Linear and Nonlinear Analysis
Volume 3, Number 3, 2017, 393–399

EVERY GENERALIZED METRIC SPACE HAS A SEQUENTIALLY COMPATIBLE TOPOLOGY

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Dedicated to Professor Makoto Tsukada on the occasion of his 65th birthday

Abstract. We prove that every 2-generalized metric space (X, d) has a sequentially compatible topology with d. We also discuss separation axioms for the topology.

1. Introduction

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

In 2000, Branciari in [3] introduced a very interesting concept whose name is 'ν-generalized metric space'.

Definition 1.1 (Branciari [3])**.** Let X be a set, let d be a function from $X \times X$ into $[0, \infty)$ and let $\nu \in \mathbb{N}$. Then (X, d) is said to be a *ν-generalized metric space* if the following hold:

(N1) $d(x, y) = 0$ iff $x = y$ for any $x, y \in X$.
(N2) $d(x, y) = d(y, x)$ for any $x, y \in X$.
(N3)$_\nu$ $d(x, y) \leq d(x, u_1) + d(u_1, u_2) + \cdots + d(u_\nu, y)$ for any $x, u_1, u_2, \ldots, u_\nu, y \in X$ such that $x, u_1, u_2, \ldots, u_\nu, y$ are all different.

In the case where $\nu = 2$, X is simply called a generalized metric space.

Definition 1.2 (Branciari [3])**.** Let X and d be as in Definition 1.1. Then (X, d) is said to be a *generalized metric space* if (N1), (N2) and the following hold:

(N3)$_2$ $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ for any $x, u, v, y \in X$ such that x, u, v, y are all different.

The concept of 'generalized metric space' is very similar to that of 'metric space'. However, it is very difficult to treat this concept because X does not necessarily have the topology which is compatible with d. It is obvious that (X, d) is a metric space iff (X, d) is a 1-generalized metric space. So every 1-generalized metric space has the compatible topology with d. In [12], we proved that every 3-generalized metric

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2010 *Mathematics Subject Classification.* 54E99.

Key words and phrases. ν-generalized metric space, sequentially compatible topology, T_1 space, T_2 space.

The author is supported in part by JSPS KAKENHI Grant Number 16K05207 from Japan Society for the Promotion of Science.

space has the compatible topology. On the other hand, for $\nu \in \{2, 4, 5, \dots\}$, there is an example of ν-generalized metric space which does not have the compatible topology; see Example 7 in [8] and Example 4.2 in [12]. See [1, 2, 5–7, 9–11, 13] and references therein for more information on this concept.

Motivated by the above, in this paper, we prove that every 2-generalized metric space (X, d) has a sequentially compatible topology with d. We also discuss separation axioms for the topology.

2. Compatibility

In order to argue precisely, we give definitions of two concepts on compatibility.

Definition 2.1. Let X be a topological space with topology τ. Let d be a function from $X \times X$ into $[0, \infty)$.

- τ is said to be *compatible* with d iff the following are equivalent for any net $\{x_\alpha\}$ in X and $x \in X$:
  - $\lim_\alpha d(x, x_\alpha) = 0$.
  - $\{x_\alpha\}$ converges to x in τ.
- τ is said to be *sequentially compatible* with d iff the following are equivalent for any sequence $\{x_n\}$ in X and $x \in X$:
  - $\lim_n d(x, x_n) = 0$.
  - $\{x_n\}$ converges to x in τ.

Remark 2.2. It is obvious that there exists at most one topology which is compatible with d. On the other hand, in general, there can be plural topologies which is sequentially compatible with d. We sometimes say that the topology τ is a compatible symmetric topology; see [4] and others.

3. Main result

In this section, we prove that every 2-generalized metric space has a sequentially compatible topology.

