Linear and Nonlinear Analysis
Volume 3, Number 3, 2017, 423–435

CONVERGENCE CHARACTERISTICS OF A SHRINKING PROJECTION ALGORITHM IN THE SENSE OF MOSCO FOR SPLIT EQUILIBRIUM PROBLEM AND FIXED POINT PROBLEM IN HILBERT SPACES*

MUHAMMAD AQEEL AHMAD KHAN

Dedicated to Professor Makoto Tsukada on the occasion of his 65th birthday

Abstract. The purpose of this paper is to propose and analyze an iterative hybrid shrinking projection algorithm which exhibits strong convergence in the sense of Mosco. The proposed algorithm is used to approximate a common element in the set of solutions of a finite family of split equilibrium problems and the set of common fixed points of a finite family of k-strict pseudo-contractions in the setting of Hilbert spaces. Our results can be viewed as a generalization and improvement of various existing results in the current literature.

1. Introduction

Throughout this paper, we work in the setting of a real Hilbert space H equipped with the inner product $\langle \cdot\, , \cdot \rangle$ and the induced norm $\|\cdot\|$. Let C be a nonempty subset of a real Hilbert space H and let $T : C \to C$ be a mapping. The set of fixed points of the mapping T is defined and denoted as: $F(T) = \{x \in C : T(x) = x\}$. The self-mapping T is said to be a k-strict pseudo contraction if there exists $k \in \mathbb{R}$ with $0 \leq k < 1$, such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k \|(I - T)x - (I - T)y\|^2 , \ \forall\ x, y \in C. \tag{1.1}$$

In 1967, Browder and Petryshyn [5] introduced the concept of a strict pseudo-contraction in Hilbert spaces which satisfies the following Lipschitz condition:

$$\|Tx - Ty\| \leq \frac{1 + k}{1 - k} \|x - y\| .$$

This class is prominent among various classes of nonlinear mappings and has powerful applications, in particular, to solve inverse problems [24]. However, the odds in the development of iterative algorithms for strict pseudo-contractions is the second term in its definition. Note that the class of k-strict pseudo-contractions contains the class of:

2010 *Mathematics Subject Classification.* Primary 47H05, 47H09, 47J05; Secondary 49H05.

Key words and phrases. Split equilibrium problem, fixed point problem, k-strict pseudo-contraction, Mosco convergence, shrinking projection algorithm.

The author would like to acknowledge the support provided by the Higher Education Commission of Pakistan for funding this work through project No. NRPU 5332.

(i) nonexpansive mappings (i.e., $\|Tx - Ty\| \leq \|x - y\|$) which are 0-strict pseudo-contraction;
(ii) firmly nonexpansive mappings

(i.e., $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$) which are -1-strict pseudo-contraction.

Moreover, the class of k-strict pseudo-contractions falls into the one between the classes of nonexpansive mappings and pseudo-contractions (i.e., $\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$). Observe that the class of strong pseudo-contractions (i.e., if $T - \delta I$ is a pseudo-contraction where $\delta \in (0, 1)$) is independent of the class of k-strict pseudo-contractions.

We remark that the iterative construction of fixed points of various nonlinear mappings have been extensively studied in the current literature. Some iterative algorithms for the class of k-strict pseudo-contractions have been proposed and analyzed, see, for example, [18, 21, 22, 28] and the references cited therein. It is therefore, natural to employ an iterative algorithm for the construction of fixed points of k-strict pseudo-contractions which exhibits strong convergence under mild control conditions on the parameters. It is worth mentioning that hybrid projection algorithm, introduced by Haugazeau [14], have been modified in different ways to ensure strong convergence of the algorithm. In 2008, Takahashi et al. [26] firstly proposed the shrinking projection method to establish strong convergence results for families of nonexpansive mappings in Hilbert spaces. We, therefore, employ the shrinking projection algorithm involving a finite family of k-strict pseudo-contractions in Hilbert spaces.

The theory of equilibrium problems is a systematic approach to study a diverse range of problems arising in the field of physics, optimization, variational inequalities, transportation, economics, network and noncooperative games, see, for example [1, 3, 12] and the references cited therein. The existence result of an equilibrium problem can be found in the seminal work of Blum and Oettli [3]. Moreover, this theory has a computational flavor and flourishes significantly due to an excellent paper of Combettes and Hirstoaga [11]. The classical equilibrium problem theory has been generalized in several interesting ways to solve real world problems. In 2012, Censor et al. [9] proposed a theory regarding split variational inequality problem (SVIP) which aims to solve a pair of variational inequality problems in such a way that the solution of a variational inequality problem, under a given bounded linear operator, solves another variational inequality.

Motivated by the work of Censor et al. [9], Moudafi [23] generalized the concept of SVIP to that of split monotone variational inclusions (SMVIP) which includes, as a special case, split variational inequality problem, split common fixed point problem, split zeroes problem, split equilibrium problem and split feasibility problem. These problems have already been studied and successfully employed as a model in intensity-modulated radiation therapy treatment planning, see [7, 8]. This formalism is also at the core of modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see, for example, [6, 10].

