Linear and Nonlinear Analysis
Volume 3, Number 3, 2017, 385–391

A SUFFICIENT CONDITION FOR A STRONG FORM OF THE EGOROV THEOREM IN NON-ADDITIVE MEASURE THEORY

TOSHIAKI MUROFUSHI AND SATORU SUJINO

Abstract. This paper shows that the conjunction of the continuity from below, the uniform subadditive continuity, and the order continuity of the non-additive measure is a sufficient condition for the consequent of the strong form of the Egorov theorem, i.e., for the statement that strong almost everywhere convergence implies strong almost uniform convergence; a sequence $\{f_n\}$ of measurable functions is said to converge *strongly almost everywhere* to a measurable function f if $\{f_n\}$ converges pointwise to f except on a strong null set, where a strong null set w.r.t. a non-additive measure μ is defined to be a measurable set N such that $\mu(N \cup B) = \mu(B)$ for every measurable set B; a sequence $\{f_n\}$ of measurable functions is said to converge *strongly almost uniformly* to a measurable function f w.r.t. a non-additive measure μ if, for every $\varepsilon > 0$, there exists a measurable set A_ε such that $\mu(A_\varepsilon \cup B) \leq \mu(B) + \varepsilon$ for every measurable set B and $\{f_n\}$ converges uniformly to f on the complement of A_ε.

1. Introduction

The Egorov theorem in the classical measure theory asserts that, if the measure is finite, then almost everywhere convergence implies almost uniform convergence. The non-additive measure theory has at least three mutually nonequivalent definitions of null set (for instance, [4, 5]). The corresponding definitions of almost everywhere convergence are also nonequivalent to one another. The same ought to apply to the definition of almost uniform convergence as well. Therefore, in non-additive measure theory, the consequent of the Egorov theorem, i.e, the implication of almost uniform convergence from almost everywhere convergence, has mutually nonequivalent forms; none of which holds without additional conditions other than the finiteness of the non-additive measure. The papers [4, 6] discuss conditions for the consequent of the Egorov theorem based on two nonequivalent definitions of null set. This paper discusses the Egorov theorem based on the definition of null set in [5], which we call the strong form of the Egorov theorem, and the paper gives a sufficient condition for the consequent of the strong form of the Egorov theorem, which has not been discussed so far.

This paper deals with the concepts of null set, almost everywhere convergence, and almost uniform convergence defined in two nonequivalent ways; we distinguish them by the adjectives "weak" and "strong."

2010 *Mathematics Subject Classification.* 28E10.

Key words and phrases. non-additive measure, the Egorov theorem, null set, almost everywhere convergence, almost uniform convergence.

2. Preliminaries: definitions and existing results

Throughout this paper, $(X, \mathcal{F})$ is assumed to be a measurable space.

Definition 2.1. A *non-additive measure* on $\mathcal{F}$ is a set function $\mu : \mathcal{F} \to [0, \infty]$ satisfying the following two conditions:

(1) $\mu(\emptyset) = 0$,
(2) $A, B \in \mathcal{F}, A \subset B \Rightarrow \mu(A) \le \mu(B)$.

In the rest of the paper, μ is assumed to be a non-additive measure on $\mathcal{F}$.

Definition 2.2. (1) μ is said to be *continuous from below* [resp. *continuous form above*] if, for every increasing [resp. decreasing] sequence $\{A_n\}$ of measurable sets, it holds that $\mu(\lim_{n\to\infty} A_n) = \lim_{n\to\infty} \mu(A_n)$.

(2) [3] μ is said to be *order continuous* if, for every decreasing sequence $\{A_n\}$ of measurable sets such that $A_n \downarrow \emptyset$, it holds that $\lim_{n\to\infty} \mu(A_n) = 0$.

(3) μ is said to be *subadditive* if the inequality

$$\mu(A \cup B) \le \mu(A) + \mu(B)$$

holds whenever A and B are measurable sets.

(4) [2, 7] μ is said to be *uniformly subadditively continuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$\mu(A \cup B) \le \mu(B) + \varepsilon$$

holds whenever A and B are measurable sets and $\mu(A) < \delta$.

(5) [7] μ is said to be *null-additive* if $\mu(A \cup B) = \mu(B)$ whenever A and B are measurable sets and $\mu(A) = 0$.

In [7], uniform subadditive continuity is called uniform autocontinuity.

By definition, subadditivity implies uniform subadditive continuity, and uniform subadditive continuity implies null-additivity.

