Linear and Nonlinear Analysis
Volume 3, Number 3, 2017, 343–348

SOME GENERALIZATIONS OF NON-HERMITIAN UNCETRAINTY RELATION DESCRIBED BY THE GENERALIZED QUASI-METRIC ADJUSTED SKEW INFORMATION

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Abstract. Recently in [10] we obtained non-hermitian extensions of Heisenberg type and Schrödinger type uncertainty relations for generalized quasi-metric adjusted skew information or generalized quasi-metric adjusted correlation measure and applied to the inequalities related to fidelity and trace distance for different two generalized states which were given by Audenaert et al; and Powers-Stϕrmer [1, 2, 5]. In this paper we give some more generalizations of these uncertainty relations and show that several results obtained in [3, 5] are given as the corollaries in our theorems.

1. Introduction

Let $M_n(\mathbb{C})$(resp. $M_{n,sa}(\mathbb{C})$) be the set of all $n \times n$ complex matrices (resp. all $n \times n$ self-adjoint matrices), endowed with the Hilbert-Schmidt scalar product $\langle X, Y \rangle = Tr[X^*Y]$. Let $M_{n,+}(\mathbb{C})$ be the set of strictly positive elements of $M_n(\mathbb{C})$ and $M_{n,+,1}(\mathbb{C})$ be the set of density matrices. A function $f : (0, +\infty) \to \mathbb{R}$ is said operator monotone if, for any $n \in \mathbb{N}$, and $A, B \in M_{n,+}(\mathbb{C})$ such that $0 \le A \le B$, the inequality $0 \le f(A) \le f(B)$ holds. An operator monotone function is said symmetric if $f(x) = xf(x^{-1})$ and normalized if $f(1) = 1$.

Definition 1.1. *Let $\mathfrak{F}_{op}$ be the class of functions $f : (0, +\infty) \to (0, +\infty)$ satisfying*

(1) $f(1) = 1$,
(2) $tf(t^{-1}) = f(t)$,
(3) f *is operator monotone.*

Example 1.2. *Examples of elements of $\mathfrak{F}_{op}$ are given by the following list, for any $x > 0$,*

$$f_{RLD}(x) = \frac{2x}{x+y},\ f_{SLD}(x) = \frac{x+1}{2},\ f_{BKM}(x) = \frac{x-1}{\log x},$$

$$f_{WY}(x) = \left(\frac{\sqrt{x}+1}{2}\right)^2,\ f_{WYD}(x) = \alpha(1-\alpha)\frac{(x-1)^2}{(x^\alpha - 1)(x^{1-\alpha}-1)}, \alpha \in (0,1).$$

2010 *Mathematics Subject Classification.* 15A45, 47A63, 94A17.
Key words and phrases. Generalized quasi-metric adjusted skew information, non-hermitian observable, uncertainty relation.
The author was partially supported by JSPS KAKENHI Grant Number 26400119.

For $f \in \mathfrak{F}_{op}$ define $f(0) = \lim_{x\to 0} f(x)$. We introduce the sets of regular and non-regular functions

$$\mathfrak{F}_{op}^r = \{f \in \mathfrak{F}_{op} | f(0) \neq 0\}, \ \ \mathfrak{F}_{op}^n = \{f \in \mathfrak{F}_{op} | f(0) = 0\}$$

and notice that trivially $\mathfrak{F}_{op} = \mathfrak{F}_{op}^r \cup \mathfrak{F}_{op}^n$. In Kubo-Ando theory of matrix means one associates a mean to each operator monotone function $f \in \mathfrak{F}_{op}$ by the formula

$$m_f(A,B) = A^{1/2} f(A^{-1/2} B A^{-1/2}) A^{1/2},$$

where $A, B \in M_{n,+}(\mathbb{C})$. By using the notion of matrix means we define the generalized monotone metrics for $X, Y \in M_n(\mathbb{C})$ by the following formula

$$\langle X, Y \rangle_f = Tr[X^* m_f(L_A, R_B)^{-1} Y],$$

where $L_A(X) = AX, R_B(X) = XB$.