Let C be a nonempty subset of a real Hilbert space H_1, Q be a nonempty subset of a real Hilbert space H_2 and let $A : H_1 \to H_2$ be a bounded linear operator. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be two bifunctions. Recall that a split equilibrium problem (SEP) is to find:

$$x^* \in C \quad \text{such that } F(x^*, x) \geq 0 \text{ for all } x \in C, \tag{1.2}$$

and

$$y^* = Ax^* \in Q \quad \text{such that } G(y^*, y) \geq 0 \text{ for all } y \in Q. \tag{1.3}$$

It is remarked that the inequality (1.2) represents the classical equilibrium problem and its solution set is denoted as $EP(F)$. Moreover, the inequalities (1.2) and (1.3) constitute a pair of equilibrium problems which aim to find a solution x^* of an equilibrium problem (1.2) such that its image $y^* = Ax^*$ under a given bounded linear operator A also solves another equilibrium problem (1.3). The set of solutions of SEP (1.2) and (1.3) is denoted as $\Omega = \{z \in EP(F) : Az \in EP(G)\}$. Some methods have been proposed and analyzed to solve SEP together with the fixed point problem in Hilbert spaces, see, for example, [4, 13, 15–17, 25] and the references cited therein.

Recently, Deepho et al. [13] studied a general iterative algorithm to solve split variational inequality problems and fixed point problems of k- strict pseudo-contractions in Hilbert spaces. Another motivation for the current research is the recent result of Wang et al. [?] concerning split equilibrium problems and fixed point problems of nonexpansive mappings in Hilbert spaces. Inspired and motivated by the above mentioned results and the ongoing research in this direction, we aim to employ a hybrid shrinking projection algorithm to find a common element in the set of solutions of a finite family of split equilibrium problems and the set of common fixed points of a finite family of k-strict pseudo-contractions in Hilbert spaces. Our results can be viewed as a generalization and improvement of various existing results in the current literature.

2. Preliminaries

Throughout this paper, we write $x_n \to x$ (resp. $x_n \rightharpoonup x$) to indicate strong convergence (resp. weak convergence) of a sequence $\{x_n\}_{n=1}^{\infty}$. Now, we recall some basic notions and results required in the sequel. Let C be a nonempty closed convex subset of a Hilbert space H_1. For each $x \in H_1$, there exists a unique nearest point of C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } y \in C.$$

Such a mapping $P_C : H_1 \to C$ is known as a metric projection or a nearest point projection of H_1 onto C. Moreover, P_C satisfies nonexpansiveness in a Hilbert space and $\langle x - P_C x, P_C x - y\rangle \geq 0$ for all $x, y \in C$. It is remarked that P_C is firmly nonexpansive mapping from H_1 onto C, that is,

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\ \rangle\,, \text{ for all } x, y \in C.$$

Recall that a nonlinear mapping $A : C \to H_1$ is λ-inverse strongly monotone if it satisfies

$$\langle x - y\ , Ax - Ay\rangle \geq \lambda \|Ax - Ay\|^2 .$$

Note that, if $A := I - T$ is a λ-inverse strongly monotone mapping, then:

(i) A is $\left(\frac{1}{\lambda}\right)$-Lipschitz continuous mapping;
(ii) if T is a nonexpansive mapping, then A is $\left(\frac{1}{2}\right)$-inverse strongly monotone mapping;
(iii) if $\eta \in (0, 2\lambda]$, then $I - \eta A$ is a nonexpansive mapping.

The following lemma collects some well-known equations in the context of a real Hilbert space.

Lemma 2.1. *Let H_1 be a real Hilbert space, then:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$, *for all* $x, y \in H_1$;
(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$, *for all* $x, y \in H_1$;
(iii) $2\langle x - y, u - v\rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2$, *for all* $x, y, u, v \in H_1$;
(iv) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$ *for all* $x, y \in H_1$ *and* $\alpha \in \mathbb{R}$.

It is well-known that H_1 satisfies Opial's condition, that is, for any sequence $\{x_n\}$ in H_1 with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| ,$$

holds for all $y \in H_1$ with $x \neq y$.

Recall that a mapping $T : H_1 \to H_1$ is said to be demiclosed at origin if for any sequence $\{x_n\}$ in H_1 with $x_n \rightharpoonup x$ and $\|x_n - Tx_n\| \to 0$, we have $x = Tx$.

The following results collect some of the characterizations of a k-strict pseudo-contraction T and the set of fixed points $F(T)$ in Hilbert spaces.

Lemma 2.2 ([21, Proposition 2.1 (iii)])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H. If $T : C \to H$ is a k-strict pseudo-contraction, then the xed point set $F(T)$ is closed and convex so that the projection $P_{F(T)}$ is well de ned.*

Lemma 2.3 ([21, Proposition 2.1 (ii)])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and $T : C \to C$ a k-strict pseudo-contraction. Then $(I - T)$ is demiclosed, that is, if $\{x_n\}$ is a sequence in C with $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x \in F(T)$.*

We now introduce the notion of Mosco convergence.