Remark 2.3. In the above-mentioned definition of uniform subadditive continuity, and in the below-mentioned definitions of strong null set and strong almost uniform convergence, the comparison between $\mu(A \cup B)$ and $\mu(B)$ can be replaced with the comparison between $\mu(B \setminus A)$ and $\mu(B)$ and with the comparison between $\mu(A \triangle B)$ and $\mu(B)$; that is, for every $\varepsilon \ge 0$ and for every measurable set A, the following holds:

$$\begin{aligned} \forall B \in \mathcal{F}\ [\mu(A \cup B) \le \mu(B) + \varepsilon] \ &\Leftrightarrow\ \forall B \in \mathcal{F}\ [\mu(B \setminus A) \ge \mu(B) - \varepsilon] \\ &\Leftrightarrow\ \forall B \in \mathcal{F}\ [\mu(B) - \varepsilon \le \mu(A \triangle B) \le \mu(B) + \varepsilon]. \end{aligned}$$

Definition 2.4. (1) A measurable set A is called a (μ-)*weak null set* if $\mu(A) = 0$, where (μ-) means that we write "μ-weak null set" when we specify the non-additive measure μ explicitly, and we write merely "weak null set" when the non-additive measure μ is clear from the context (the same goes in what follows).

(2) [1, 5] A measurable set A is called a (μ-)*strong null set* if $\mu(A \cup B) = \mu(B)$ for every measurable set B.

By definition, Proposition 2.5 below is obvious.

Proposition 2.5. (1) *Every strong null set is a weak null set.*

(2) *Let μ be null-additive. Then a measurable set N is a weak null set iff N is a strong null set.*

Proposition 2.6 below, which gives properties of weak null sets, is derived directly from the definition.

Proposition 2.6. (1) *The empty set is a weak null set.*

(2) *Every measurable subset of a weak null set is a weak null set.*

The following proposition gives properties of strong null sets.

Proposition 2.7 ([5])**.** (1) *The empty set is a strong null set.*

(2) *Every measurable subset of a strong null set is a strong null set.*
(3) *Every finite union of strong null sets is a strong null set.*
(4) *If μ is continuous from below or continuous from above, then every countable union of strong null sets is a strong null set.*

Generally, weak null sets do not have the properties corresponding to Proposition 2.7 (3) and (4).

Throughout the paper, every measurable function is assumed to be finite real-valued.

Definition 2.8. Let $\{f_n\}$ be a sequence of measurable functions, and f a measurable function.

(1) $\{f_n\}$ is said to converge (μ-)*weakly almost everywhere* [resp. (μ-)*strongly almost everywhere*] to f if there exists a weak [resp. strong] null set N such that $\{f_n(x)\}$ converges to $f(x)$ for every $x \in X \setminus N$.
(2) $\{f_n\}$ is said to converge (μ-)*weakly almost uniformly* to f if, for every $\varepsilon > 0$, there exists a measurable set A_ε such that $\mu(A_\varepsilon) \leq \varepsilon$ and $\{f_n\}$ converges uniformly to f on $X \setminus A_\varepsilon$.
(3) $\{f_n\}$ is said to converge (μ-)*strongly almost uniformly* to f if, for every $\varepsilon > 0$, there exists a measurable set A_ε such that for every measurable set B

$$\mu(A_\varepsilon \cup B) \leq \mu(B) + \varepsilon$$

and that $\{f_n\}$ converges uniformly to f on $X \setminus A_\varepsilon$.

Proposition 2.9 below is derived directly from the definitions.

Proposition 2.9. *Let $\{f_n\}$ be a sequence of measurable functions, and f a measurable function.*

(1) *If $\{f_n\}$ converges strongly almost everywhere to f, then $\{f_n\}$ converges weakly almost everywhere to f.*
(2) *If $\{f_n\}$ converges strongly almost uniformly to f, then $\{f_n\}$ converges weakly almost uniformly to f.*

By definition, the following proposition is also obvious.

Proposition 2.10. (1) *If μ is null-additive, then weak almost everywhere convergence is equivalent to strong almost everywhere convergence.*
(2) *If μ is uniformly subadditively continuous, then weak almost uniform convergence is equivalent to strong almost uniform convergence.*

Earlier studies in non-additive measure theory (for example, [4,6]) discussed conditions for the consequent of the weak form of the Egorov theorem, i.e, for the implication of weak almost uniform convergence from weak almost everywhere convergence, and the following theorem was obtained.

Theorem 2.11 ([4])**.** *If μ is continuous from below and above, then weak almost everywhere convergence implies weak almost uniform convergence.*

This paper discusses conditions for the consequent of the strong form of the Egorov theorem, i.e, the implication of strong almost uniform convergence from strong almost everywhere convergence.

3. Supremum increment ${}^{\Delta}\mu$ and the main result

Definition 3.1. The *supremum increment* ${}^{\Delta}\mu : \mathcal{F} \to [0, \infty]$ of μ is defined by

$$ {}^{\Delta}\mu(A) := \sup\{\mu(A \cup B) - \mu(B) \mid B \in \mathcal{F}_{\mu<\infty}\}, $$

where $\mathcal{F}_{\mu<\infty} := \{B \in \mathcal{F} \mid \mu(B) < \infty\}$.

