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CUTTER MAPPINGS AND SUBGRADIENT PROJECTIONS IN BANACH SPACES

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Dedicated to Professor Makoto Tsukada on the occasion of his 65th birthday.

Abstract. We study the problem of approximating common fixed points of a family of cutter mappings of type (P) in the sense of Kimura and Saejung [Linear Nonlinear Anal. **1** (2015), 53–65] in Banach spaces. Using the concept of subgradient projections associated with metric projections in Banach spaces, we also discuss some applications of our results to convex feasibility problems in such spaces.

1. Introduction

The aim of this paper is twofold. Firstly, we obtain weak and strong convergence theorems for a family of cutter mappings of type (P) in the sense of Kimura and Saejung [22] in Banach spaces. Secondly, using the concept of subgradient projections associated with metric projections, we discuss some applications of our results to convex feasibility problems in Banach spaces.

The fixed point theory for firmly nonexpansive mappings is strongly related to many nonlinear problems such as convex minimization problems, variational inequality problems, minimax problems, and equilibrium problems in Hilbert spaces. In fact, to each of these problems, there corresponds a firmly nonexpansive mapping whose fixed point set coincides with its solution set. Recall that a mapping $T\colon C \to H$ is said to be firmly nonexpansive [17, 18, 25, 26] if

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle \tag{1.1}$$

for all $x, y \in C$, where C is a nonempty subset of a Hilbert space H.

Aoyama, Kohsaka, and Takahashi [10] introduced the concept of mappings of type (P) in Banach spaces, which is a generalization of the concept of firmly nonexpansive mappings in Hilbert spaces. Recall that a mapping $T\colon C \to E$ is said to be of type (P) in the sense of [10] if

$$\langle Tx - Ty, J(x - Tx) - J(y - Ty)\rangle \ge 0 \tag{1.2}$$

for all $x, y \in C$, where C is a nonempty subset of a smooth Banach space E and J is the normalized duality mapping of E. If E is a Hilbert space, then J is the identity mapping on E and (1.2) is reduced to (1.1). It is known [10, Theorems 7.3

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and 7.4] that the following hold: If E is a smooth, strictly convex, and reflexive Banach space and C is a nonempty bounded closed convex subset of E, then every mapping $T\colon C \to C$ of type (P) has a fixed point. Further, if E has the Kadec–Klee property, then every mapping $T\colon C \to E$ of type (P) is norm-to-norm continuous. The following are typical examples of mappings of type (P) in a smooth, strictly convex, and reflexive Banach space E:

- If C is a nonempty closed convex subset of E, then the metric projection P_C of E onto C is of type (P) and $\mathcal{F}(P_C) = C$, where $\mathcal{F}(P_C)$ is the set of fixed points of P_C;
- if $A\colon E \to 2^{E^*}$ is maximal monotone, then the resolvent $J_A\colon E \to E$ of A given by $J_A = (I + J^{-1}A)^{-1}$ is of type (P) and $\mathcal{F}(J_A) = A^{-1}0$, where I is the identity mapping on E.

See also [3, 5, 8] for more details on several results related to mappings of type (P).

Later, Kimura and Saejung [22] proposed the concept of cutter mappings of type (P) in Banach spaces and studied the approximation of fixed points of such mappings. Recall that $T\colon C \to E$ is said to be a cutter mapping of type (P) in the sense of [22] if $\mathcal{F}(T)$ is nonempty and

$$\langle Tx - y, J(x - Tx)\rangle \geq 0 \tag{1.3}$$

for all $x \in C$ and $y \in \mathcal{F}(T)$, where C is a nonempty subset of a smooth Banach space E. Every mapping of type (P) with a fixed point is clearly a cutter mapping of type (P).

On the other hand, as we see in Section 5, the subgradient projections associated with metric projections in Banach spaces are cutter mappings of type (P). This projection is a generalization of metric projections in Banach spaces. It is known [4, Section 4] that this projection is not generally idempotent and can be discontinuous even in Hilbert spaces. The notion of subgradient projections in Hilbert spaces was studied in [4, 12, 13]. Bauschke and Combettes [14] also studied the subgradient projections associated with Bregman projections in Banach spaces.