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of Hilbert space H_1. We denote the set of all strong limit points of $\{C_n\}$ by s-Li_nC_n, that is, $x \in s$-Li_nC_n if and only if there exists $\{x_n\} \subset H_1$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all n. Similarly, we define the set of all weak subsequential limit points by w-Ls_nC_n, that is, $y \in w$-Ls_nC_n if and only if there exist a subsequence

$\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H_1$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all i. If C_0 satisfies

$$C_0 = s\text{-}Li_n C_n = w\text{-}Ls_n C_n,$$

then we say that $\{C_n\}$ converges to C_0 in the sense of Mosco and we write $C_0 = M\text{-}\lim_n C_n$. By definition, it always holds that $s\text{-}Li_n C_n \subset w\text{-}Ls_n C_n$. Therefore, to prove $C_0 = M\text{-}\lim_n C_n$, it suffices to show that $w\text{-}Ls_n C_n \subset C_0 \subset s\text{-}Li_n C_n$. One of the simplest examples of Mosco convergence is a decreasing sequence $\{C_n\}$ with respect to inclusion. The Mosco limit of such a sequence is $\bigcap_{n=1}^{\infty} C_n$. For more details, we refer to [2, 19, 20].

For a relation between a sequence of closed and convex sets and the corresponding metric projections, we state the following lemma which can be deduced from the theorem for a strictly convex reflexive Banach space satisfying Kadec-Klee property.

Lemma 2.4 ([27])**.** *Let $\{C_n\}$ be a sequence of nonempty closed convex subset of H_1. If $C_0 = \bigcap_{n=1}^{\infty} C_n$ is nonempty, then $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$ for each $x \in C$.*

In order to solve an equilibrium problem, we consider a bifunction $F : C \times C \to \mathbb{R}$ satisfying the following conditions(c.f. [3] and [11]):

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, that is, $F(x, y) + F(y, x) \le 0$ *for all* $x, y \in C$;
(A3) for each $y \in C$, the function $x \mapsto F(x, y)$ is upper hemicontinuous, that is, for each $x, z \in C$,

$$\lim_{t \to 0} F(tz + (1-t)x, y) \le F(x, y);$$

(A4) for each $x \in C$, the function $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Lemma 2.5 ([11])**.** *Let C be a closed convex subset of a real Hilbert space H_1 and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). For $r > 0$ and $x \in H_1$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \ \textit{for all } y \in C.$$

Moreover, de ne a mapping $T_r^F : H_1 \to C$ by

$$T_r^F(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \quad \textit{for all } y \in C \right\},$$

for all $x \in H_1$. Then, the following hold:

(i) T_r^F *is single-valued;*
(ii) $T_r^{F_1}$ *is rmly nonexpansive, i.e., for every $x, y \in H$,*

$$\left\|T_r^F x - T_r^F y\right\|^2 \le \langle T_r^F x - T_r^F y, x - y\rangle$$

(iii) $F(T_r^F) = EP(F)$;
(iv) $EP(F)$ *is closed and convex.*

It is remarked that if $G : Q \times Q \to \mathbb{R}$ is a bifunction satisfying conditions (A1)-(A4), then for $s > 0$ and $w \in H_2$ we can define a mapping:

$$T_s^G(w) = \left\{ d \in C : G(d,e) + \frac{1}{s}\langle e-d, d-w\rangle \geq 0, \quad \text{for all } e \in Q \right\},$$

which is, nonempty, single-valued and firmly nonexpansive. Moreover, $EP(G)$ is closed and convex, and $F(T_s^G) = EP(G)$.

3. Main results

In this section, we establish results concerning shrinking projection algorithm which exhibits strong convergence in the sense of Mosco. As a consequence, we approximate a common element in the set of solutions of a finite family of split equilibrium problems and the set of common fixed points of a finite family of k-strict pseudo-contractions in the setting of Hilbert spaces.

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $F_i : C \times C \to \mathbb{R}$ and $G_i : Q \times Q \to \mathbb{R}$ be two nite families of bifunctions satisfying conditions (A1)-(A4) such that G_i is upper hemicontinuous for each $i \in \{1,2,3,\cdots,N\}$. Let $S_i : C \to C$ be a nite family of k-strict pseudo contractions and let $A_i : H_1 \to H_2$ be a nite family of bounded linear operators for each $i \in \{1,2,3,\cdots,N\}$. Suppose that $\mathbb{F} := \left[\bigcap_{i=1}^N F(S_i)\right] \cap \Theta \neq \emptyset$, where $\Theta = \left\{ z \in C : z \in \bigcap_{i=1}^N EP(F_i) \text{ and } A_i z \in EP(G_i) \text{ for } 1 \leq i \leq N \right\}$. Let $\{x_n\}$ be a sequence generated by:*

$$\begin{aligned} & x_1 \in C_1 = C, \\ & u_n = T_{r_n}^{F_n}\left(I - \gamma A^*_{n(\operatorname{mod} N)}\left(I - T_{s_n}^{G_n}\right) A_{n(\operatorname{mod} N)}\right) x_n, \\ & y_n = \alpha_n u_n + (1-\alpha_n) S_{n(\operatorname{mod} N)} u_n, \\ & C_{n+1} = \left\{ z \in H_1 \colon \|y_n - z\|^2 \leq \|x_n - z\|^2 \right\} \cap C_n, \\ & x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1, \end{aligned} \tag{3.1}$$

where $\{r_n\}, \{s_n\}$ are two positive real sequences and $\{\alpha_n\}$ is a sequence in $(0,1)$. Let $\gamma \in \left(0, \frac{1}{L}\right)$, where $L = \max\{L_1, L_2, \cdots, L_N\}$ and L_i is the spectral radius of the operator $A_i^* A_i$ and A_i^* is the adjoint of A_i for each $i \in \{1,2,3,\cdots,N\}$. Assume that $\{\alpha_n\}, \{r_n\}$ and $\{s_n\}$ satisfy the following restrictions:

(C1) $0 \leq k < a \leq \alpha_n \leq b < 1$;
(C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\liminf_{n\to\infty} s_n > 0$;

then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $x = P_{\mathbb{F}} x_1$.