By definition, Proposition 3.2 below is obvious.

Proposition 3.2. *The supremum increment ${}^{\Delta}\mu$ is a subadditive non-additive measure, and for every measurable set A, it holds that $\mu(A) \leq {}^{\Delta}\mu(A)$. Furthermore, if μ is subadditive, then ${}^{\Delta}\mu = \mu$.*

Proposition 3.3 below is derived directly from Definitions 2.4 and 3.1.

Proposition 3.3. *Let N be an arbitrary measurable set. N is a μ-strong null set iff N is a ${}^{\Delta}\mu$-weak null set.*

The following corollary is obtained by Proposition 3.3 and Definition 2.8.

Corollary 3.4. *Let $\{f_n\}$ be a sequence of measurable functions and f a measurable function. The sequence $\{f_n\}$ converges μ-strongly almost everywhere to f iff $\{f_n\}$ converges ${}^{\Delta}\mu$-weakly almost everywhere to f.*

The following proposition is derived directly from Definitions 2.8 and 3.1.

Proposition 3.5. *Let $\{f_n\}$ be a sequence of measurable functions and f a measurable function. The sequence $\{f_n\}$ converges μ-strongly almost uniformly tof iff $\{f_n\}$ converges ${}^{\Delta}\mu$-weakly almost uniformly to f.*

Corollary 3.6 below is obtained by Corollary 3.4 and Proposition 3.5.

Corollary 3.6. *The implication of μ-strong almost uniform convergence from μ-strong almost everywhere convergence is equivalent to the implication of ${}^{\Delta}\mu$-weak almost uniform convergence from ${}^{\Delta}\mu$-weak almost everywhere convergence.*

The following corollary follows from Corollary 3.6 and Theorem 2.11.

Corollary 3.7. *If $^\Delta\mu$ is continuous from below and above, then μ-strong almost everywhere convergence implies μ-strong almost uniform convergence.*

We consider to represent sufficient conditions for the continuity of $^\Delta\mu$ from below and above by only using μ. Propositions 3.8 and 3.9 below, which give the conditions represented by μ, are the main results of this paper. The proofs of Propositions 3.8 and 3.9 and Remark 3.10 will be given in Appendix.

Proposition 3.8. *If μ is continuous from below, then so is $^\Delta\mu$.*

Proposition 3.9. *If μ is uniformly subadditively continuous and order continuous, then $^\Delta\mu$ is continuous from above.*

Remark 3.10. Even if μ is continuous from above, $^\Delta\mu$ is not necessarily continuous from above.

The following main theorem is obtained by Corollary 3.7, Propositions 3.8 and 3.9.

Theorem 3.11. *If μ is continuous from below, uniformly subadditively continuous, and order continuous, then strong almost everywhere convergence implies strong almost uniform convergence.*

Theorem 3.11 above can be proved directly without the supremum increment $^\Delta\mu$, i.e., without using Corollary 3.7, Propositions 3.8 and 3.9; the proof will be shown in Appendix.

4. Conclusions

We have given a sufficient condition for the consequent of the strong form of the Egorov theorem; we have applied the existing result that the continuity from below and above is a sufficient condition for the consequent of the weak form of the Egorov theorem to the fact that the consequent of the strong form with respect to μ is equivalent to the consequent of the weak form with respect to the supremum increment $^\Delta\mu$.

Other than the continuity from below and above, there are several conditions for the consequent of the weak form of the Egorov theorem [6]. By rewriting with μ the conditions described with the supremum increment $^\Delta\mu$, we expect to obtain conditions for the consequent of the strong form other than that in Theorem 3.11.

References

[1] R. J. Aumann and L. S. Shapley, *Values of Non-Atomic Games*, Princeton Univ. Press, Princeton, NJ, 1974.

[2] I. Dobrakov, *On submeasures I*, Dissertationes Math. (Rozprawy Mat.), **112** (1974), 5–35.

[3] L. Drewnowski, *Topological rings of sets, continuous set functions, integration I*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **20** (1972), 269–276.

[4] J. Li, *On Egoroff theorems on fuzzy measure spaces*, Fuzzy Sets and Systems, **135** (2003), 367–375.

[5] T. Murofushi and M. Sugeno, *A theory of fuzzy measures: representations, the Choquet integral, and null sets*, J. Math. Anal. Appl., **159** (1991), 532–549.
[6] T. Murofushi, K. Uchino, and S. Asahina, *Conditions for Egoroff's theorem in non-additive measure theory*, Fuzzy Sets and Systems, **146** (2004), 135–146.
[7] Z. Wang, *The autocontinuity of set function and the fuzzy integral*, J. Math. Anal. Appl., **99** (1984), 195–218.

