# ROLES OF ALTERNATIVE THEOREMS IN LINEAR COMPLEMENTARITY 

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#### Abstract

In this paper, we give a concise review of some strong relationships between alternative theorems and the linear complementarity problems. Espe cially, we discuss about the properties the matrix de ning linear complementarity problems and their solvability together with alternative theorems. Some new results are also included.


## 1. Introduction

Among many variations of alternative theorems in linear system, the following is the most classic and fundamental one. Let $A$ be a real $m \times n$ matrix and $b$ be an $m$-dimensional real vector. Farkas' lemma [4] states that one and only one of the following holds: (a) there exists $x \in \mathbb{R}^{n}$ satisfying that $A x=b, x \geq 0$ or (b) there exists $y \in \mathbb{R}^{m}$ satisfying that $A^{T} y \geq 0, b^{T} y<0$. It is well-known that the duality theorem in linear optimization can be proved by Farkas' lemma. For detail, see [10]. Other alternative theorems in linear system which is equivalent to Farkas' lemma are listed in the textbook [8]. Furthermore, some generalizations of alternative theorems in the setting of convex cones and linear subspaces exist [12]. These generalized alternative theorems can be applied to the duality of cone programming [1].

A natural extension of the alternative theorem to the setting of linear complementarity exists [5]. Let $A$ be an $n \times n$ real square matrix, and $b$ be an $n$ dimensional real vector. The linear complementarity problem (LCP), is the problem of the form:

$$
\begin{array}{l|ll}
\operatorname{LCP}(A, b) & \text { find } & x, y \in \mathbb{R}^{n}  \tag{1.1}\\
\text { satisfying that } & y=A x+b, x \geq 0, y \geq 0, \\
& x_{i} \cdot y_{i}=0(i=1,2, \ldots, n)
\end{array}
$$

Any linear optimization, any quadratic convex optimization can be formulated to LCPs. If a given matrix $A$ satisfies a certain condition, then exactly one of (a) there exists a solution to $\operatorname{LCP}(A, b)$ and (b) there exist vectors $w$ and $z$ such that $w=-A^{T} z, w, z \geq 0, b^{T} z<0, z_{i} \cdot w_{i}=0(i=1,2, \ldots, n)$ holds. Note that the problem of finding vectors requested in (b) is regarded as the dual LCP.

[^0]The solvability of the LCP, in general, depends on the properties of square matrix $A$ and the vector $b$ defining the problem. Typical example is that if $A$ belongs to the class of P-matrix, then there exists a unique solution to $\operatorname{LCP}(A, b)$ for any vector $b$. Here, a square matrix is called P-matrix if every principal minors of $A$ are positive. Thus many researchers are interested in detecting the given matrix $A$ to be P -matrix or not, which is called P-matrix problem. Unfortunately, the P-matrix problem is co- $N P$-complete [3], that means we can not expect polynomial algorithms for it.

For the P-matrix problem, Morris and Namiki [9] proposed hidden matrix class that contains P -matrix class and that is contained in P -matrix class. Here we call $A$ to be hidden $\mathcal{C}$ if there exist matrices $C$ and $B$ such that $A C=B$, where $\mathcal{C}$ is a matrix class. Alternative theorems for $A$ to be hidden matrix also exist, and they are used for detecting a membership for $A$ to be hidden matrix class.

Another story about solvability of LCPs is due to the paper [7]. Mangasarian gave sufficient conditions of $A$ and $b$ for $\operatorname{LCP}(A, b)$ to be solvable as a linear optimization problem. Note that the linear optimization in general is polynomially solvable. We can see that one of hidden matrix classes satisfy this sufficient conditions.

The paper is organized as follows. In section 2, we list several forms of alterna-
(b) There exists $y \in \mathbb{R}^{m}$ satisfying that $A^{T} y \geq 0, b^{T} y<0$.

Proof. A proof is immediate once we take $A=[A,-b], N=\{1,2, \ldots, n, n+1\}$, $I=\{1,2, \ldots, n\}, J=K=\emptyset, L=\{n+1\}$.

Alternative theorems in linear systems can be generalized in the context of convex cones and their dual cones. A subset $\mathcal{C}$ of $\mathbb{R}^{n}$ is called a convex cone if $\alpha x+\beta y \in \mathcal{C}$ for any $x, y \in \mathcal{C}$ and $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta>0$. For a subset $\mathcal{C} \subseteq \mathbb{R}^{n}$, we define $\mathcal{C}^{*}=\left\{y \in \mathbb{R}^{n} \mid x^{T} y \geq 0\right.$ for all $\left.x \in \mathcal{C}\right\}$, which is called dual of $\mathcal{C}$. For any $\mathcal{C} \subseteq \mathbb{R}^{n}$, $\mathcal{C}^{*}$ is closed convex cone. Moreover, if $\mathcal{C}$ is a closed convex cone then $\mathcal{C}^{* *}=\mathcal{C}$. A linear subspace $\mathcal{L} \subseteq \mathbb{R}^{n}$ is also closed convex cone and $\mathcal{L}^{*}=\mathcal{L}^{\perp}$, where $\mathcal{L}^{\perp}$ is the orthogonal complement of $\mathcal{L}$.

Now, we introduce generalized alternative theorem in the context of cone and linear subspace.
Theorem 2.3 (Namiki and Tsukada [12]). Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^{n}, \mathcal{C}$ be a convex cone such that image of $\mathcal{C}$ by the orthogonal projection onto $\mathcal{L}$ is closed, and $b \in \mathcal{L}$. Then one and only one of the following holds:
(a) There exists $x \in \mathcal{L}^{\perp}$ such that $x+b \in \mathcal{C}$.
(b) There exists $y \in \mathcal{L}$ such that $y \in C^{*}$ and $b^{T} y<0$.