Proof. For the sake of simplicity, we define $A_n = A_{n(\operatorname{mod} N)}$ and $S_n = S_{n(\operatorname{mod} N)}$ for all $n \geq 1$.We now start our proof to show that the sequence $\{x_n\}$ defined in (3.1) is well-defined. For this, we first show by mathematical induction that $\mathbb{F} \subset C_{n+1}$ for all $n \geq 1$. Obviously, $\mathbb{F} \subset C_1$ as if $p \in \mathbb{F}$ implies that $T_{r_n}^{F_n} p = p$ and $(I - \gamma A_n^*(I - T_{s_n}^{G_n})A_n)p = p$, then $p \in C = C_1$. Now, assume that $\mathbb{F} \subset C_i$ for some

$i \geq 1$. Then, we estimate

$$\begin{aligned}\|u_i - p\|^2 &= \left\|T_{r_i}^{F_i}\left(I - \gamma A_i^*\left(I - T_{s_i}^{G_i}\right)A_i\right)x_i - T_i^{F_i}p\right\|^2\\ &\leq \left\|x_i - \gamma A_i^*\left(I - T_{s_i}^{G_i}\right)A_i x_i - p\right\|^2\\ &\leq \|x_i - p\|^2 + \gamma^2\left\|A_i^*\left(I - T_{s_i}^{G_i}\right)A_i x_i\right\|^2 + 2\gamma\left\langle p - x_i, A_i^*\left(I - T_{s_i}^{G_i}\right)A_i x_i\right\rangle.\end{aligned}$$

Thus, we have

$$\begin{aligned}\|u_i - p\|^2 &\leq \|x_i - p\|^2 + \gamma^2\left\langle A_i x_i - T_{s_i}^{G_i}A_i x_i, A_i A_i^*\left(I - T_{s_i}^{G_i}\right)A_i x_i\right\rangle\\ &\quad + 2\gamma\left\langle p - x_i, A_i^*\left(I - T_{s_i}^{G_i}\right)A_i x_i\right\rangle\\ &\leq \|x_i - p\|^2 + L\gamma^2\left\langle A_i x_i - T_{s_i}^{G_i}A_i x_i, A_i x_i - T_{s_i}^{G_i}A_i x_i\right\rangle\\ &\quad + 2\gamma\left\langle p - x_i, A_i^*\left(I - T_{s_i}^{G_i}\right)A_i x_i\right\rangle \qquad (3.2)\\ &= \|x_i - p\|^2 + L\gamma^2\left\|A_i x_i - T_{s_i}^{G_i}A_i x_i\right\|^2 + 2\gamma\left\langle p - x_i, A_i^*\left(I - T_{s_i}^{G_i}\right)A_i x_i\right\rangle.\end{aligned}$$

On the other hand

$$\begin{aligned}&2\gamma\left\langle p - x_i, A_i^*\left(I - T_{s_i}^{G_i}\right)A_i x_i\right\rangle\\ &= 2\gamma\left\langle A_i\left(p - x_i\right), A_i x_i - T_{s_i}^{G_i}A_i x_i\right\rangle\\ &= 2\gamma\left\langle A_i\left(p - x_i\right) + \left(A_i x_i - T_{s_i}^{G_i}A_i x_i\right) - \left(A_i x_i - T_{s_i}^{G_i}A_i x_i\right), A_i x_i - T_{s_i}^{G_i}A_i x_i\right\rangle\\ &= 2\gamma\left\{\left\langle A_i p - T_{s_i}^{G_i}A_i x_i, A_i x_i - T_{s_i}^{G_i}A_i x_i\right\rangle - \left\|A_i x_i - T_{s_i}^{G_i}A_i x_i\right\|^2\right\} \qquad (3.3)\\ &\leq 2\gamma\left\{\frac{1}{2}\left\|A_i x_i - T_{s_i}^{G_i}A_i x_i\right\|^2 - \left\|A_i x_i - T_{s_i}^{G_i}A_i x_i\right\|^2\right\}\\ &= -\gamma\left\|A_i x_i - T_{s_i}^{G_i}A_i x_i\right\|^2.\end{aligned}$$

Substituting (3.2) in (3.3) and simplifying, we have

$$\|u_i - p\|^2 \leq \|x_i - p\|^2 + \gamma\left(L\gamma - 1\right)\left\|A_i x_i - T_{s_i}^{G_i}A_i x_i\right\|^2. \qquad (3.4)$$