Throughout this section we let (X, d) be a 2-generalized metric space. Define $S(x, \delta)$ and $T(x, \delta)$ by

$$S(x, \delta) = \{y \in X : \ d(x, y) < \delta\} \quad \text{and} \quad T(x, \delta) = S(x, \delta) \setminus \{x\}$$

for $x \in X$ and $\delta > 0$. Define a set $F(x, \delta)$ as follows: $f \in F(x, \delta)$ iff f is a function from $T(x, \delta)$ into $(0, \infty)$ satisfying

$$d(x, y) + f(y) < \delta \quad \text{for any } y \in T(x, \delta).$$

For $x \in X$, $\delta > 0$ and $f \in F(x, \delta)$, we define $U(x, \delta, f)$ by

$$U(x, \delta, f) = \{x\} \cup \bigcup \big[S\big(y, f(y)\big) : y \in T(x, \delta)\big].$$

Let τ be a topology on X induced by a subbase

$$\{U(x, \delta, f) : x \in X, \delta > 0, f \in F(x, \delta)\}.$$

Lemma 3.1. *Let $x \in X$, $\delta > 0$ and $f \in F(x, \delta)$. Then the following hold*:

(i) *For any $z \in U(x, \delta, f)$, there exists $\varepsilon > 0$ satisfying $S(z, \varepsilon) \subset U(x, \delta, f)$.*

(ii) *For any* $z \in U(x, \delta, f)$*, there exist* $\varepsilon > 0$ *and* $g \in F(z, \varepsilon)$ *satisfying* $U(z, \varepsilon, g) \subset U(x, \delta, f)$*.*

Proof. We first show (i). We fix $z \in U(x, \delta, f)$ and consider the following three cases:

- $z = x$
- $z \in T(x, \delta)$
- $z \in U(x, \delta, f) \setminus S(x, \delta)$

In the first case, putting $\varepsilon = \delta$, we have

$$S(z, \varepsilon) = S(x, \delta) \subset U(x, \delta, f).$$

In the second case, putting $\varepsilon = f(z)$, we have

$$S(z, \varepsilon) = S\big(z, f(z)\big) \subset U(x, \delta, f).$$

In the third case, there exists $y \in T(x, \delta)$ satisfying $d(y, z) < f(y)$. Put

$$\varepsilon := \delta - d(x, y) - f(y) > 0.$$

Let $w \in S(z, \varepsilon)$. In the case where $w \in \{x, y, z\}$, it is obvious that $w \in U(x, \delta, f)$. In the other case, where $w \notin \{x, y, z\}$, we note that x, y, z, w are all different. So we have by $(\mathrm{N3})_2$

$$d(x, w) \leq d(x, y) + d(y, z) + d(z, w) < d(x, y) + f(y) + \varepsilon = \delta.$$

Thus $w \in S(x, \delta) \subset U(x, \delta, f)$. We have shown (i).

Let us prove (ii). By (i), there exists a function h from $U(x, \delta, f)$ into $(0, \infty)$ satisfying $S\big(y, h(y)\big) \subset U(x, \delta, f)$ for any $y \in U(x, \delta, f)$. Fix $z \in U(x, \delta, f)$ and put $\varepsilon = h(z)$. Define a function g from $T(z, \varepsilon)$ into $(0, \infty)$ by

$$g(y) = \min\big\{\big(\varepsilon - d(z, y)\big)/2, h(y)\big\}.$$

Then we have $g \in F(z, \varepsilon)$ and

$$S\big(y, g(y)\big) \subset S\big(y, h(y)\big) \subset U(x, \delta, f)$$

and hence $U(z, \varepsilon, g) \subset U(x, \delta, f)$. $\square$

Lemma 3.2. *Let* U *be an open subset of* (X, τ)*. Then the following hold:*

(i) *For any* $x \in U$*, there exists* $\delta > 0$ *satisfying* $S(x, \delta) \subset U$*.*

(ii) *For any* $x \in U$*, there exist* $\delta > 0$ *and* $f \in F(x, \delta)$ *satisfying* $U(x, \delta, f) \subset U$*.*

Remark 3.3. From (ii), $\{U(x, \delta, f) : \delta > 0, f \in F(x, \delta)\}$ is a neighborhood basis at x.