The above theorem can be slightly modified to adapt alternative theorem on hidden matrices in the context of cone of matrices.
Theorem 2.4. Let $\mathcal{L}$ be a linear subspace of $\mathbb{R}^{n}$ and $\mathcal{C}$ be non-empty open convex cone such that image of $\operatorname{cl}(\mathcal{C})$ by the orthogonal projection onto $\mathcal{L}$ is closed. Then one and only one of the following holds:
(a) There exists $x \in \mathcal{L}^{\perp}$ such that $x \in \operatorname{ri}(\mathcal{C})$.
(b) There exists $y \in \mathcal{L}$ such that $y \in C^{*}$ and $y \neq 0$.

Where $\operatorname{ri}(\mathcal{C})$ is the set of relative interior point of $\mathcal{C}$ and $\operatorname{cl}(\mathcal{C})$ is the closure of the set $\mathcal{C}$.

Proof. It is easily seen that never both occur simultaneously. Assume that $\mathcal{L}^{\perp} \cap$ $\operatorname{ri}(\mathcal{C})=\emptyset$. Then there exists $z \in \operatorname{ri}(\mathcal{C})$ such that $z \notin \mathcal{L}^{\perp}$. Let $b$ to be the orthogonal projection of $-z$ onto $\mathcal{L}$. Then $b \neq 0$ and we have $x+b \notin \operatorname{cl}(\mathcal{C})$ for all $x \in \mathcal{L}^{\perp}$. Now we can apply Theorem 2.3 for $\mathcal{L}$ and $\operatorname{cl}(\mathcal{C})$. We get $y \in \operatorname{cl}(\mathcal{C})^{*}=\mathcal{C}^{*}$, and clearly $y \neq 0$.

In order to extend alternative theorems in more general setting, we define the notion of hidden matrix class for square matrices.
Definition 2.5 (hidden matrices). Let $\mathcal{C}$ be a class of square matrix. A square matrix $A$ is called hidden $\mathcal{C}$ if there exist square matrices $C$ and $B$ in $\mathcal{C}$ satisfying that $A C=B$.

For two square matrices $R$ and $C$, we we define the inner product of $R$ and $C$ as $R \cdot S:=\operatorname{trace}\left(R^{T} S\right)$. This is the usual inner product if one considers the matrix to be the vector of length $n^{2}$. If $C, B$ is in the class of convex cone of matrices $\mathcal{C}$, then the pair $(C, B)$ is in $\mathcal{C} \times \mathcal{C}$. The set of matrices $(R, S)$ that satisfy $(R, S) \cdot(C, B) \geq 0$ for all $C, B$ in a convex cone $\mathcal{C}$ is called the dual cone of $\mathcal{C} \times \mathcal{C}$ and is denoted $(\mathcal{C} \times \mathcal{C})^{*}=\left(\mathcal{C}^{*} \times \mathcal{C}^{*}\right)$.

For a given matrix $A \in \mathbb{R}^{n \times n}$, let us define $\mathcal{L}_{A}$ to be the set of matrices $(C, B)$ satisfying that $C, B \in \mathbb{R}^{n \times n}$ and $A C=B$. Then, $\mathcal{L}_{A}$ is a subspace of $\mathbb{R}^{n \times 2 n}$ spanned by the matrices ( $E_{i j}, A E_{i j}$ ), where $E_{i j}$ is the square matrix with $(i, j)$ element equal to one and zeros everywhere else. It follows that $(R, S) \cdot\left(E_{i j}, A E_{i j}\right)=R_{i j}+\left(A^{T} S\right)_{i j}$. The set of matrices $(R, S)$ that satisfy $R+A^{T} S=0$ is therefore the orthogonal complement $\mathcal{L}_{A}^{\perp}$ of $\mathcal{L}_{A}$. In the rest of this paper, we assume that matrices $C$ and $B$ belong to some class consisting of convex cone of matrices.

Now we are ready to introduce alternative theorem of hidden matrix which already appeared in [9] but the proof is different.
Theorem 2.6. Let $A \in \mathbb{R}^{n \times n}, \mathcal{L}_{A}=\left\{(C, B) \mid C, B \in \mathbb{R}^{n \times n}, A C=B\right\}, \mathcal{L}_{A}^{\perp}=$ $\left.\{R, S) \mid R, S \in \mathbb{R}^{n \times n}, R+A^{T} S=0\right\}$ and $\mathcal{C}$ be a nonempty, open convex cone in $\mathbb{R}^{n \times n}$ such that the image of $\operatorname{cl}(\mathcal{C})$ by the orthogonal projection onto $\mathcal{L}_{A}^{\perp}$ is closed. Then one and only one of the following hold:
(a) There exists $(C, B)$ in $(\mathcal{C} \times \mathcal{C}) \cap \mathcal{L}_{A}$.
(b) There is a nonzero $(R, S)$ in $\mathcal{L}_{A}^{\perp} \cap(\mathcal{C} \times \mathcal{C})^{*}$.

Proof. A proof is immediate from Theorem 2.4.
Note that the existence of matrices in (a) represents that $A$ is hidden $\mathcal{C}$ matrix and that it is hard to check the closedness of the image of $\operatorname{cl}(\mathcal{C})$, However, we don't need to check it if the cone $\mathcal{C}$ is polyhedral. Hidden matrices with polyhedral cone of matrices have nice properties and plays a central role in polynomial solvability in LCPs.

## 3. Matrix classes and their properties

In this section we introduce several classes of real square matrix which play important roles in linear complementarity problems.

The first one is a P-matrix, which is the most important for the linear complementarity.
Definition 3.1 (P-matirx). A real $n \times n$ square matrix $C$ is called a $P$-matrix if every principal minor of $C$ is positive, that is, $\operatorname{det}\left(C_{L L}\right)>0$ for all subsets $L$ of index set $N=\{1,2, \ldots, n\}$.

