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AN APPROXIMATION METHOD WITH NONSUMMABLE ERRORS FOR CONVEX MINIMIZATION PROBLEMS

YASUNORI KIMURA

Dedicated to Professor Makoto Tsukada on the occasion of his 65th birthday

Abstract. In this work, we propose an iterative scheme generated by a shrinking projection method to approximate a solution to the convex minimization problem on a Hilbert space by using the notion of resolvent. The proposed method contains nonsummable error terms. We also obtain an upper bound of the error of the approximate value.

1. Introduction

Let H be a real Hilbert space and $f : H \to]-\infty, +\infty]$ a proper lower semicontinuous convex function, where $]-\infty, +\infty] = \mathbb{R} \cup \{+\infty\}$. We consider the convex minimization problem for f, that is, the problem finding $x_0 \in H$ such that

$$f(x_0) = \inf_{x \in H} f(x).$$

The notion of resolvent for a function f is known as one of the most powerful tools for this problem. For $\rho > 0$, the resolvent $J_\rho y$ of $y \in H$ is defined as a unique minimizer of the function

$$g(x) = \rho f(x) + \frac{1}{2} \|x - y\|^2$$

on H and we know that the set of fixed points of J_ρ coincides with that of solutions to the convex minimization problem for f. From this fact, we may apply various kinds of techniques used in fixed point theory to solve this problem.

One of the most popular methods to approximate a solution to this problem using resolvent of the function is the proximal point algorithm, introduced by Martinet [8] and Rockafellar [10]. The iterative sequence generated by this scheme is guaranteed to be convergent weakly to a solution. Since this algorithm was introduced, a large number of variations of this result has been investigated and proposed.

On the other hand, strong convergence algorithms have been also studied. The following result proposed by Takahashi, Takeuchi, and Kubota [13] is known as the shrinking projection method. Notice that the original result of this theorem deals with a family of nonexpansive mappings.

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Theorem 1.1 (Takahashi-Takeuchi-Kubota [13]). *Let H be a real Hilbert space and C a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into itself such that $\operatorname{Fix} T = \{z \in C : z = Tz\}$ is nonempty. Let $\{\alpha_n\}$ be a sequence in $[0, a]$, where $0 < a < 1$. For a point $x \in H$ chosen arbitrarily, generate a sequence $\{x_n\}$ by the following iterative scheme: $x_1 \in C$, $C_0 = C$, and*

$$\begin{aligned} &y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ &C_n = \{z \in C : \|z - y_n\| \leq \|z - x_n\|\} \cap C_{n-1}, \\ &x_{n+1} = P_{C_n} x \end{aligned}$$

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to $P_{\operatorname{Fix} T} x \in C$, where P_K is the metric projection of H onto a nonempty closed convex subset K of H.

Applying the resolvent operator to the mapping T, we obtain a strongly convergent approximation result to the solution of the convex minimization problem. Related results are also given in [5, 11].

Recently, the author [6, 7] proposed iterative sequences generated by a shrinking projection method containing nonsummable error terms. This technique is applied to the convex minimization problem by Ibaraki [4].

In this work, we deal with a shrinking projection method to approximate a solution to the convex minimization problem on a Hilbert space. The proposed method also contains nonsummable error terms, however, the structure of the sequence of convex sets in the method is different from that obtained by Ibaraki.

2. Preliminaries

In what follows, the symbol H is always a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. The set of positive integers is denoted by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. We know the following basic equality: For $x, y \in H$ and $\tau \in \mathbb{R}$,

$$\|\tau x + (1 - \tau y)\|^2 = \tau \|x\|^2 + (1 - \tau) \|y\|^2 - \tau(1 - \tau) \|x - y\|^2 .$$

In particular, when $\tau = 1/2$, it is known as the parallelogram law.

Let C be a nonempty closed convex subset of H. We say a mapping $T : C \to H$ is nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for every $x, y \in C$. A point $z \in C$ satisfying that $z = Tz$ is called a fixed point of T and we denote by $\operatorname{Fix} T$ the set of all fixed points of T. We know that, if T is nonexpansive, then $\operatorname{Fix} T$ is a closed convex subset of C.

Let $f : H \to]-\infty, +\infty]$. We say f is proper if there exists $x \in H$ such that $f(x) < \infty$. We say f is convex if

$$f(\tau x + (1 - \tau) y) \leq \tau f(x) + (1 - \tau) f(y)$$

for $x, y \in H$ and $\tau \in]0, 1[$. f is said to be lower semicontinuous if

$$f(x) \leq \liminf_{n \to \infty} f(x_n),$$

whenever a sequence $\{x_n\} \subset H$ converges strongly to $x \in H$. The set of minimizers of f on H is denoted by $\operatorname{argmin}_H f$.