Proof of Lemma 3.2. There exist $k \in \mathbb{N}$, $y_j \in X$, $\varepsilon_j > 0$ and $g_j \in F(y_j, \varepsilon_j)$ $(j = 1, \ldots, k)$ satisfying

$$x \in \bigcap_{j=1}^{k} U(y_j, \varepsilon_j, g_j) \subset U.$$

By Lemma 3.1 (ii), there exist $\delta_j > 0$ and $f_j \in F(x, \delta_j)$ satisfying $U(x, \delta_j, f_j) \subset U(y_j, \varepsilon_j, g_j)$. Put

$$\delta := \min\{\delta_j : j = 1, \ldots, k\} > 0$$

and define a function from $T(x, \delta)$ into $(0, \infty)$ by

$$f(y) = \min\{f_j(y) : j = 1, \dots, k\}.$$

Then $f \in F(x, \delta)$ obviously holds. We have

$$U(x, \delta, f) \subset \bigcap_{j=1}^{k} U(x, \delta_j, f_j) \subset \bigcap_{j=1}^{k} U(y_j, \varepsilon_j, g_j) \subset U.$$

We have shown (ii). It is obvious that (i) follows from (ii). □

Lemma 3.4. *Let U be a subset of X. Then U is open in τ iff the following holds*:

(A) *For any $x \in U$, there exists $\delta > 0$ satisfying $S(x, \delta) \subset U$.*

Proof. We assume that U is open in τ. Then by Lemma 3.2 (i), (A) holds. Let us prove the converse implication. We assume that (A) holds, that is, for any $x \in U$, there exists $\delta_x > 0$ satisfying $S(x, \delta_x) \subset U$. Define $f_x \in F(x, \delta_x)$ by

$$f_x(y) = \min\big\{\big(\delta_x - d(x, y)\big)/2, \delta_y\big\}$$

for $y \in T(x, \delta_x)$. Then we have

$$\bigcup \big[U(x, \delta_x, f_x) : x \in U\big] = U.$$

Therefore U is open in τ. □

Now we can prove the following:

Theorem 3.5. *τ is sequentially compatible with d.*

Proof. Let $\{x_n\}$ be a sequence in X and let $x \in X$. Let us prove that the following are equivalent:

(i) $\lim_n d(x_n, x) = 0$.
(ii) $\{x_n\}$ converges to x in τ.

We first prove (i) $\Rightarrow$ (ii). We assume $\lim_n d(x_n, x) = 0$. Let U be an open subset of (X, τ) containing x. Then by Lemma 3.2 (i), there exists $\delta > 0$ satisfying $S(x, \delta) \subset U$. From (i), there exists $\mu \in \mathbb{N}$ such that $n \geq \mu$ implies $d(x_n, x) < \delta$. So

$$x_n \in S(x, \delta) \subset U$$

holds for $n \in \mathbb{N}$ with $n \geq \mu$. Therefore $\{x_n\}$ converges to x in τ. Next, in order to prove (ii) $\Rightarrow$ (i), we assume that $\limsup_n d(x_n, x) > 0$. We will show the following (B):

(B) There exist a subsequence $\{y_n\}$ of $\{x_n\}$, $\delta > 0$ and $f \in F(x, \delta)$ satisfying $U(x, \delta, f) \cap \{y_n : n \in \mathbb{N}\} = \varnothing$.

In order to show (B), we choose $\beta > 0$ and a subsequence $\{y_n\}$ of $\{x_n\}$ such that $d(y_n, x) > \beta$ for any $n \in \mathbb{N}$. We consider the following three cases:

- There exists $z \in X$ such that $\#\{n \in \mathbb{N} : y_n = z\} = \infty$.
- There exists $z \in X$ such that $\#\{n \in \mathbb{N} : y_n = z\} < \infty$ and $\liminf_n d(y_n, z) = 0$.
- $\liminf_n d(y_n, z) > 0$ holds for any $z \in X$.