Note that $C_{L L}$ is called principal sub-matrix of $C$. As the most impressive characterizations of the P-matrix, we have the following fact related to the linear complementarity.
Theorem 3.2. Let $A$ be an $n \times n$ a $P$-matrix. Then $\operatorname{LCP}(A, b)$ has a unique solution for any $b$.

This is fundamental and the reader can find it in the standard LCP textbook [2]. Detecting, generating and characterizing the class of P-matrix are attractive topic of the LCP researchers.
Definition 3.3. Let $C$ be an $n \times n$ real square matrix.
(1) $C$ is $p r d d-1$ matrix if $C_{i i}>\sum_{j \neq i}\left|C_{i j}\right|$ for all $i=1,2, \ldots, n$.
(2) $C$ is $p r d d-\infty$ matrix if $C_{i i}>\left|C_{i j}\right|$ for all $i, j=1,2, \ldots, n$ with $i \neq j$.
(3) $C$ is $Z^{o}$ - matrix if $C_{i j}<0$ for all $i, j=1,2, \ldots, n$ with $i \neq j$ and $C e>0$.
(4) $C$ is $Z^{o}+$ matrix if $C_{i j}<0$ for all $i, j=1,2, \ldots, n$ with $i \neq j, C_{i i}>0$ for all $i$ and $C e<0$.
Where $e$ be the $n$-dimensional vector with all entries to be ones.
Note that prdd is an abbreviation form of positive row diagonally dominant and usually we use $Z$-matrix if every off-diagonal element is non-positive. Let $\mathcal{K}_{\text {prdd-1 }}$ be the set of prdd-1 matrix, $\mathcal{K}_{\text {prdd- }}, \mathcal{K}_{Z^{\circ}-}$ and $\mathcal{K}_{Z^{\circ}+}$, respectively. Inclusion $\mathcal{K}_{Z^{\circ}-} \subseteq \mathcal{K}_{\text {prdd }-1} \subseteq \mathcal{K}_{\text {prdd }-\infty}$ and the relation $\mathcal{K}_{Z^{\circ}-} \cap \mathcal{K}_{Z^{\circ}+}$ hold.

It is easily checked that each of the sets listed in Definition 3.3 is an open convex polyhedral cones in the space of real $n \times n$ matrices. The following proposition ensure us this fact and gives us extreme rays of the cone of each matrix class.

Proposition 3.4. Let $C \in \mathbb{R}^{n \times n}$ and $N=\{1,2, \ldots, n\}$.
(1) $C \in \mathcal{K}_{\text {prdd-1 }}$ if and only if there exist $y_{i j}>0$ and $z_{i j}>0$ for all $i, j \in N, i \neq j$ and such that $C=\sum_{i \neq j}\left(y_{i j}\left(E_{i i}-E_{i j}\right)+z_{i j}\left(E_{i i}+E_{i j}\right)\right)$.
(2) $C \in \mathcal{K}_{\text {prdd- }}$ if and only if there exists $y_{i, J}>0$ for every pair $(i, J)$ with $J \subseteq N, i \notin J$, such that $C=\sum_{i \notin J} y_{i, J}\left(\sum_{j \notin J} E_{i j}-\sum_{j \in J} E_{i j}\right)$.
(3) $C \in \mathcal{K}_{Z^{\circ}-}$ if and only if there exist $y_{i j}>0$ for $i, j \in N, i \neq j$ and $z_{i}>0$ for all $i \in N$ such that $C=\sum_{i \neq j} y_{i j}\left(E_{i i}-E_{i j}\right)+\sum_{i=1}^{n} z_{i} E_{i i}$.
(4) $C \in \mathcal{K}_{Z^{\circ}+}$ if and only if there exist $y_{i j}>0$ and $z_{i j}>0$ for $i, j \in N, i \neq j$ such that $C=\sum_{i \neq j}\left(y_{i j}\left(E_{i i}-E_{i j}\right)+z_{i j}\left(-E_{i j}\right)\right)$.
Where $E_{i j}$ is the square matrix with $(i, j)$ element equal to one and zeros everywhere else.

By using above matrix classes, we can establish the class of hidden matrices. Before doing this, we analyze the properties of base matrix classes themselves.

For an $C \in \mathbb{R}^{n \times n}$ matrix, the comparison matrix of $C$, denoted by is $M(C)$, is the $n \times n$ matrix satisfying that $(M(C))_{i i}=\left|C_{i i}\right|$ for $i=1,2, \ldots, n$, and $(M(C))_{i j}=$ $-\left|C_{i j}\right|$ for $i, j(\neq i)=1,2, \ldots, n$. It is easily checked that if $C$ is $Z^{o}$ - matrix then $C=M(C)$. Moreover, for a matrix $C$ and a vector $v$, we denote the matrix and the vector whose entries are equal to the absolute of their original entries, by $|C|$ or $|v|$ respectively.

Now we have the following properties according to the inverse of prdd-1 matrix and $Z^{o}$ - matrix.

Lemma 3.5. Let $C \in \mathbb{R}^{n \times n}$ be prdd-1 matrix and $M \in \mathbb{R}^{n \times n}$ be the comparison matrix of $C$. Then $C^{-1}$ and $M^{-1}$ exist and the following inequalities hold:
(i) $M^{-1} \geq\left|C^{-1}\right|$ and $C_{i i}^{-1}>0$ for $i=1,2, \ldots, n$.
(ii) $C_{i i}^{-1}>\left|C_{j i}^{-1}\right|, M_{i i}^{-1}>M_{j i}^{-1}$ for $i, j=1,2 \ldots, n$ with $i \neq j$.