It now follows from the definition of γ, we have

$$\|u_i - p\|^2 \leq \|x_i - p\|^2. \qquad (3.5)$$

Now, utilizing (3.1) and condition (C1), we have

$$\begin{aligned}\|y_i - p\|^2 &= \|\alpha_i u_i + (1 - \alpha_i)S_i u_i - p\|^2\\ &\leq \alpha_i\|u_i - p\|^2 + (1 - \alpha_i)\|S_i u_i - p\|^2 - \alpha_i(1 - \alpha_i)\|u_i - S_i u_i\|^2\\ &\leq \alpha_i\|u_i - p\|^2 + (1 - \alpha_i)\left\{\|u_i - p\|^2 + k\|u_i - S_i u_i\|^2\right\}\\ &\quad - \alpha_i(1 - \alpha_i)\|u_i - S_i u_i\|^2\\ &= \|u_i - p\|^2 - (1 - \alpha_i)(\alpha_i - k)\|u_i - S_i u_i\|^2\\ &\leq \|u_i - p\|^2. \qquad (3.6)\end{aligned}$$

As a consequence of (3.5) and (3.6), we get

$$\|y_i - p\|^2 \leq \|x_i - p\|^2.$$

Hence $p \in C_{i+1}$ this implies $\mathbb{F} \subset C_n$ for all $n \geq 1$.
Moreover, since

$$\left\{z \in H_1 : \|y_n - z\|^2 \leq \|x_n - z\|^2\right\} = \left\{z \in H_1 : \|y_n\|^2 - \|x_n\|^2 \leq 2\langle y_n - x_n, z\rangle\right\},$$

it is closed and convex. Summing up these facts, we conclude that C_n is nonempty, closed and convex for all $n \geq 1$, and hence the sequence $\{x_n\}$ is well-defined. Note that $\{C_n\}$ is a nonincreasing sequence of nonempty, closed and convex subsets of H_1 with respect to inclusion, it follows that $\bigcap_{n=1}^{\infty} C_n$ is nonempty. That is,

$$\emptyset \neq \mathbb{F} \subset M\text{-}\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n.$$

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then by Lemma 2.4, $\{x_n\} = \{P_{C_n}x_1\}$ converges strongly to $x = P_{C_0}x_1$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

It remains to show that $x \in \mathbb{F}$. For this, we divide the remaining part of the proof into the following steps:
Step 1. Show that:

(i) $\lim_{n\to\infty} \|y_n - x_{n+1}\| = \lim_{n\to\infty} \|y_n - x_n\| = 0$,
(ii) $\lim_{n\to\infty} \|u_n - x_n\| = 0$,
(iii) $\lim_{n\to\infty} \|y_n - u_n\| = 0$,
(iv) $\lim_{n\to\infty} \|S_n u_n - u_n\| = 0$.

Since $x_{n+1} \in C_{n+1} \subset C_n$, we have $\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|$. Therefore, using (3.7), we get that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{3.8}$$

Also, from (3.7), (3.8) and the following triangular inequality:

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\|,$$

we have

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.9}$$

Note that, the estimates (3.4) and (3.6) imply that

$$\begin{aligned} \gamma(1-\gamma L)\left\|A_n x_n - T_{s_n}^{G_n} A_n x_n\right\|^2 &\leq \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\leq (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\|. \end{aligned}$$

Since $\gamma(1-\gamma L) > 0$, therefore letting $n \to \infty$ in the above estimate and utilizing (3.9), we have

$$\lim_{n\to\infty} \left\|A_n x_n - T_{s_n}^{G_n} A_n x_n\right\|^2 = 0. \tag{3.10}$$

Since $T_{r_n}^{F_n}$ is firmly nonexpansive, then

$$\begin{aligned}\|u_n - p\|^2 &= \left\|T_{r_n}^{F_n}\left(I - \gamma A_n^*\left(I - T_{s_n}^{G_n}\right)A_n\right)x_n - T_{r_n}^{F_n}p\right\| \\ &\leq \left\langle u_n - p, x_n - \gamma A_n^*\left(I - T_{s_n}^{G_n}\right)A_n x_n - p\right\rangle \\ &= \frac{1}{2}\{\|u_n - p\|^2 + \left\|x_n - \gamma A_n^*\left(I - T_{s_n}^{G_n}\right)A_n x_n - p\right\|^2 \\ &\quad - \left\|u_n - x_n + \gamma A_n^*\left(I - T_{s_n}^{G_n}\right)A_n x_n\right\|^2\} \\ &\leq \frac{1}{2}\left\{\|u_n - p\|^2 + \|x_n - p\|^2 - \left\|u_n - x_n + \gamma A_n^*\left(I - T_{s_n}^{G_n}\right)A_n x_n\right\|^2\right\} \\ &= \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - (\|u_n - x_n\|^2 + \gamma^2\left\|A_n^*\left(I - T_{s_n}^{G_n}\right)A_n x_n\right\|^2 \\ &\quad -2\gamma\left\langle u_n - x_n, A_n^*\left(I - T_{s_n}^{G_n}\right)A_n x_n\right\rangle)\}.\end{aligned}$$

So, we obtain

$$\begin{aligned}\|u_n - p\|^2 &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\gamma\left\langle u_n - x_n, A_n^*\left(I - T_{s_n}^{G_n}\right)A_n x_n\right\rangle \\ &\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 \\ &\quad +2\gamma\|u_n - x_n\|\left\|A_n^*\left(I - T_{s_n}^{G_n}\right)A_n x_n\right\|.\end{aligned}$$