2. Generalized Quasi-Metric Adjusted Skew Information and Correlation Measure

Definition 2.1. *Let $g, f \in \mathfrak{F}_{op}^r$ satisfy*

$$g(x) \geq k \frac{(x-1)^2}{f(x)}$$

for some $k > 0$. We define

$$\Delta_g^f(x) = g(x) - k\frac{(x-1)^2}{f(x)} \in \mathfrak{F}_{op}. \tag{2.1}$$

Definition 2.2. *Notation as in Definition 2.1. For $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$, we define the following quantities:*

(1) $\Gamma_{A,B}^{(g,f)}(X,Y) = k\langle (L_A - R_B)X, (L_A - R_B)Y\rangle_f$
$= kTr[X^*(L_A - R_B) m_f(L_A, R_B)^{-1}(L_A - R_B)Y]$
$= Tr[X^* m_g(L_A, R_B)Y] - Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y]$,

(2) $I_{A,B}^{(g,f)}(X) = \Gamma_{A,B}^{(g,f)}(X,X)$,

(3) $\Psi_{A,B}^{(g,f)}(X,Y) = Tr[X^* m_g(L_A, R_B)Y] + Tr[X^* m_{\Delta_g^f}(L_A, R_B)Y]$,

(4) $J_{A,B}^{(g,f)}(X) = \Psi_{A,B}^{(g,f)}(X,X)$,

(5) $U_\rho^{(g,f)}(X) = \sqrt{I_{A,B}^{(g,f)}(X) J_{A,B}^{(g,f)}(X)}$.

The quantities $I_{A,B}^{(g,f)}(X)$ and $\Gamma_{A,B}^{(g,f)}(X,Y)$ are said generalized quasi-metric adjusted skew information and generalized quasi-metric adjusted correlation measure, respectively.

Theorem 2.3. *For $f \in \mathfrak{F}_{op}^r$, it holds*

$$I_{A,B}^{(g,f)}(X) \cdot I_{A,B}^{(g,f)}(Y) \geq |\Gamma_{A,B}^{(g,f)}(X,Y)|^2 \geq \frac{1}{16}\left(I_{A,B}^{(g,f)}(X+Y) - I_{A,B}^{(g,f)}(X-Y)\right)^2,$$

where $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$.

Proof. Since the first inequality was proved in [10], we prove the second inequality. Since

$$\begin{aligned} I_{A,B}^{(g,f)}(X+Y) &= Tr[(X^*+Y^*)m_g(L_A,R_B)(X+Y)] \\ &\quad -Tr[(X^*+Y^*)m_{\Delta_g^f}(L_A,R_B)(X+Y)], \\ I_{A,B}^{(g,f)}(X-Y) &= Tr[(X^*-Y^*)m_g(L_A,R_B)(X-Y)] \\ &\quad -Tr[(X^*-Y^*)m_{\Delta_g^f}(L_A,R_B)(X-Y)], \end{aligned}$$

we have

$$\begin{aligned} I_{A,B}^{(g,f)}(X+Y)-I_{A,B}^{(g,f)}(X-Y) &= 2Tr[X^*m_g(L_A,R_B)Y]+2TrY^*m_g(L_A,R_B)X] \\ &\quad -2Tr[X^*m_{\Delta_g^f}(L_A,R_B)Y]-2Tr[Y^*m_{\Delta_g^f}(L_A,R_B)X] \\ &= 2\Gamma_{A,B}^{(g,f)}(X,Y)+2\Gamma_{A,B}^{(g,f)}(Y,X) \\ &= 4Re\{\Gamma_{A,B}^{(g,f)}(X,Y)\}. \end{aligned}$$

Similarly we have

$$I_{A,B}^{(g,f)}(X+Y)+I_{A,B}^{(g,f)}(X-Y)=2(I_{A,B}^{(g,f)}(X)+I_{A,B}^{(g,f)}(Y)).$$

Then

$$\begin{aligned} \Gamma_{A,B}^{g,f)}(X,Y) &= Re\{\Gamma_{A,B}^{(g,f)}(X,Y)\}+iIm\{\Gamma_{A,B}^{(g,f)}(X,Y)\} \\ &= \frac{1}{4}(I_{A,B}^{(g,f)}(X+Y)-I_{A,B}^{(g,f)}(X-Y))+iIm\{\Gamma_{A,B}^{(g,f)}(X,Y). \end{aligned}$$

Since

$$\begin{aligned} |\Gamma_{A,B}^{(g,f)}(X,Y)|^2 &= \frac{1}{16}(I_{A,B}^{(g,f)}(X+Y)-I_{A,B}^{(g,f)}(X-Y))^2+(Im\{\Gamma_{A,B}^{(g,f)}(X,Y)\})^2 \\ &\geq \frac{1}{16}(I_{A,B}^{(g,f)}(X+Y)-I_{A,B}^{(g,f)}(X-Y))^2, \end{aligned}$$

we have the result. □

By setting $g=f_{SLD},\ f=f_{WY},\ k=\frac{1}{4},\ A=B=\rho\in M_{n,+,1}(\mathbb{C})$, we have the following corollary.