Note that $M$ is a $Z^{o}$ - matrix. So these are also properties about $Z^{o}$ - matrix.
Proof. We shall prove the existence of $C^{-1}$ and $M^{-1}$ and inequalities (i) by induction on $n$. For $n=1$, clearly the statements holds.

For $n>1$, let $C^{\prime}$ be a prdd- 1 matrix and $M^{\prime}$ be its comparison matrix and assume that the claim holds for any smaller matrices than $C^{\prime}$. We distinguish the last rows
and columns of $C^{\prime}$ and $M^{\prime}$ from the the others illustrated as follows.

$$
C^{\prime}=\left[\begin{array}{c|c}
C & a \\
\hline b^{T} & d
\end{array}\right], \quad M^{\prime}=\left[\begin{array}{c|c}
M & -|a| \\
\hline-|b|^{T} & d
\end{array}\right],
$$

where $C, M, a, b$ and $d$ are appropriate size of matrices, vectors and a scalar. Since $C$ is a prdd-1 matrix again, $C^{-1}$ and $M^{-1}$ exists and the inequalities (i) hold for them. Thus we have $e>M^{-1}|a|$ and $d-|b|^{T} e>0$. Let $\delta_{M}=d-|b|^{T} M^{-1}|a|$ and $\delta_{C}=d-b^{T} C^{-1} a$. We can see that $\delta_{C} \geq \delta_{M}>0$ as follows: $\delta_{M}=d-$ $|b|^{T} M^{-1}|a|=d-|b|^{T} e+|b|^{T}\left(e-M^{-1}|a|\right)>0, \delta_{C}=d-b^{T} C^{-1} a \geq d-|b|^{T}\left|C^{-1}\right||a|$ $\geq d-|b|^{T} M^{-1}|a|=\delta_{M}>0$. Therefore there exist $C^{\prime-1}$ and $M^{\prime-1}$ having the following actual forms:

$$
\begin{gathered}
C^{\prime-1}=\frac{1}{\delta_{C}}\left[\begin{array}{c|c}
\delta_{C} C^{-1}+C^{-1} a b^{T} C^{-1} & -C^{-1} a \\
-b^{T} C^{-1} & 1
\end{array}\right] \\
M^{\prime-1}=\frac{1}{\delta_{M}}\left[\begin{array}{c|c}
\delta_{M} M^{-1}+M^{-1}|a||b|^{T} M^{-1} & M^{-1}|a| \\
|b|^{T} M^{-1} & 1
\end{array}\right] .
\end{gathered}
$$

We have the following series of inequalities to prove inequalities (i).

$$
\begin{aligned}
\left|C_{n n}^{\prime-1}\right| & =\frac{1}{\delta_{C}} \leq \frac{1}{\delta_{M}}=M_{n n}^{\prime-1}, \quad C_{n n}^{\prime-1}=\frac{1}{\delta_{C}}>0, \\
\left|C_{i n}^{\prime-1}\right| & =\frac{1}{\delta_{C}}\left|-C^{-1} a\right|_{i} \leq \frac{1}{\delta_{M}}\left(M^{-1}|a|\right)_{i}=M_{i n}^{\prime-1} \quad \text { for } i=1, \ldots, n-1, \\
\left|C_{n j}^{\prime-1}\right| & =\frac{1}{\delta_{C}}\left|-b^{T} C^{-1}\right|_{n j} \leq \frac{1}{\delta_{M}}\left(|b|^{T} M^{-1}\right)_{j}=M_{n j}^{\prime-1} \text { for } j=1, \ldots, n-1, \\
\left|C_{i j}^{\prime-1}\right| & =\left|C_{i j}^{-1}+\frac{1}{\delta_{\delta}}\left(C^{-1} a b^{T} C^{-1}\right)_{i j}\right| \\
& \leq\left|C_{i j}^{-1}\right|+\frac{1}{\delta_{C}}\left(\left|C^{-1}\right||a||b|^{T}\left|C^{-1}\right|\right)_{i j} \\
& \leq M_{i j}^{-1}+\frac{1}{\delta_{M}}\left(M^{-1}|a||b|^{T} M^{-1}\right)_{i j}=M_{i j}^{\prime-1} \text { for } i, j=1,2, \ldots, n-1 .
\end{aligned}
$$

Next we show (ii). For $C_{i i}^{-1}>\left|C_{j i}^{-1}\right|$, suppose the contrary, that is, there exist $i, j$ such that $C_{i i}^{-1} \leq\left|C_{j i}^{-1}\right|$. Let $r$ be an index that attains $\max _{k \neq i}\left|C_{k i}^{-1}\right|$. If $C_{r i}^{-1}>0$, then for $i$-th column of $C^{-1}$, we have $C_{r i}^{-1} \geq\left|C_{j i}^{-1}\right|$ for all $j \neq i$. Because $C$ is prdd-1 matrix, $\sum_{j=1}^{n} C_{r j} C_{j i}^{-1}>0$. If $C_{r i}^{-1}<0$, then we have $-C_{r i}^{-1} \geq\left|-C_{j i}^{-1}\right|$ for all $j \neq i$. Because $C$ is prdd-1 matrix, $\sum_{j=1}^{n} C_{r j}\left(-C_{j i}^{-1}\right)>0$. These contradict that $C^{-1}$ is the inverse of $C$. We can prove in the same way for the $Z^{o}-$ matrix $M$.

We use these properties about the inverse of $Z^{o}$ - matrix to show polynomial solvability of LCP with hidden $Z^{o}$ - matrix in Section 4.

From here, we discuss about the properties of hidden matrix classes. By combining the definitions of base matrix and the notion of hidden matrices, we get the following concrete hidden matrix class which plays important role in the linear complementarity.