Utilizing the above estimate and (3.6), we have

$$\begin{aligned}\|u_n - x_n\|^2 &\leq (\|x_n - p\| + \|y_n - p\|)\|x_n - y_n\| \\ &\quad +2\gamma\|u_n - x_n\|\left\|A_n^*\left(I - T_{s_n}^{G_n}\right)A_n x_n\right\|.\end{aligned}$$

Taking lim sup on both sides of the above estimate, it then follows by using (3.9) and (3.10) that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.11}$$

From (3.9), (3.11) and the following triangular inequality:

$$\|y_n - u_n\| \leq \|y_n - x_n\| + \|x_n - u_n\|,$$

we get

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \tag{3.12}$$

Further $\|y_n - u_n\| = (1 - \alpha_n)\|S_n u_n - u_n\|$. It follows from (3.12) and the fact that $0 \leq k < a \leq \alpha_n \leq b < 1$ (by C1), we get

$$\lim_{n\to\infty}\|S_n u_n - u_n\| = 0. \tag{3.13}$$

Step 2. Show that $x \in \bigcap_{i=1}^N F(S_i)$.

We show that $x \in \bigcap_{i=1}^N F(S_i)$, that is, $x \in F(S_i)$ for $1 \leq i \leq N$. Since $x_{Nn+i} \to x$ as $n \to \infty$, it follows from (3.11) that $u_{Nn+i} \to x$ and consequently we have $u_{Nn+i} \rightharpoonup x$. Since $S_{Nn+i} = S_i$ for all $n \geq 1$, therefore from (3.13) and Lemma 2.3, we have that $x \in F(S_i)$ for each $1 \leq i \leq N$.

Step 3. Show that $x \in \Theta$, i.e., $x \in \bigcap_{i=1}^N EP(F_i)$ and $A_i x \in EP(G_i)$ for each $1 \leq i \leq N$.

In order to show that $x \in \bigcap_{i=1}^{N} EP(F_i)$, that is, $x \in EP(F_i)$ for each $1 \leq i \leq N$, we define subsequence $\{n_j\}$ of index $\{n\}$ such that $n_j = Nj + i$ for all $n \geq 1$. As a consequence, we can write $F_{n_j} = F_i$ for $1 \leq i \leq N$. From $u_{n_j} = T_{r_{n_j}}^{F_i}\left(I - \gamma A_{n_j}^*\left(I - T_{s_{n_j}}^{G_{n_j}}\right)A_{n_j}\right)x_{n_j}$ for all $n \geq 1$, we have

$$F_i(u_{n_j}, y) + \frac{1}{r_{n_j}}\left\langle y - u_{n_j}, u_{n_j} - x_{n_j} - \gamma A_{n_j}^*\left(I - T_{s_{n_j}}^{G_{n_j}}\right)A_{n_j}x_{n_j}\right\rangle \geq 0, \text{ for all } y \in C.$$

This implies that

$$F_i(u_{n_j}, y) + \frac{1}{r_{n_j}}\left\langle y - u_{n_j}, u_{n_j} - x_{n_j}\right\rangle - \frac{1}{r_{n_j}}\left\langle y - u_{n_j}, \gamma A_{n_j}^*\left(I - T_{s_{n_j}}^{G_{n_j}}\right)A_{n_j}x_{n_j}\right\rangle \geq 0$$

From (A2), we have

$$\frac{1}{r_{n_j}}\left\langle y - u_{n_j}, u_{n_j} - x_{n_j}\right\rangle - \frac{1}{r_{n_j}}\left\langle y - u_{n_j}, \gamma A_{n_j}^*\left(I - T_{s_{n_j}}^{G_{n_j}}\right)A_{n_j}x_{n_j}\right\rangle \geq F_i(y, u_{n_j}),$$

for all $y \in C$. Since $\liminf_{j\to\infty} r_{n_i} > 0$ (by (C2)), therefore it follows from (3.10) and (3.11) that

$$F_i(y, x) \leq 0, \quad \text{for all } y \in C \text{ and for } 1 \leq i \leq N.$$

Let $y_t = ty + (1-t)x$ for some $0 < t < 1$ and $y \in C$. Since $x \in C$, this implies that $y_t \in C$. Using (A1) and (A4), the following estimate:

$$0 = F_i(y_t, y_t) \leq tF_i(y_t, y) + (1-t)F_i(y_t, x) \leq tF_i(y_t, y),$$

implies that

$$F_i(y_t, y) \geq 0, \text{ for } 1 \leq i \leq N.$$

Letting $t \to 0$, we have $F_i(x, y) \geq 0$ for all $y \in C$. Thus, $x \in EP(F_i)$ for $1 \leq i \leq N$. That is, $x \in \bigcap_{i=1}^{N} EP(F_i)$. Reasoning as above, we show that $A_i x \in EP(G_i)$ for each $1 \leq i \leq N$. Since $x_{n_l} \longrightarrow x$ and A_{n_l} is a bounded linear operator, therefore $A_{n_l}x_{n_l} \longrightarrow A_{n_l}x$. Hence, it follows from (3.10) that

$$T_{s_{n_l}}^{G_{n_l}}A_{n_l}x_{n_l} \longrightarrow A_{n_l}x \quad \text{as} \quad l \to \infty.$$