Appendix

Proof of Proposition 3.8. Assume $A_n \uparrow A$. We prove that ${}^{\Delta}\mu(A_n) \uparrow {}^{\Delta}\mu(A)$. Let α be an arbitrary real number for which $\alpha < {}^{\Delta}\mu(A)$. By the definition of supremum increment ${}^{\Delta}\mu$, there exists a measurable set B such that $\alpha < \mu(A \cup B) - \mu(B)$. Then, since $(A_n \cup B) \uparrow (A \cup B)$, and since μ is continuous from below, there exists n_0 such that for every $n \geq n_0$

$$\alpha < \mu(A_n \cup B) - \mu(B) \leq {}^{\Delta}\mu(A_n).$$

Therefore it follows that ${}^{\Delta}\mu(A_n) \uparrow {}^{\Delta}\mu(A)$.

Proof of Proposition 3.9. Assume $A_n \downarrow A$. We prove that ${}^{\Delta}\mu(A_n) \downarrow {}^{\Delta}\mu(A)$. Let $\varepsilon > 0$. Since μ is uniformly subadditively continuous, there exists $\delta > 0$ such that, if a measurable set E satisfies $\mu(E) \leq \delta$, then, for an arbitrary measurable set F, it holds that

$$\mu(E \cup F) \leq \mu(F) + \varepsilon.$$

Since $(A_n \setminus A) \downarrow \emptyset$, and since μ is order continuous, there exists n_0 such that $\mu(A_n \setminus A) \leq \delta$ for every $n \geq n_0$. Hence, for every $n \geq n_0$, we have

$$\begin{aligned}
{}^{\Delta}\mu(A_n) &= \sup_{B \in \mathcal{F}_{\mu<\infty}} [\mu(A_n \cup B) - \mu(B)] \\
&= \sup_{B \in \mathcal{F}_{\mu<\infty}} [\mu((A_n \setminus A) \cup A \cup B) - \mu(B)] \\
&\leq \sup_{B \in \mathcal{F}_{\mu<\infty}} [\mu(A \cup B) + \varepsilon - \mu(B)] \\
&= \sup_{B \in \mathcal{F}_{\mu<\infty}} [\mu(A \cup B) - \mu(B)] + \varepsilon \\
&= {}^{\Delta}\mu(A) + \varepsilon.
\end{aligned}$$

Therefore it follows that ${}^{\Delta}\mu(A_n) \downarrow {}^{\Delta}\mu(A)$.

Proof of Remark 3.10: a counterexample. Consider a measurable space $(X, \mathcal{F})$ where there exists a sequence $\{N_n\}$ of nonempty measurable sets such that $N_n \downarrow \emptyset$; for example, the space where X is an infinite set and $\mathcal{F}$ is the power set of X. Define a non-additive measure μ on $(X, \mathcal{F})$ by

$$\mu(A) := \begin{cases} 1 & \text{if } A = X, \\ 0 & \text{if } A \neq X. \end{cases}$$

Then obviously μ is continuous from above. The supremum increment ${}^{\Delta}\mu$ of μ is given by

$$ {}^{\Delta}\mu(A) \ = \ \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases} $$

Consider a sequence $\{N_n\}$ of nonempty measurable sets such that $N_n \downarrow \emptyset$. Since ${}^{\Delta}\mu(N_n) = 1$ for all n, and since ${}^{\Delta}\mu(\bigcap_{n=1}^{\infty} N_n) = {}^{\Delta}\mu(\emptyset) = 0$, it follows that ${}^{\Delta}\mu$ is not continuous from above.

Direct proof of Theorem 3.11. Let the non-additive measure μ be continuous from below, uniformly subadditively continuous, and order continuous. Assume that a sequence $\{f_n\}$ of measurable functions converges strongly almost everywhere to a measurable function f. Then, from Proposition 2.9 (1) it follows that $\{f_n\}$ converges weakly almost everywhere to f. Since μ is uniformly subadditively continuous and order continuous, it is continuous from above [2]. Hence, Theorem 2.11 implies that $\{f_n\}$ converges weakly almost uniformly to f. Proposition 2.10 (2) says that, under the uniform subadditive continuity, weak almost uniform convergence is equivalent to strong almost uniform convergence. Therefore, $\{f_n\}$ converges strongly almost uniformly to f.

Manuscript received 31 December 2016

T. Murofushi
School of Computing, Tokyo Institute of Technology, 4259-G3-47 Nagatsuta, Midori-ku, Yokohama 226-8502, Japan
E-mail address: murofusi@c.titech.ac.jp

S. Sujino
Department of Computational Intelligence and Systems Science, Tokyo Institute of Technology, 4259-G3-47 Nagatsuta, Midori-ku, Yokohama 226-8502, Japan