Let $f : H \to]-\infty, +\infty]$ be a proper lower semicontinuous convex function. It is not necessarily true that f has a minimizer, however, if we define $g_y : H \to]-\infty, +\infty]$ for fixed $y \in H$ by

$$g_y(x) = f(x) + \frac{1}{2} \|x - y\|^2$$

for any $x \in H$, then it is guaranteed that g_y always has a unique minimizer $u_y \in H$. Since this minimizer u_y depends on $y \in H$, we may consider the operator $J : H \to H$ defined by $Jy = u_y$ and we call this J the resolvent of f. Further, for $\rho > 0$, the resolvent of ρf is denoted by J_ρ.

The resolvent operator J_ρ of ρf has the following properties:

- It is firmly nonexpansive in the sense of Browder [3];
$$\|J_\rho x - J_\rho y\|^2 \leq \langle J_\rho x - J_\rho y, x - y\rangle$$
for every $x, y \in H$;
- the set of its fixed points coincides with the set of minimizers of f;
$$\operatorname{Fix} J_\rho = \operatorname*{argmin}_H f.$$

Let C be a nonempty closed convex subset of H. For fixed $y \in H$, define $d_y : H \to \mathbb{R}$ by $d_y(x) = \|x - y\|^2$. Then we know that d_y has a unique minimizer $x_y \in C$. We define the metric projection P_C by $P_C x = x_y$ for every $x \in H$. Since we may see that the metric projection P_C coincides with the resolvent of the indicator function $i_C : H \to]-\infty, +\infty]$, which is defined by

$$i_C(x) = \begin{cases} 0 & (x \in C), \\ +\infty & (x \notin C), \end{cases}$$

P_C is also firmly nonexpansive and $\operatorname{Fix} P_C = \operatorname{argmin}_H i_C = C$. In what follows, the metric projection onto a nonempty closed convex subset C is denoted by P_C.

We may also define the resolvents and metric projections by using the subdifferential of convex functions. For their details, see [1, 12] and others.

The following theorem given by Tsukada [14] plays an important role in our main result. It is obvious that this result is available when the space E is a real Hilbert space.

Theorem 2.1 (Tsukada [14])**.** *Let E be a smooth, reflexive, and strictly convex real Banach space having the Kadec-Klee property. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of E and suppose that $C_0 = \text{M-lim}_{n\to\infty} C_n$ exists and is nonempty, then $\{P_{C_n} x\}$ converges strongly to $P_{C_0} x$ for each $x \in E$.*

Remark 2.2. For a sequence of nonempty closed convex subsets $\{C_n\}$ of H, the Mosco limit $\text{M-lim}_{n\to\infty} C_n$ of $\{C_n\}$ is defined as follows: We first define a subset $\text{s-Li}_n C_n$ of H as the set of all limit points of $\{C_n\}$. Namely, $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $\text{w-Ls}_n C_n$ of H is the set of all subsequential weak limit points of $\{C_n\}$; $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If $C_0 \subset H$ satisfies that $C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n$, we say that $\{C_n\}$

converges to C_0 in the sense of Mosco [9] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. For more details, see [2].

In particular, if $\{C_n\}$ is a sequence of closed convex subsets of E which is decreasing with respect to inclusion, then

$$\operatorname*{M-lim}_{n\to\infty} C_n = \bigcap_{n\in\mathbb{N}} C_n.$$

3. Approximating a minimizer of convex functions

In this section, we propose an iterative scheme for approximating a minimizer of the proper lower semicontinuous convex functions defined on a Hilbert space. The scheme contains error terms when we generate a sequence and we show that the sequence has sufficiently nice properties even if the sequence of error terms does not converges to zero.

Theorem 3.1. *Let H be a Hilbert space and $f : H \to]-\infty, +\infty]$ a proper lower semicontinuous convex function satisfying $\operatorname{argmin}_H f \neq \emptyset$. Let $\{\delta_n\}$ and $\{\rho_n\}$ be real sequences such that $\delta_n \geq 0$ and $\inf_{n\in\mathbb{N}} \rho_n > 0$. Let $\delta_0 = \limsup_{n\to\infty} \delta_n$ and $\rho_0 = \liminf_{n\to\infty} \rho_n$. For given $u, x_1 \in H$ with $\|u - x_1\| \leq \delta_1$, generate an iterative sequence $\{x_n\}$ as follows: $C_1 = H$,*