Corollary 2.4 ([4], Theorem 3.3)**.** *Let $X,Y\in M_n(\mathbb{C})$ and $\rho\in M_{n,+,1}(\mathbb{C})$ be a quantum state. Then*

$$|I_\rho|(X)\cdot|I_\rho|(Y)\geq\frac{1}{16}(|I_\rho|(X+Y)-|I_\rho|(X-Y))^2,$$

where $|I_\rho|(X)=-\frac{1}{2}Tr[[\rho^{1/2},X^*][\rho^{1/2},X]]$ and $[X,Y]=XY-YX$.

We note the equation

$$|L_A-R_B|=\sum_{i=1}^n\sum_{j=1}^n|\lambda_i-\mu_j|L_{|\phi_i\rangle\langle\phi_i|}R_{|\psi_j\rangle\langle\psi_j|},$$

where $A=\sum_{i=1}^n\lambda_i|\phi_i\rangle\langle\phi_i|,\ B=\sum_{j=1}^n\mu_j|\psi_j\rangle\langle\psi_j|$ are the spectral decompositions.

Theorem 2.5. *For $f \in \mathfrak{F}_{op}^r$, if*

$$g(x) + \Delta_g^f(x) \geq \ell f(x) \tag{2.2}$$

for some $\ell > 0$, then the followings hold for $X, Y \in M_n(\mathbb{C})$ and $A, B \in M_{n,+}(\mathbb{C})$

(1) $U_{A,B}^{(g,f)}(X) \cdot U_{A,B}^{(g,f)}(Y) \geq k\ell |Tr[X^*|L_A - R_B|Y]|^2$.

(2) $U_{A,B}^{(g,f)}(X) \cdot U_{A,B}^{(g,f)}(Y) \geq \frac{f(0)^2\ell}{k} |\Gamma_{A,B}^{(g,f)}(X,Y)|^2$.

Proof. Since (1) was proved in [10], we prove (2). By Lemma 3.3 and Lemma 3.4 in [3]

$$m_g(x,y)^2 - m_{\Delta_g^f}(x,y)^2 \geq k\ell(x-y)^2 \geq k\ell \frac{f(0)^2}{k^2}(m_g(x,y) - m_{\Delta_g^f}(x,y))^2.$$

Then

$$m_g(x,y) + m_{\Delta_g^f}(x,y) \geq \frac{f(0)^2\ell}{k}(m_g(x,y) - m_{\Delta_g^f}(x,y)).$$

Hence we have

$$\begin{aligned} J_{A,B}^{(g,f)}(Y) &= \sum_{i,j}\{m_g(\lambda_i,\mu_j) + m_{\Delta_g^f}(\lambda_i,\mu_j)\}|\langle\phi_i|Y|\psi_j\rangle|^2 \\ &\geq \frac{f(0)^2\ell}{k}\sum_{i,j}\{m_g(\lambda_i,\mu_j) - m_{\Delta_g^f}(\lambda_i,\mu_j)\}|\langle\phi_i|Y|\psi_j\rangle|^2 \\ &= \frac{f(0)^2\ell}{k} I_{A,B}^{(g,f)}(Y). \end{aligned}$$

By the first inequality in Theorem 2.3

$$|\Gamma_{A,B}^{(g,f)}(X,Y)|^2 \leq I_{A,B}^{(g,f)}(X) \cdot I_{A,B}^{(g,f)}(Y) \leq I_{A,B}^{(g,f)}(X) \cdot \frac{k}{f(0)^2\ell} J_{A,B}^{(g,f)}(Y).$$

Then

$$I_{A,B}^{(g,f)}(X) \cdot J_{A,B}^{(g,f)}(Y) \geq \frac{f(0)^2\ell}{k}|\Gamma_{A,B}^{(g,f)}(X.Y)|^2.$$

Similarly we have

$$J_{A,B}^{(g,f)}(X) \cdot I_{A,B}^{(g,f)}(Y) \geq \frac{f(0)^2\ell}{k}|\Gamma_{A,B}^{(g,f)}(X.Y)|^2.$$

Therefore we get the result. □

By setting $A = B = \rho \in M_{n,+,1}(\mathbb{C})$ we have the following corollary.