Definition 3.6 (hidden prdd-1, prdd- $\infty, Z^{o}-$ and $Z^{o}+$ matrices). A matrix $A \in$ $\mathbb{R}^{n \times n}$ is called hidden prdd-1 (prdd- $\infty, Z^{o}-, Z^{\circ}+$, resp.) if there are matrices $C$ and $B$ in $\mathcal{K}_{\text {prdd-1 }}\left(\mathcal{K}_{p r d d-\infty}, \mathcal{K}_{Z^{\circ}-}, \mathcal{K}_{Z^{\circ}+}\right)$ so that $A C=B$.

By applying Theorem 2.6 to base matrices consisting of cone in $\mathbb{R}^{n \times n}$, we also get the following alternative theorems for hidden matrix classes. To save space, we only give for hidden prdd-1 and $Z^{o}-$ matrices.

Corollary 3.7. Let $A \in \mathbb{R}^{n \times n}$.
(1) One and only one of the following holds:
(a) $A$ is hidden prdd-1.
(b) There exist matrices $R$ and $S$, not both 0 , so that $R+A^{T} S=0, R_{i i} \geq$ $\left|R_{i j}\right|, S_{i i} \geq\left|S_{i j}\right|$ for all $i \neq j$.
(2) One and only one of the following holds:
(a) $A$ is hidden $Z^{o}$-.
(b) There exist matrices $R$ and $S$, not both 0 , so that $R+A^{T} S=0, R_{i i} \geq$ $R_{i j}, S_{i i} \geq S_{i j}$ for all $i \neq j, R_{i i} \geq 0, S_{i i} \geq 0$ for all $i$.

We can utilize the above alternative theorem to detect $A$ to be a hidden matrix or not efficiently. That is, we can get matrices $(C, B)$ or $(R, S)$ in the statemetns by solving appropriate linear optimization, which is polynomially solvable [6].

There are inclusion relations discovered in [9]. Here we introduce only the results.
Theorem 3.8 ( [9]). The following inclusion relations hold.
(1) Every hidden $Z^{o}$ - matrix is a hidden prdd-1 matrix.
(2) Every hidden prdd-1 matrix is a P-matrix.
(3) Every P-matrix is a hidden $Z^{o}+$ matrix.
(4) Every hidden $Z^{o}+$ matrix is a hidden prdd- $\infty$ matrix.

We also have some matrix transformations preserving hidden prdd-1 matrix and hidden $Z^{o}$ - matrix memberships.
Theorem 3.9. Let $A \in \mathbb{R}^{n \times n}$ be a hidden prdd-1 matrix (hidden $Z^{o}$ - matrix, respectively). Then the following hold.
(1) $Q A Q^{T}$ is hidden prdd-1 matrix for every permutation matrix $Q$.
(2) $S A S$ is hidden prdd-1 matrix for every signature matrix $S$.
(3) $D A E$ is hidden prdd-1 matrix for all diagonal matrices $D, E$ such that $D_{i i} E_{i i}>0$ for all $i=1,2, \ldots n$.
(4) $A+D$ is hidden prdd-1 matrix for all diagonal matrices $D$ with all nonnegative diagonal elements. diagonal.
(5) $A^{-1}$ is hidden prdd-1 matrix.
(6) $I+F_{A}, I-F_{A}$ are hidden prdd-1 matrices, where $F_{A}=(I+A)^{-1}(I-A)$.

Proof. It is easily proved from 1. to 4 . For 5 , let $A$ be hidden prdd-1. From Lemma 3.5, there exist prdd-1 matrices $C, B$ such that $A=B C^{-1} . A^{-1}=C B^{-1}$ implies $A^{-1} B=C$. Thus $A^{-1}$ is hidden prdd-1. For 6 , let $A$ be hidden prdd-1. We have $(I+A)\left(I+F_{A}\right)=2 I$. Then $I+F_{A}=2(I+A)^{-1}$. From 4 and $5, I+F_{A}$ is hidden prdd-1 again. For $I-F_{A}$, let $A$ be hidden prdd-1. Then $A$ is singular. Because $I+A=A\left(I+A^{-1}\right)$ is singular, we have $(I+A)^{-1}=\left(I+A^{-1}\right)^{-1} A^{-1}$. Moreover, $I-F_{A}=I-(I+A)^{-1}(I-A)=I-\left(I+A^{-1}\right)^{-1} A^{-1}(I-A)=I+\left(I+A^{-1}\right)^{-1}\left(I-A^{-1}\right)$. We get $\left(I+A^{-1}\right)\left(I-F_{A}\right)=2 I$. Finally, we get $I-F_{A}=2\left(I+A^{-1}\right)^{-1}$, which is hidden prdd-1 again.

We can utilize Corollary 3.7 and Theorem 3.9 in order to generate P-matrix. For example, if $C$ and $B$ are both prdd-1 matrices then $A=B C^{-1}$ is a P-matrix. Since every transformations of Theorem 3.9 preserve the hidden prdd-1 property, then the new P-matrix will be obtained.

We should also note that the above theorem hold for hidden $Z^{o}-$ matrix and P -matrix. Moreover, if $A$ is P -matrix the trasposition of $A$ is again P -matrix. Howerver, the transposition does not preserve memberships of hidden $Z^{o}-$ class and hidden prdd-1 matrix class.

Here we introduce a hidden $Z^{o}$ - matrix whose transpose is not even a hidden prdd-1 matrix. One can easily check the membership of hidden $Z^{o}-$ matrix class and hidden prdd- 1 class by Corollary 3.7.