In the first case, without loss of generality, we may assume that $y_n = z$ for any $n \in \mathbb{N}$. Since $d(z,x) > \beta$, $x \neq z$ holds. Put $\delta = \beta$ and let $y \in T(x,\delta)$. Since

$$0 < d(x,y) < \delta < d(x,z),$$

$y \neq z$ holds. We can choose $f(y)$ satisfying

$$0 < f(y) < \min\big\{\delta - d(x,y), d(y,z)\big\}.$$

Then it is obvious that $f \in F(x,\delta)$ and $S\big(y,f(y)\big) \cap \{y_n : n \in \mathbb{N}\} = \varnothing$ hold. Thus we obtain (B). In the second case, without loss of generality, we may assume that $\{y_n\}$ itself satisfies $\lim_n d(y_n,z) = 0$, y_n are all different and $y_n \neq z$ for any $n \in \mathbb{N}$. Since $d(y_n,x) > \beta$, $x \notin \{y_n : n \in \mathbb{N}\}$ and $z \neq x$ hold, thus, $d(x,z) > 0$ holds. Put $\delta := \min\big\{\beta, d(x,z)\big\} > 0$ and let $y \in T(x,\delta)$. We have $y \notin \{y_n : n \in \mathbb{N}\}$ because $d(x,y) < \delta \leq \beta$ holds. We also have $y \neq z$. Noting that the four elements x, z, y, y_n are all different for any $n \in \mathbb{N}$, we have by $(\mathrm{N3})_2$

$$d(x,z) \leq d(x,y) + d(y,y_n) + d(y_n,z)$$

and hence

$$\liminf_{n\to\infty} d(y,y_n) \geq d(x,z) - d(x,y) \geq \delta - d(x,y) > 0.$$

So, we can choose $f(y)$ satisfying

$$0 < f(y) < \delta - d(x,y) \quad \text{and} \quad S\big(y,f(y)\big) \cap \{y_n : n \in \mathbb{N}\} = \varnothing.$$

Therefore we obtain (B). In the third case, we put $\delta = \beta$ and let $y \in T(x,\delta)$. Then we note $y \notin \{y_n : n \in \mathbb{N}\}$. Since $\liminf_n d(y_n,y) > 0$, we can prove (B) as in the second case. Therefore we have shown (B) in all cases. Since $U(x,\delta,f)$ is an open neighborhood of x, $\{x_n\}$ does not converge to x in τ. Therefore we have shown (ii) $\Rightarrow$ (i). We have shown that τ is sequentially compatible with d. $\square$

Remark 3.6. Theorem 3.5 is independent of Lemma 9.3 in [4] because in Page 426 of [4] we assume that the whole space is regular and T_1.

Theorem 3.7. *Every 2-generalized metric space (X,d) has a sequentially compatible topology with d.*

4. Separation Axioms

In this section, we discuss separation axioms.

We recall that a topological space X is said to be a *T_1 space* iff for any two distinct points x and y in X, there exist open subsets U and V of X satisfying $x \in U$, $y \in V$, $y \notin U$ and $x \notin V$. A T_1 space is also called an *accessible space* or a *Fréchet space*. It is well known that X is T_1 iff for any $x \in X$, the singleton set $\{x\}$ is closed.

We recall that X is said to be a *T_2 space* iff for any two distinct points x and y in X, there exist open subsets U and V of X satisfying $x \in U$, $y \in V$ and $U \cap V = \varnothing$. A T_2 space is also called a *separated space* or a *Hausdorff space*.

The following theorem tells that for a generalized metric space X, (X,τ) is T_1, where τ is given in Section 3.