Now, from Lemma 2.5 we have

$$G_i\left(T_{s_{n_l}}^{G_{n_l}}A_{n_l}x_{n_l}, z\right) + \frac{1}{s_{n_l}}\left\langle z - T_{s_{n_l}}^{G_{n_l}}A_{n_l}x_{n_l}, T_{s_{n_l}}^{G_{n_l}}A_{n_l}x_{n_l} - A_{n_l}x_{n_l}\right\rangle \geq 0, \text{ for all } z \in Q.$$

Since G_i is upper hemicontinuous in the first argument for each $1 \leq i \leq N$, therefore taking lim sup on both sides of the above estimate as $l \to \infty$ and utilizing (C2) and (3.10), we get

$$G_i\left(A_{n_l}x, z\right) \geq 0, \quad \text{for all } z \in Q \text{ and for each } 1 \leq i \leq N.$$

Hence $A_i x \in EP(G_i)$ for each $1 \leq i \leq N$. This together with the conclusion of Step 2, we have that $x \in \mathbb{F}$. This completes the proof. □

In particular, if S_i - - in algorithm (3.1) - - is a finite family of nonexpansive mappings, then we have the the following useful result:

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $F_i : C \times C \to \mathbb{R}$ and $G_i : Q \times Q \to \mathbb{R}$ be two nite families of bifunctions satisfying conditions (A1)-(A4) such that G_i is upper hemicontinuous for each $i \in \{1, 2, 3, \cdots, N\}$. Let $S_i : C \to C$ be a nite family of nonexpansive mappings and let $A_i : H_1 \to H_2$ be a nite family of bounded linear operators for each $i \in \{1, 2, 3, \cdots, N\}$. Suppose that $\mathbb{F} := \left[\bigcap_{i=1}^{N} F(S_i)\right] \cap \Theta \neq \emptyset$, where $\Theta = \left\{z \in C : z \in \bigcap_{i=1}^{N} EP(F_i) \text{ and } A_i z \in EP(G_i) \text{ for } 1 \leq i \leq N\right\}$. Let $\{x_n\}$ be a sequence generated by:*

$$
\begin{aligned}
& x_1 \in C_1 = C, \\
& u_n = T_{r_n}^{F_n}\left(I - \gamma A_{n(\operatorname{mod} N)}^{*}\left(I - T_{s_n}^{G_n}\right) A_{n(\operatorname{mod} N)}\right) x_n, \\
& y_n = \alpha_n u_n + (1 - \alpha_n) S_{n(\operatorname{mod} N)} u_n, \\
& C_{n+1} = \left\{z \in H_1 : \|y_n - z\|^{2} \leq \|x_n - z\|^2\right\} \cap C_n, \\
& x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1,
\end{aligned} \tag{3.14}
$$

where $\{r_n\}, \{s_n\}$ are two positive real sequences and $\{\alpha_n\}$ is a sequence in $(0, 1)$. Let $\gamma \in \left(0, \frac{1}{L}\right)$, where $L = \max\{L_1, L_2, \cdots, L_N\}$ and L_i is the spectral radius of the operator $A_i^* A_i$ and A_i^* is the adjoint of A_i for each $i \in \{1, 2, 3, \cdots, N\}$. Assume that $\{\alpha_n\}, \{r_n\}$ and $\{s_n\}$ satisfy the following restrictions:
$(C1) : 0 \leq k < a \leq \alpha_n \leq b < 1$;
$(C2) : \liminf_{n\to\infty} r_n > 0$ and $\liminf_{n\to\infty} s_n > 0$;
then the sequence $\{x_n\}$ generated by (3.14) converges strongly to $x = P_{\mathbb{F}} x_1$.

In order to solve the classical equilibrium problem together with the fixed point problem, we prove the following result:

Theorem 3.3. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $F_i : C \times C \to \mathbb{R}$ and $G_i : Q \times Q \to \mathbb{R}$ be two nite families of bifunctions satisfying conditions (A1)-(A4) such that G_i is upper hemicontinuous for each $i \in \{1, 2, 3, \cdots, N\}$. Let $S_i : C \to C$ be a nite family of k-strict pseudo contractions and let $A_i : H_1 \to H_2$ be a nite family of bounded linear operators for each $i \in \{1, 2, 3, \cdots, N\}$. Suppose that $\mathbb{F} := \left[\bigcap_{i=1}^{N} F(S_i)\right] \cap \Theta \neq \emptyset$, where $\Theta = \left\{z \in C : z \in \bigcap_{i=1}^{N} EP(F_i) \text{ and } A_i z \in EP(G_i) \text{ for } 1 \leq i \leq N\right\}$. Let $\{x_n\}$ be a sequence generated by:*

$$
\begin{aligned}
& x_1 \in C_1 = C, \\
& u_n = T_{r_n}^{F_n}\left(I - \gamma A_{n(\operatorname{mod} N)}^{*}\left(I - T_{s_n}^{G_n}\right) A_{n(\operatorname{mod} N)}\right) x_n, \\
& y_n = \alpha_n u_n + (1 - \alpha_n) S_{n(\operatorname{mod} N)} u_n, \\
& C_{n+1} = \left\{z \in H_1 : \|y_n - z\|^{2} \leq \|x_n - z\|^2\right\} \cap C_n, \\
& x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1,
\end{aligned} \tag{3.15}
$$

where $\{r_n\}, \{s_n\}$ are two positive real sequences and $\{\alpha_n\}$ is a sequence in $(0, 1)$. Let $\gamma \in \left(0, \frac{1}{L}\right)$, where $L = \max\{L_1, L_2, \cdots, L_N\}$ and L_i is the spectral radius of *the operator* $A_i^*A_i$ *and* A_i^* *is the adjoint of* A_i *for each* $i \in \{1, 2, 3, \cdots, N\}$*. Assume that* $\{\alpha_n\}, \{r_n\}$ *and* $\{s_n\}$ *satisfy the following restrictions:*