Example 3.10. Let $A=\left[\begin{array}{cccc}1 & -6 & 0 & -2 \\ 4 & 4 & 1 & 3 \\ -2 & -4 & 1 & 1 \\ 4 & -1 & -2 & 2\end{array}\right]$. Then $A$ is hidden $Z^{o}-$. Because
$C=\left[\begin{array}{cccc}342 & -2 & -312 & -2 \\ -2 & 228 & -51 & -173 \\ -2 & -2 & 1456 & -1396 \\ -688 & -2 & -2 & 698\end{array}\right]$ and $B=\left[\begin{array}{cccc}1730 & -1366 & -2 & -360 \\ -706 & 896 & -2 & -2 \\ -1366 & -912 & 2282 & -2 \\ -2 & -236 & -4113 & 4353\end{array}\right]$
satisfy the conditions of $Z^{o}-$ matrix and $A C=B$. However, $A^{T}$ is not even hidden prdd-1 because $R=\left[\begin{array}{cccc}56 & 56 & 28 & 56 \\ -96 & 96 & -72 & -40 \\ 84 & 84 & 84 & -84 \\ -35 & 63 & 63 & 63\end{array}\right]$ and $S=\left[\begin{array}{cccc}14 & -14 & 14 & -14 \\ 13 & 15 & 15 & -1 \\ 0 & -28 & 28 & 28 \\ -4 & -24 & -24 & 24\end{array}\right]$ satisfy the conditions for $A^{T}$ not to be hidden prdd-1 matrix ((b) of 1 in Corollary $2)$.

## 4. Polynomial solvability of the LCP

In the last part, we discuss about polynomial solvability of the LCP. In fact, we can see that if the input matrix $A$ is hidden $Z^{o}$ - matrix, then both $\operatorname{LCP}(A, b)$ and $\mathrm{LCP}\left(A^{T}, b\right)$ are polynomially solvable, but algorithms for solving them are different.

We define the feasible set of the $\operatorname{LCP}(A, b)$ by $\mathcal{X}(A, b)=\left\{x \in \mathbb{R}^{n} \mid A x+b \geq 0, x \geq\right.$ $0\}$. The following proposition ensures the existence of candidates for $\operatorname{LCP}(A, b)$ if $A$ belongs to the class of P-matrix.

Proposition 4.1. Let $A$ be a P-matrix. Then $\mathcal{X}(A, b) \neq \emptyset$ and $L C P(A, b)$ has a unique solution for any $b \in \mathbb{R}^{n}$.

We omit the proof since one can find it in the textbook [2]
Now we introduce a powerful properties about the polynomial solvability of $\operatorname{LCP}(A, b)$ by Mangasarian [7].

Theorem $4.2([7])$. If $\mathcal{X}(A, b) \neq \emptyset$ and there exists $\delta \in \mathbb{R}, r, s \in \mathbb{R}^{n}, Z_{1}, Z_{2} \in \mathbb{R}^{n \times n}$ such that

$$
\begin{aligned}
& A Z_{1}=Z_{2}+\delta b e^{T} \\
& r^{T} Z_{1}+s^{T} Z_{2} \geq 0 \\
& r+s>0, \delta>0 \\
& Z_{1}, Z_{2} \in Z, r, s \geq 0
\end{aligned}
$$

then the $\operatorname{LCP}(A, b)$ has a solution which can be obtained by solving the linear optimization: $\min \left\{c^{T} x \mid A x+b \geq 0, x \geq 0\right\}$ with $c=r+A^{T} s$.

We note that several sufficient conditions of $A$ and $b$ for the $\operatorname{LCP}(A, b)$ to be solved by linear optimization, are proposed in [7]. The above theorem is the most useful for our discussion.

In considering the properties of $Z^{o}-$ matrix and hidden $Z^{o}$ matrix, we can see the following theorem.

Theorem 4.3. Let $A$ be hidden $Z^{o}$ - matrix. Then for any $b \in \mathbb{R}^{n}, \mathcal{X}(A, b) \neq \emptyset$ and there exist a vector $c \in \mathbb{R}^{n}$ such that the solution to $\operatorname{LCP}(A, b)$ can be obtained by solving linear optimization: $\min \left\{c^{T} x \mid A x+b \geq 0, x \geq 0\right\}$.

Proof. From Proposition 4.1, $\mathcal{X}(A, b) \neq \emptyset$ for any $b \in \mathbb{R}^{n}$. Let $A$ be hidden $Z^{o}-$ matrix and $b \in \mathbb{R}^{n}$. Then there exist $Z^{o}-$ matrix $C, B$ such that $A C=B$. From Lemma 3.5, both $C$ and $B$ has non-negative inverse and thus $A=B C^{-1}$ is nonsingular. Let $Z_{1}=C+\delta A^{-1} b e^{T}$ and $Z_{2}=B$. For sufficiently small positive number $\delta, Z_{1}$ and $Z_{2}$ still remain $Z^{o}-$ matrix and $A Z_{1}=Z_{2}+\delta b e^{T}$ holds. Take $r^{T}=u^{T} Z_{1}^{-1} \geq 0$ and $s^{T}=v^{T} Z_{2}^{-1} \geq 0$ for some positive vectors $u, v$. Then $r+s>0$ because the inverse of $Z^{o}-$ matrix always non-negative matrix. Therefore, from Theorem 4.2, the solution of $\operatorname{LCP}(A, b)$ can be obtained by solving linear optimization: $\min \left\{c^{T} x \mid A x+b \geq 0, x \geq 0\right\}$ for $c=r+A^{T} s$.

The above theorem means that $\operatorname{LCP}(A, b)$ can be polynomially solvable when the input matrix $A$ is hidden $Z^{o}$ - matrix, because the linear optimization problem is solved in polynomial time with some interior point methods.

Here is an example of the above theorem.
Example 4.4. Let $A=\left[\begin{array}{cccc}1 & -6 & 0 & -2 \\ 4 & 4 & 1 & 3 \\ -2 & -4 & 1 & 1 \\ 4 & -1 & -2 & 2\end{array}\right]$ and $b=\left[\begin{array}{c}-1 \\ -2 \\ -3 \\ -4\end{array}\right]$ (same $A$ in Example 3.10). Then $A$ is hidden $Z^{o}$ - with the same $C, B$ in Example 3.10.