$$C_{n+1} = \{z \in H : \|J_{\rho_n} x_n - z\| \leq \|x_n - z\|\} \cap C_n, \text{ and}$$
$$x_{n+1} \in C_{n+1} \text{ such that } \|u - x_{n+1}\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}^2$$

for each $n \in \mathbb{N}$, where J_{ρ_n} is the resolvent for $\rho_n f$. Then,

$$\limsup_{n\to\infty} \|J_{\rho_n} x_n - x_n\| \leq \delta_0$$

and

$$\limsup_{n\to\infty} f(J_{\rho_n} x_n) - \min_{y\in H} f(y) \leq \frac{4\delta_0(2d\,(u, \operatorname{argmin}_H f) + \delta_0)}{\rho_0}.$$

Moreover, if $\delta_0 = 0$, then $\{x_n\}$ converges strongly to $P_{\operatorname{argmin}_H f} u$.

Proof. We first show that the sequence $\{x_n\}$ is well defined and $\operatorname{argmin}_H f \subset \bigcap_{n\in\mathbb{N}} C_n$ by induction. Fix $n \in \mathbb{N}$ arbitrarily and suppose that $x_n \in H$ is defined and $\operatorname{argmin}_H f \subset C_n$. Then, since resolvents are nonexpansive, we have that

$$C_{n+1} \supset \operatorname{Fix} J_{\rho_n} = \operatorname*{argmin}_H \rho_n f = \operatorname*{argmin}_H f \neq \emptyset.$$

Thus we may find $x_{n+1} \in C_{n+1}$ such that $\|u - x_{n+1}\|^2 \leq d(u, C_{n+1})^2 + \delta_{n+1}^2$, and hence $\{x_n\}$ is well defined and $\operatorname{argmin}_H f \subset \bigcap_{n\in\mathbb{N}} C_n$.

Let $D_n = \{z \in H : \|J_{\rho_n} x_n - z\| \leq \|x_n - z\|\}$ for $n \in \mathbb{N}$. Then, since

$$\begin{aligned} D_n &= \{z \in H : \|J_{\rho_n} x_n - z\|^2 \leq \|x_n - z\|^2\} \\ &= \{z \in H : \langle z, 2(J_{\rho_n} x_n - x_n)\rangle \geq \|J_{\rho_n} x_n\|^2 - \|x_n\|^2\}, \end{aligned}$$

we have that D_n is a closed convex subset of H, and so is C_n.

Let $C_0 = \bigcap_{n\in\mathbb{N}} C_n$. By definition, $\{C_n\}$ is a decreasing sequence of sets with respect to inclusion. By Theorem 2.1 with Remark 2.2, we have that the sequence

$\{p_n\}$ converges strongly to $p_0 \in H$, where $p_n = P_{C_n} u$ for $n \in \mathbb{N}$ and $p_0 = P_{C_0} u$. We also have that

$$\|u - x_n\|^2 \leq d(u, C_n)^2 + \delta_n^2 = \|u - p_n\|^2 + \delta_n^2$$

and, for $\tau \in]0, 1[$,

$$\begin{aligned} \|u - p_n\|^2 &\leq \|u - (\tau p_n + (1 - \tau)x_n)\|^2 \\ &= \tau \|u - p_n\|^2 + (1 - \tau) \|u - x_n\|^2 - \tau(1 - \tau) \|x_n - p_n\|^2, \end{aligned}$$

which implies that

$$\tau \|x_n - p_n\|^2 \leq \|u - x_n\|^2 - \|u - p_n\|^2.$$

Tending $\tau \uparrow 1$, we have that

$$\|x_n - p_n\|^2 \leq \|u - x_n\|^2 - \|u - p_n\|^2 \leq \delta_n^2.$$

Since $p_{n+1} \in C_{n+1}$, it follows that

$$\begin{aligned} \|J_{\rho_n} x_n - x_n\| &\leq \|J_{\rho_n} x_n - p_{n+1}\| + \|p_{n+1} - x_n\| \\ &\leq 2 \|p_{n+1} - x_n\| \\ &\leq 2(\|p_{n+1} - p_n\| + \|p_n - x_n\|) \\ &\leq 2(\|p_{n+1} - p_n\| + \delta_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, we obtain that

$$\limsup_{n \to \infty} \|J_{\rho_n} x_n - x_n\| \leq 2\delta_0.$$

We also have that, for $p = P_{\operatorname{argmin}_H f} u$ and $\tau \in]0, 1[$,

$$\begin{aligned} &\rho_n f(J_{\rho_n} x_n) + \|J_{\rho_n} x_n - x_n\|^2 \\ &\leq \rho_n f(\tau J_{\rho_n} x_n + (1 - \tau)p) + \|\tau J_{\rho_n} x_n + (1 - \tau)p - x_n\|^2 \\ &\leq \tau \rho_n f(J_{\rho_n} x_n) + (1 - \tau)\rho_n f(p) \\ &\quad + \tau \|J_{\rho_n} x_n - x_n\|^2 + (1 - \tau) \|p - x_n\|^2 - \tau(1 - \tau) \|J_{\rho_n} x_n - p\|^2. \end{aligned}$$