(C1) $0 \leq k < a \leq \alpha_n \leq b < 1$;
(C2) $\liminf_{n\to\infty} r_n > 0$ *and* $\liminf_{n\to\infty} s_n > 0$;

then the sequence $\{x_n\}$ *generated by (3.15) converges strongly to* $x = P_{\mathbb{F}}x_1$.

Proof. Set $H_1 = H_2,\ C = Q$ and $A = I$(the identity mapping) then the desired result then follows from Theorem 3.1 immediately. □

Remark 3.4. *It is instructive to compare the results of Theorems 3.1 & 3.3 in the current literature, in particular to those as mentioned above, that our results can be viewed as a generalization and improvement of various existing results in the current literature.*

Acknowledgment. The author wish to thank the anonymous reviewer, handling editor and Prof. Dr. Yasunori Kimura for careful reading and valuable suggestions to improve the quality of the paper.

References

[1] J. P. Aubin, *Optima and Equilibria: An Introduction to Nonlinear Analysis,* Springer, New York, NY, 1998.

[2] G. Beer, *Topologies on closed and convex sets,* Kluwer Academic Publishers Group, Dordrecht, 1993.

[3] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems,* Math. Stud. **63** (1994), 123–145.

[4] A. Bnouhachem, *Strong convergence algorithm for split equilibrium problems and hierarchical fixed point problems,* Sci. World J. **2014** (2014):390956, 16 pages.

[5] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space,* J. Math. Anal. Appl. **20** (1967), 197–228.

[6] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem,* Inverse Probl. **18** [illegible])Y(Ana6])-ensoe20(T(Ana6])Elfv(ma27(e20(N(Ana6i)-opfAna

[16] A. Kangtunyakarn, *Hybrid algorithm for finding common elements of the set of generalized equilibrium problems and the set of fixed point problems of strictly pseudo-contractive mapping,* Fixed Point Theory Appl. **2011** (2011): 274820, 19 pages.
[17] K. R. Kazmi and S. H. Rizvi, *Iterative approximation of a common solution of a split equilibrium problems, a variational inequality problem and fixed point problem,* J. Egypt. Math. Soc. **21** (2013), 44–51.
[18] M. A. A. Khan and H. Fukhar-ud-din, *Strong convergence by the shrinking effect of two half-spaces,* Fixed Point Theory Appl. **2013** (2013):30, 13 pages.
[19] Y. Kimura and W. Takahashi, *On a hybrid method for a family of relatively nonexpansive mappings in a Banach space.* J. Math. Anal. Appl. **357** (2009), 356–363.
[20] H. Komiya and W. Takahashi, *Strong convergence theorems by shrinking projection methods for generalized split feasibility problems in Hilbert spaces,* Pacific J. Optim. **12** (2016), 1–17.
[21] G. Marino and H. K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces,* J. Math. Anal. Appl. **329** (2007), 336–346.
[22] G. Marino, B. Scardamaglia and E. Karapinar, *Strong convergence theorem for strict pseudo-contractions in Hilbert spaces*, J. Inequal. Appl. **2016** (2016):134, 12 pp.
[23] A. Moudafi, *Split monotone variational inclusions,* J. Optim. Theory Appl. **150** (2011), 275–283.
[24] O. Scherzer, *Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems*, J. Math. Anal. Appl. **194** (1991), 911–933.
[25] S. Suantai, P. Cholamjiak, Y. J. Cho and W. Cholamjiak, *On solving split equilibrium problems and fixed point problems of nonspreading multi-valued mappings in Hilbert space,* Fixed Point Theory Appl. **2016** (2016):35, 16 pp.
[26] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces,* J. Math. Anal. Appl. **341** (2008), 276–286.
[27] M. Tsukada, *Convergence of best approximations in a smooth Banach space,* J. Approx. Theory **40** (1984), 301–309.
[28] K. Ungchittrakool, *Existence and convergence of fixed points for a strict pseudo-contraction via an iterative shrinking projection technique,* J. Nonlinear Convex Anal. **15** (2014), 693–710.
[29] S. Wang, X. Gong, A. An Abdou and Y. J. Cho, *Iterative algorithm for a family of split equilibrium problems and fixed point problems in Hilbert spaces with applications,* Fixed Point Theory Appl. **2016** (2016):4, 22 pp.

Manuscript received 21 April 2017
revised 20 November 2017

Muhammad Aqeel Ahmad Khan
Department of Mathematics, COMSATS Institute of Information Technology Lahore, 54000, Pakistan
E-mail address: itsakb@hotmail.com, maqeelkhan@ciitlahore.edu.pk