For $\delta=1$, let $Z_{1}=C+\delta A^{-1} b e^{T}$ and $Z_{2}=B$. Then $Z_{1}$ and $Z_{2}$ remain $Z^{o}-$ matrix and $A Z_{1}=Z_{2}+\delta b e^{T}$. We define vectors $r, s$ as $r^{T}=e^{T} Z_{1}^{-1}$, $s^{T}=e^{T} Z_{2}^{-1}$. Note that, from Lemma $3.5 r, s>0$ hold. Let $c=r+A^{T} s$, then approximately $c^{T}=[0.1993,0.01207,0.04469,0.09591]$. We get the optimal solution $x^{* T}=[6,0,25 / 2,5 / 2]$ and $y=A x^{*}+b=[0,42,0,0]$, which is the solution to $\operatorname{LCP}(A, b)$.

Next, we consider the $\operatorname{LCP}\left(A^{T}, b\right)$ when the input matrix $A$ is hidden $Z^{o}$ - matrix. Let $p$ be positive $n$-vector. We consider the parametric LCP denoted by
$\operatorname{pLCP}(A, b, p)$ as follows.

$$
\begin{array}{l|ll}
\mathrm{pLCP}(A, b, q) & \text { find } & x, y \in \mathbb{R}^{n} \text { for } t \geq 0 \\
\text { satisfying that } & y=b+t p+A x, \\
& x, y \geq 0 \text { and } \\
& x_{i} \cdot y_{i}=0(i=1,2, \ldots, n) .
\end{array}
$$

Since $p$ is a positive vector, $\operatorname{pLCP}(A, b, p)$ always has a solution $\left(x=0, y=b+t_{0} p\right)$ for some sufficiently large number $t_{0}>\max \left\{-b_{i} / p_{i} \mid p_{i}>0\right\}$. We can see the positive vector $p$ as a guide to the solution.

The parametric principal pivot method generates the equality systems of the following form in each iteration:

$$
\bar{y}=\bar{b}+\bar{t} \bar{p}+\bar{A} \bar{x}
$$

which is equivalent to the initial system $y=b+t p+A x$. The left hand side variable $\bar{y}$ is called basic variable and $i$-th basic variable $\bar{y}_{i}$ is either $y_{i}$ or $x_{i}$. And the right hand side variable is called non-basic variable and $i$-th non-basic variable $\bar{x}_{i}$ is either $x_{i}$ or $y_{i}$. If one take non-basic variable to be zeros, then the values of basic variable will be uniquely determined, which is called basic solution. We will assume that the basic solution is complementary, that is $x_{i}$ is basic variable if and only if $y_{i}$ is non-basic variable.

The parametric algorithm starts with the system: $y=b+t p+A x$ and the corresponding complementary basic solution $(x, y)=(0, b)$ In each iteration, the algorithm calculates $\bar{t}=\max \left\{-\bar{b}_{i} / \bar{p}_{i} \mid \bar{p}_{i}>0\right\}$ and let $k=\arg \max \left\{-\bar{b}_{i} / \bar{p}_{i} \mid \bar{p}_{i}>0\right\}$. By replacing the positions and roles of the variable $x_{k}$ and $y_{k}$ in the system, which is called principal pivot, new complementary system and the corresponding basic solution are obtained.
If the given input matrix $A$ is a P-matrix, then the parametric principal pivot method terminates finitely and solves the $\operatorname{LCP}(A, b)$. Furthermore $A$ is a hidden $Z^{0}$ - matrix, that is there exist $Z^{o}$ matrix $B, C$ such that $A C=B$, then the parametric principal pivoting method terminates in at most $n$ steps in solving the $\operatorname{pLCP}\left(A^{T}, b, p=\left(C^{T}\right)^{-1} e\right)$ for any $b$.
Theorem 4.5 ( [11]). Let $A$ be hidden $Z^{o}-$ matrix. Then for any $b \in \mathbb{R}^{n}$, $\mathcal{X}\left(A^{T}, b\right) \neq \emptyset$. The parametric principal pivoting method finds a solution to $\operatorname{LCP}(A, b)$ in at most $n$ steps.

Here is a numerical example of solving parametric LCP with a hidden matrix.
Example 4.6. Let $A=\left[\begin{array}{cccc}1 & -6 & 0 & -2 \\ 4 & 4 & 1 & 3 \\ -2 & -4 & 1 & 1 \\ 4 & -1 & -2 & 2\end{array}\right]$. The matrix $A$ is hidden $Z^{o}-$ with the same $Z^{o}$ - matrix $C$ and $B$. We consider the $\operatorname{pLCP}\left(A^{T}, b, p\right)$ with the following $b$ and $p=\left(C^{T}\right)^{-1} e$ :

$$
b=\left[\begin{array}{l}
-1 \\
-2 \\
-3 \\
-4
\end{array}\right], p=\left(\left[\begin{array}{cccc}
342 & -2 & -312 & -2 \\
-2 & 228 & -51 & -173 \\
-2 & -2 & 1456 & -1396 \\
-688 & -2 & -2 & 698
\end{array}\right]^{T}\right)^{-1}\left[\begin{array}{l}
1 \\
1 \\
1 \\
1
\end{array}\right] \cong\left[\begin{array}{c}
0.0962 \\
0.00583 \\
0.0216 \\
0.0463
\end{array}\right]
$$

