$ where $a, b > 0$. Then we have $\alpha a^p + \beta b \geq ab^{\frac{1}{q}}$, but again as $b \to \infty$, we will reach a contradiction. Similarly, it is impossible that $\alpha \leq 0$ and $\beta > 0$. Hence, we have $\alpha, \beta > 0$. Also, by putting $y = 1$, we have $\alpha x^p + \beta \geq x$ for $\forall x > 0$ and so $p \geq 1$. Similarly we have $q \geq 1$. If p were equal to 1, then we would have $\beta y^q \geq (y-\alpha)x$ for $\forall x, y > 0$. However, when $y = \alpha + 1$, the inequality would become $\beta(\alpha+1)^q \geq x$, and as $x \to \infty$ we would arrive at a contradiction. Thus we have $p > 1$, and similarly $q > 1$.

Next let us consider the function $f : \mathbb{R}_+^2 \to \mathbb{R}$ where $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, defined by

$$f(x, y) = \alpha x^p + \beta y^q - xy.$$

Since $\frac{\partial f}{\partial x}(x, y) = \alpha p x^{p-1} - y$, for an arbitrary fixed $y > 0$, we have

$$\frac{\partial f}{\partial x}(x,y) \begin{cases} < 0 & \text{if } 0 < x < \left(\dfrac{y}{\alpha p}\right)^{\frac{1}{p-1}} \\ = 0 & \text{if } x = \left(\dfrac{y}{\alpha p}\right)^{\frac{1}{p-1}} \\ > 0 & \text{if } \left(\dfrac{y}{\alpha p}\right)^{\frac{1}{p-1}} < x < +\infty. \end{cases}$$

Hence, it follows that $\mathrm{Y}_{(\alpha,\beta)}$ holds if and only if

$$f\left(\left(\frac{y}{\alpha p}\right)^{\frac{1}{p-1}}, y\right) \geq 0 \text{ for } \forall y > 0.$$

By an easy calculation this is rewritten to

$$\mathrm{Z}_{(\alpha,\beta)} : \ \beta y^q \geq \left(\left(\frac{1}{\alpha p}\right)^{\frac{1}{p-1}} - \alpha\left(\frac{1}{\alpha p}\right)^{\frac{p}{p-1}}\right) y^{\frac{p}{p-1}} \text{ for } \forall y > 0. \tag{2.2}$$

Therefore, $\mathrm{Y}_{(\alpha,\beta)}$ holds if and only if $\mathrm{Z}_{(\alpha,\beta)}$ holds. However, since

$$\left(\frac{1}{\alpha p}\right)^{\frac{1}{p-1}} - \alpha\left(\frac{1}{\alpha p}\right)^{\frac{p}{p-1}} = \alpha^{\frac{1}{1-p}} p^{\frac{1}{1-p}}\left(1 - \frac{1}{p}\right) > 0,$$

$\mathrm{Z}_{(\alpha,\beta)}$ holds if and only if

$$\begin{cases} p, q > 1, \alpha, \beta > 0 \\ q = \dfrac{p}{p-1} \\ \beta \geq p^{\frac{1}{1-p}}\left(1 - \dfrac{1}{p}\right)\alpha^{\frac{1}{1-p}}. \end{cases}$$

Thus, this is a necessary and sufficient condition for $\mathrm{Y}_{(\alpha,\beta)}$ to hold. Consequently, if we fix p and q, then the point (α, β) on the curve:

$$C_H : \ Y = p^{\frac{1}{1-p}}\left(1 - \frac{1}{p}\right) X^{\frac{1}{1-p}} \text{ where } X > 0$$

in the X-Y plane gives the best inequality:

$$Y_\alpha : \ \alpha x^p + p^{\frac{1}{1-p}}\left(1 - \frac{1}{p}\right)\alpha^{\frac{1}{1-p}} y^q \geq xy \text{ where } x, y > 0 \text{ and } \alpha > 0.$$

In particular, the point $\left(\frac{1}{p}, \frac{1}{q}\right)$ on C_H provides the original Young's inequality.

Now, since α is an arbitrary positive real number, there seems to be infinitely many different best inequalities Y_α's. However, since $\frac{1}{q} = 1 - \frac{1}{p}$, Y_α is rewritten to

$$\alpha p^{\frac{1}{p-1}} \alpha^{\frac{1}{p-1}} x^p + \frac{y^q}{q} \geq p^{\frac{1}{p-1}} \alpha^{\frac{1}{p-1}} xy \text{ where } x, y > 0,$$

and so Y_α is reduced to Young's inequality

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy \text{ for } \forall x, y > 0,$$

by replacing $p^{\frac{1}{p-1}} \alpha^{\frac{1}{p-1}} x$ by x. This is why Young's inequality is special and might be a "Heaven's blessing".

Up to here we have primarily discussed Young's inequality from an analytic viewpoint. Let us consider it from a somewhat algebraic viewpoint by making the number of variables general. Let n be an arbitrary positive integer.