Corollary 2.6 ([3, Theorem 3.5])**.** *If $f, g \in \mathfrak{F}_{op}$ satisfy (2.2), then*

$$U_\rho^{(g,f)}(X) \cdot U_\rho^{(g,f)}(Y) \geq \frac{f(0)^2\ell}{k}|Corr_\rho^{(g,f)}(X,Y)|^2,$$

where $X, Y \in M_n(\mathbb{C})$ and $\rho \in M_{n,+,1}(\mathbb{C})$. Here $U_\rho^{(g,f)}(X)$ and $Corr_\rho^{(g,f)}(X,Y)$ are defined in [3].

Remark 2.7. *When $X = Y \in M_n(\mathbb{C})$, the following holds.*

$$\frac{f(0)^2\ell}{k}|\Gamma_{A,B}^{(g,f)}(X,X)|^2 = \frac{f(0)^2\ell}{k}|I_{A,B}^{(g,f)}(X)|^2 \leq k\ell|Tr[X^*|L_A - R_B|X]|^2.$$

However it is unknown the relationship between $\frac{f(0)^2\ell}{k}|\Gamma_{A,B}^{(g,f)}(X,Y)|$ *and* $k\ell|Tr[X^*|L_A - R_B|Y]|$.

Example 2.8. *Let* $x_{ij} = \langle\phi_i|X|\psi_j\rangle,\ y_{ij} = \langle\phi_i|Y|\psi_j\rangle$.
(1) *When* $g = f_{LSD},\ f = f_{WYD},\ k = \frac{f(0)}{2},\ \ell = 2$,

$$k\ell|Tr[X|L_A - R_B|Y]|^2 = \alpha(1-\alpha)|\sum_{i,j}|\lambda_i - \mu_j|\overline{x_{ij}}y_{ij}|^2.$$

$$\frac{f(0)^2\ell}{k}|\Gamma_{A,B}^{(g,f)}(X,Y)|^2 = \alpha(1-\alpha)|\sum_{i,j}(\lambda_i^{\alpha} - \mu_j^{\alpha})(\lambda_i^{1-\alpha} - \mu_j^{1-\alpha})\overline{x_{ij}}y_{ij}|^2.$$

(2) *When* $g = f_{WY},\ f = f_{WYD},\ k = \frac{f(0)}{8},\ \ell = \frac{3}{2}$,

$$k\ell|Tr[X|L_A - R_B|Y]|^2 = \frac{3}{16}\alpha(1-\alpha)|\sum_{i,j}|\lambda_i - \mu_j|\overline{x_{ij}}y_{ij}|^2,$$

$$\frac{f(0)^2\ell}{k}|\Gamma_{A,B}^{(g,f)}(X,Y)|^2 = \frac{3}{16}\alpha(1-\alpha)|\sum_{i,j}(\lambda_i^{\alpha} - \mu_j^{\alpha})(\lambda_i^{1-\alpha} - \mu_j^{1-\alpha})\overline{x_{ij}}y_{ij}|^2.$$

(3) *When* $g = \left(\frac{x^{\gamma}+1}{2}\right)^{1/\gamma}\ (\frac{3}{4} \leq \gamma \leq 1),\ f = f_{WY},\ k = \frac{1}{16},\ \ell = 2$,

$$k\ell|Tr[X|L_A - R_B|Y]|^2 = \frac{1}{8}|\sum_{i,j}|\lambda_i - \mu_j|\overline{x_{ij}}y_{ij}|^2,$$

$$\frac{f(0)^2\ell}{k}|\Gamma_{A,B}^{(g,f)}(X,Y)|^2 = \frac{1}{8}|\sum_{i,j}(\sqrt{\lambda_i} - \sqrt{\mu_j})^2\overline{x_{ij}}y_{ij}|^2.$$

(4) *When* $g = \left(\frac{x^{\gamma}+1}{2}\right)^{1/\gamma}\ (\frac{5}{8} \leq \gamma \leq 1),\ f = f_{WY},\ k = \frac{1}{32},\ \ell = 2$,

$$k\ell|Tr[X|L_A - R_B|Y]|^2 = \frac{1}{16}|\sum_{i,j}|\lambda_i - \mu_j|\overline{x_{ij}}y_{ij}|^2,$$

$$\frac{f(0)^2\ell}{k}|\Gamma_{A,B}^{(g,f)}(X,Y)|^2 = \frac{1}{16}|\sum_{i,j}(\sqrt{\lambda_i} - \sqrt{\mu_j})^2\overline{x_{ij}}y_{ij}|^2.$$

Acknowledgements. The author dedicates to Professor Makoto Tsukada who retired from Toho University.

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Manuscript received 25 December 2016

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