For the simplicity of the description, we use approximated value of the element of the vector $p$. The principal pivoting method starts with the initial complementary system:

$$
\left[\begin{array}{l}
y_{1} \\
y_{2} \\
y_{3} \\
y_{4}
\end{array}\right]=\left[\begin{array}{l}
-1 \\
-2 \\
-3 \\
-4
\end{array}\right]+\left[\begin{array}{c}
0.0962 \\
0.00583 \\
0.0216 \\
0.0463
\end{array}\right] t+\left[\begin{array}{cccc}
1 & 4 & -2 & 4 \\
-6 & 4 & -4 & -1 \\
0 & 1 & 1 & -2 \\
-2 & 3 & 1 & 2
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3} \\
x_{4}
\end{array}\right]
$$

And the corresponding basic solution is $x=\left[\begin{array}{llll}0 & 0 & 0 & 0\end{array}\right]^{T}, y=\left[\begin{array}{llll}-1 & -2 & -3 & -4\end{array}\right]^{T}$. In the first iteration, the algorithm calculates: $\bar{t}=\max \left\{-\bar{b}_{i} / \bar{p}_{i} \mid \bar{p}_{i}>0\right\}$ and $k=$ $\arg \max \left\{-\bar{b}_{i} / \bar{p}_{i} \mid \bar{p}_{i}>0\right\}$. Then make a principal pivot on $\left(y_{k}, x_{k}\right)$. In this case, $\bar{t}=343.3$ and $k=2$. By exchanging the role of the variable $y_{2}$ and $x_{2}$ in the system, we get the following complementary system:

$$
\left[\begin{array}{l}
y_{1} \\
x_{2} \\
y_{3} \\
y_{4}
\end{array}\right]=\left[\begin{array}{c}
1 \\
1 / 2 \\
-5 / 2 \\
-5 / 2
\end{array}\right]+\left[\begin{array}{c}
0.0904 \\
-0.00146 \\
0.0201 \\
0.0419
\end{array}\right] t+\left[\begin{array}{cccc}
7 & 1 & 2 & 5 \\
3 / 2 & 1 / 4 & 1 & 1 / 4 \\
3 / 2 & 1 / 4 & 2 & -7 / 4 \\
5 / 2 & 3 / 4 & 4 & 11 / 4
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
y_{2} \\
x_{3} \\
x_{4}
\end{array}\right]
$$

The corresponding basic solution is $x=\left[\begin{array}{llll}0 & 1 / 2 & 0 & 0\end{array}\right]^{T}$ and $y=\left[\begin{array}{lll}1 & 0 & -5 / 2\end{array}-5 / 2\right]^{T}$.
In the second iteration, $\bar{t}=\max \left\{-\bar{b}_{i} / \bar{p}_{i} \mid \bar{p}_{i}>0\right\}=124.3$ and $k=\arg \max \left\{-\bar{b}_{i} / \bar{p}_{i} \mid \bar{p}_{i}>\right.$ $0\}=3$. The algorithm makes a principal pivot on $\left(y_{3}, x_{3}\right)$ and we have the following complementary system:

$$
\left[\begin{array}{l}
y_{1} \\
x_{2} \\
x_{3} \\
y_{4}
\end{array}\right]=\left[\begin{array}{l}
7 / 2 \\
7 / 4 \\
5 / 4 \\
5 / 2
\end{array}\right]+\left[\begin{array}{c}
0.0703 \\
-0.0115 \\
-0.0101 \\
0.00170
\end{array}\right] t+\left[\begin{array}{cccc}
11 / 2 & 3 / 4 & 1 & 27 / 4 \\
3 / 4 & 1 / 8 & 1 / 2 & 9 / 8 \\
-3 / 4 & -1 / 8 & 1 / 2 & 7 / 8 \\
-1 / 2 & 1 / 4 & 2 & 25 / 4
\end{array}\right]\left[\begin{array}{l}
x_{1} \\
y_{2} \\
y_{3} \\
x_{4}
\end{array}\right]
$$

The corresponding basic solution is $x=\left[\begin{array}{llll}0 & 7 / 2 & 5 / 4 & 0\end{array}\right]^{T}$ and $y=\left[\begin{array}{llll}7 / 2 & 0 & 0 & 5 / 2\end{array}\right]^{T}$ which solve the $\operatorname{LCP}\left(A^{T}, b\right)$.

Mangasarian's sufficient conditions of $A$ and $b$ for the $\mathrm{LCP}(A, b)$ can be solved by linear optimization are not easy to check, because they contains non-linear terms. We should note that here, once matrices $Z_{1}$ and $Z_{2}$ are verified to be $Z^{o}-$ matrices, the existence of positive vectors $r$ and $s$ satisfying conditions are guaranteed. In order to check the existence of $Z^{o}-$ matrices $Z_{1}$ and $Z_{2}$, we can propose the following affine form of alternative theorem for hidden $Z^{o}-$ matrix.

Theorem 4.7. Let $A$ be $n \times n$ matrix and $b$ be $n$-vector. Then one and olny one of the following holds:
(1) There exist $Z^{o}-$ matrices $Z_{1}, Z_{2}$ and a positive number $\delta$ such that $A Z_{1}=$ $A Z_{2}+\delta b e^{T}$
(2) There exist matrices $R$ and $S$, not both 0 , such that $R+A^{T} S=0$, $\operatorname{trace}\left(S, b e^{T}\right)>$ $0, R_{i i} \geq\left|R_{i j}\right|$ and $S_{i i} \geq\left|S_{i j}\right|$ for all $i \neq j$.

Proof. We can prove this by recognizing $n \times n$ matrix as a vector with length $n \times n$ and using Farkas' lemma.

Clearly, if $A$ is hidden $Z^{o}$ - then $A$ satisfy the condition 1 . of the above thorem. Is there any matrices that is not hidden $Z^{\circ}$ - matrix which satisfy the condition 1 . of the above theorem? This is still open and we would like to know.

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