Dividing by $1 - \tau$ and tending $\tau \uparrow 1$, we have that

$$\begin{aligned} \rho_n f(J_{\rho_n} x_n) - \rho_n f(p) &\leq \|p - x_n\|^2 - \|J_{\rho_n} x_n - x_n\|^2 - \|p - J_{\rho_n} x_n\|^2 \\ &\leq \|J_{\rho_n} x_n - x_n\| (\|p - x_n\| + \|p - J_{\rho_n} x_n\|) - \|J_{\rho_n} x_n - x_n\|^2 \\ &= \|J_{\rho_n} x_n - x_n\| (\|p - x_n\| + \|p - J_{\rho_n} x_n\| - \|J_{\rho_n} x_n - x_n\|) \\ &\leq 2 \|J_{\rho_n} x_n - x_n\| \|p - x_n\| \\ &\leq 4(\|p_{n+1} - p_n\| + \delta_n)(\|p - u\| + \|u - p_n\| + \|p_n - x_n\|) \\ &\leq 4(\|p_{n+1} - p_n\| + \delta_n)(2 \|u - p\| + \delta_n). \end{aligned}$$

Since $\{p_n\}$ converges strongly to p_0, we have that

$$\begin{aligned}\limsup_{n\to\infty} f(J_{\rho_n}x_n) - \min_{y\in H} f(y) &= \limsup_{n\to\infty} f(J_{\rho_n}x_n) - f(p) \\ &\le \limsup_{n\to\infty} \frac{4(\|p_{n+1}-p_n\| + \delta_n)(2\|u-p\| + \delta_n)}{\rho_n} \\ &= \frac{4\delta_0(2\|u-p\| + \delta_0)}{\rho_0},\end{aligned}$$

which is the desired result.

For the latter part of the theorem, suppose that $\delta_0 = 0$. Then we have that $\lim_{n\to\infty} \|J_{\rho_n}x_n - x_n\| = 0$. We also have that

$$\lim_{n\to\infty} \|x_n - p_n\| \le \lim_{n\to\infty} \delta_n = \delta_0 = 0,$$

and that $\{p_n\}$ converges strongly to p_0. Therefore, both $\{x_n\}$ and $\{J_{\rho_n}x_n\}$ converges strongly to p_0. Using lower semicontinuity of f, we get that

$$\begin{aligned}f(p_0) - \min_{y\in H} f(y) &\le \liminf_{n\to\infty} f(J_{\rho_n}x_n) - \min_{y\in H} f(y) \\ &\le \limsup_{n\to\infty} f(J_{\rho_n}x_n) - \min_{y\in H} f(y) \\ &= \frac{4\delta_0(2\|u-p\| + \delta_0)}{\rho_0} \\ &= 0.\end{aligned}$$

That is, $p_0 \in \operatorname{argmin}_H f$. Since $p_0 = P_{C_0}u$ and $\operatorname{argmin}_H f \subset C_0$, we have that $p_0 = P_{\operatorname{argmin}_H f}u$. □

This result shows that the generated iterative sequence approximates a fixed point of the resolvent operators for f in the sense that $\{\|J_{\rho_n}x_n - x_n\|\}$ becomes sufficiently small by repeating the iteration. Since the set of fixed points of the resolvent coincides with the set of minimizers of f, we may say that the iterative sequence approximates a minimizer of f. Moreover, the inequality

$$\limsup_{n\to\infty} f(J_{\rho_n}x_n) - \min_{y\in H} f(y) \le \frac{4\delta_0(2d(u, \operatorname{argmin}_H f) + \delta_0)}{\rho_0}$$

proposes that, if we know the approximate distance between the point u and the set of minimizers of f, then we may control the upper bound of the value $f(J_{\rho_n}x_n) - \min_{y\in H} f(y)$ by choosing appropriate sequences $\{\delta_n\}$ and $\{\rho_n\}$.

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Yasunori Kimura
Department of Information Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan
E-mail address: `yasunori@is.sci.toho-u.ac.jp`