Theorem 4.1. *Let (X,d) be a 2-generalized metric space and let τ be a topology on X as in Section 3. Then (X,τ) is T_1.*

Proof. For any $x \in X$, since

$$X \setminus \{x\} = \bigcup \{ S\big(y, d(x,y)\big) : y \in X \setminus \{x\} \},$$

by Lemma 3.4, $X \setminus \{x\}$ is open. Hence $\{x\}$ is closed. □

We give a sufficient and necessary condition for that (X, τ) is T_2.

Theorem 4.2. *Let X, d and τ be as in Theorem 4.1. Then the following are equivalent*:

(i) (X, τ) *is* T_2.
(ii) *If a sequence* $\{x_n\}$ *in* X *converges to* x *in* (X, d), *then* $\liminf_n d(x_n, y) > 0$ *holds for any* $y \in X \setminus \{x\}$.
(iii) *If a sequence* $\{x_n\}$ *in* X *converges to* x *and* y *in* (X, d), *then* $x = y$ *holds.*

Proof. We first show (i) ⇒ (ii). We assume that (i) holds, a sequence $\{x_n\}$ in X converges to x in (X, d) and y is an element of $X \setminus \{x\}$. Then there exist open subsets U and V of X satisfying $x \in U$, $y \in V$ and $U \cap V = \varnothing$. We can choose $\delta > 0$ such that $S(y, \delta) \subset V$. Since $x_n \in U$ for sufficiently large $n \in \mathbb{N}$, we have $x_n \notin S(y, \delta)$ for sufficiently large $n \in \mathbb{N}$. Thus

$$\liminf_{n \to \infty} d(x_n, y) \geq \delta > 0.$$

We have shown (i) ⇒ (ii). We can easily show (ii) ⇒ (iii). We shall show (iii) ⇒ (i). We assume that (iii) holds and x and y are two distinct points of X. We put

$$\delta = \inf\{d(x,z) + d(y,z) : z \in X\}/4.$$

It follows from (iii) that $\delta > 0$ holds. We note $S(x, \delta) \cap S(y, \delta) = \varnothing$. Let $f \in F(x, \delta)$ and $g \in F(y, \delta)$ and put $U = U(x, \delta, f)$ and $V = U(y, \delta, g)$. Arguing by contradiction, we assume $w \in U \cap V$. Without loss of generality, we may assume $d(x, w) \leq d(y, w)$. We consider the following three cases:

- $0 = d(x, w) < \delta \leq d(y, w)$
- $0 < d(x, w) < \delta \leq d(y, w)$
- $\delta \leq d(x, w) \leq d(y, w)$

We note that there exists $z \in T(y, \delta)$ such that $w \in T\big(z, g(z)\big)$. It follows from $S(x, \delta) \cap S(y, \delta) = \varnothing$ that $d(x, z) \geq \delta$ holds. In the first case, we have

$$4\,\delta \leq d(x,z) + d(y,z) = d(w,z) + d(y,z) < g(z) + d(y,z) < \delta,$$

which implies a contradiction. In the second case, noting that x, y, z, w are all different, we have by $(\mathrm{N3})_2$

$$4\,\delta \leq d(x,x) + d(x,y) = d(x,y) \leq d(x,w) + d(w,z) + d(z,y) < 2\,\delta,$$

which implies a contradiction. In the third case, we note that there exists $v \in T(x, \delta)$ such that $w \in T\big(v, f(v)\big)$. Noting that x, y, z, v, w are all different, we have by $(\mathrm{N3})_2$

$$\begin{aligned} 4\,\delta &\leq d(x,v) + d(y,v) \leq d(x,v) + d(v,w) + d(w,z) + d(z,y) \\ &\leq d(x,v) + f(v) + g(z) + d(z,y) < 2\,\delta, \end{aligned}$$

which implies a contradiction. We have shown (iii) ⇒ (i). □

Acknowledgment

The author is very grateful to the referee for his/her careful reading.

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Manuscript received 4 January 2017

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