Given real numbers $p_1, \ldots, p_n$ with $p_1, \ldots, p_n > 1$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_n} = 1$, we have

$$Y_n : \frac{x_1^{p_1}}{p_1} + \cdots + \frac{x_n^{p_n}}{p_n} \geq x_1 \cdots x_n \text{ for } \forall x_1, \ldots, x_n > 0.$$

Needless to say, Y_1 is a trivial inequality and Y_2 is the original Young's inequality. Similarly to Y_2, Y_n can be proved by the concavity of the logarithmic function, but here to show the greatness of Y_2 we will use induction on n. Let Y_2 be the base case of induction and $n \geq 3$. Assume that Y_k is true for $\forall k < n$. First apply Y_2 to x_1 and $x_2 \cdots x_n$ with p_1 and $\frac{p_1}{p_1 - 1}$ to get

$$\frac{x_1^{p_1}}{p_1} + \frac{(x_2 \cdots x_n)^{\frac{p_1}{p_1-1}}}{\frac{p_1}{p_1-1}} \geq x_1(x_2 \cdots x_n).$$

Then apply Y_{n-1} to $x_2^{\frac{p_1}{p_1-1}}, \ldots, x_n^{\frac{p_1}{p_1-1}}$ with $q_2 = \frac{p_1-1}{p_1} p_2, \ldots, q_n = \frac{p_1-1}{p_1} p_n$, respectively, where $\frac{1}{q_2} + \cdots + \frac{1}{q_n} = 1$ since $\frac{1}{p_1} + \left(\frac{1}{p_2} + \cdots + \frac{1}{p_n} \right) = 1$. Then we have

$$\frac{x_1^{p_1}}{p_1} + \frac{p_1 - 1}{p_1} \left(\frac{(x_2^{\frac{p_1}{p_1-1}})^{q_2}}{q_2} + \cdots + \frac{(x_n^{\frac{p_1}{p_1-1}})^{q_n}}{q_n} \right) \geq x_1 x_2 \cdots x_n.$$

Hence, we obtain the desired Y_n.

Notice that Y_2 cannot be derived from Y_1, and so Y_2 is the simplest inequality that generates Y_n. This might be another viewpoint from which Young's inequality is special.

Young's inequality was named after the English mathematician William Henry Young (1863-1942), who must be saying "You have just realized that?" with a wry smile in his grave.

3. Generalization and the exponent condition

Let us consider a general Young-type inequality

$$f(x) + g(y) \geq xy \text{ for } \forall x, y > 0, \tag{3.1}$$

where f and g are mappings from $\mathbb{R}_+$ to $\mathbb{R}_+$. Now if we put

$$H(x, y) = f(x) + g(y) - xy \text{ where } x, y > 0,$$

then the given inequality (3.1) is equivalent to

$$H(x, y) \geq 0 \text{ for } \forall x, y > 0.$$

Therefore, if we have

$$\begin{cases} \inf_{x>0} H(x, y) = 0 \text{ for } \forall y > 0 \\ \inf_{y>0} H(x, y) = 0 \text{ for } \forall x > 0, \end{cases} \tag{3.2}$$

(3.1) can be thought of as the inequality with no redundancy. In this sense, (3.2) serves as the best condition for (3.1).

If f and g have a certain *good* property, we shall present a necessary and sufficient condition for them to satisfy (3.2). Let us strictly define the good property. The function f is *a good function* if and only if f is a C^2 function on $\mathbb{R}_+$ such that

$$\lim_{x \to +0} f(x) = 0$$

and its derivative f' is a strictly monotone increasing function such that

$$\lim_{x \to +0} f'(x) = 0$$

and

$$\lim_{x \to +\infty} f'(x) = +\infty.$$

Hence, f' has the inverse function f'^{-1} which is a C^2 function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then we have the following lemma.

Lemma 3.1. *Assume that f and g are good functions. Then, f and g satisfy the first condition of (3.2) if and only if $g' = f'^{-1}$ holds.*

Proof. First, suppose that f and g satisfy the first condition of (3.2). Since $\frac{\partial H}{\partial x} = f'(x) - y$, the function $H(\cdot, y)$ takes the minimum

$$m(y) := f(f'^{-1}(y)) + g(y) - f'^{-1}(y)y \tag{3.3}$$

at $x = f'^{-1}(y)$. Thus, by assumption, we have

$$m(y) := f(f'^{-1}(y)) + g(y) - f'^{-1}(y)y = 0 \text{ for } \forall y > 0.$$

By differentiating $m(y)$ with respect to y, we have

$$\begin{aligned} 0 = m'(y) &= f'(f'^{-1}(y))(f'^{-1})'(y) + g'(y) - (f'^{-1})'(y)y - f'^{-1}(y) \\ &= y(f'^{-1})'(y) + g'(y) - (f'^{-1})'(y)y - f'^{-1}(y) \\ &= g'(y) - f'^{-1}(y) \end{aligned}$$

for all $y > 0$, and so $g' = f'^{-1}$.

Conversely, suppose that $g' = f'^{-1}$. Define $m(y)$ by (3.3). As we have observed above, we have

$$g'(y) - f'^{-1}(y) = m'(y) \text{ for } \forall y > 0,$$

namely, $m(y)$ is a constant c. Then we have

$$f(f'^{-1}(y)) + g(y) - f'^{-1}(y)y = c \text{ for } \forall y > 0.$$

As $y \to +0$, from the goodness of f and g, we will have $c = 0$ and so they satisfy the first condition of (3.2). □

Remark. If f and g are good functions, from Lemma 3.1, the first condition of (3.2) is equivalent to the second.

In particular, take $f(x) = \alpha x^p$ and $g(x) = \beta x^q$ where $x > 0$ as good functions. Here $\alpha, \beta > 0$ and $p, q > 1$. Then, it immediately follows from Lemma 3.1 that f and g satisfy (3.2) if and only if $\frac{1}{p} + \frac{1}{q} = 1$ and the point (α, β) is on the curve C_H. Accordingly, we realize the necessity of the exponent condition $\frac{1}{p} + \frac{1}{q} = 1$.

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