This paper is organized as follows. In Section 2, we recall some definitions and results needed in this paper. In Section 3, we obtain some fundamental results for cutter mappings of type (P) in Banach spaces. We then obtain weak and strong convergence theorems for a finite family of such mappings. In Section 4, we obtain two strong convergence theorems for a sequence of cutter mappings of type (P) in Banach spaces. We also apply our results to the fixed point problem for NST mappings in the sense of [21, 30]. In Section 5, we obtain some basic results for subgradient projections associated with metric projections in Banach spaces. Several applications to convex feasibility problems in Banach spaces are also included in Section 6.

2. Preliminaries

Throughout the present paper, E denotes a real Banach space, E^* the dual of E, $\|\cdot\|$ the norms of E and E^*, S_E the unit sphere of E, B_E the closed unit ball of E, $\langle x, x^*\rangle$ the value of $x^* \in E^*$ at $x \in E$, I the identity mapping on E, $\mathbb{R}$ the set of real numbers, and $\mathbb{N}$ the set of positive integers. Strong convergence of a sequence $\{x_n\}$ in E to $x \in E$ is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Let C be a nonempty subset of a Banach space E and $T\colon C \to E$ a mapping. We denote by $\mathcal{F}(T)$ the set of fixed points of T. A point $p \in C$ is said to be an asymptotic fixed point [32] of T if C contains a sequence $\{x_n\}$ such that it is weakly convergent to p and $x_n - Tx_n \to 0$. The set of asymptotic fixed points of T is denoted by $\hat{\mathcal{F}}(T)$.

Let E be a Banach space. It is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S_E$. In this case, the duality mapping J of E given by

$$Jx = \left\{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\right\}$$

for $x \in E$ is single valued and the function $\phi\colon E \times E \to [0, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for all $x, y \in E$. The space E is said to be uniformly smooth if the limit (2.1) attains uniformly in $x, y \in S_E$. The duality mapping J of a smooth Banach space E is said to be weakly sequentially continuous if $\{Jx_n\}$ is weakly* convergent to Jx whenever $\{x_n\}$ is a sequence in E which is weakly convergent to $x \in E$. The space E is said to be strictly convex if $\|x + y\| < 2$ whenever $x, y \in S_E$ and $x \neq y$. It is known that if E is smooth, strictly convex, and reflexive, then $J\colon E \to E^*$ is a bijection. The space E is said to have the Kadec–Klee property if $x_n \to x$ whenever $\{x_n\}$ is a sequence in E, $x_n \rightharpoonup x \in E$, and $\|x_n\| \to \|x\|$. The space E is said to be uniformly convex if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$, where

$$\delta_E(\varepsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : x, y \in B_E,\ \|x - y\| \geq \varepsilon\right\}$$

for all $\varepsilon \in (0, 2]$. If E is uniformly convex, then it is both strictly convex and reflexive, and has the Kadec–Klee property. The space E is also said to be 2-uniformly convex if there exists $c > 0$ such that $\delta_E(\varepsilon) \geq c\varepsilon^2$ for all $\varepsilon \in (0, 2]$. In this case, the 2-uniform convexity constant μ_E is defined as the minimum value of the set of all $\mu \geq 1$ such that

$$\frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2\right) \geq \|x\|^2 + \left\|\mu^{-1} y\right\|^2$$

for all $x, y \in E$. It is known that $\mu_E = 1$ whenever E is a Hilbert space. We know the following lemma:

Lemma 2.1 ([5, Lemma 2.2])**.** *If E is a smooth and 2-uniformly convex Banach space, then $(\|x - y\|/\mu_E)^2 \leq \phi(x, y)$ for all $x, y \in E$.*

Let E be a strictly convex and reflexive Banach space and C a nonempty closed convex subset of E. It is known that for each $x \in E$, there exists a unique point $\hat{x} \in C$ such that $\|\hat{x} - x\| \leq \|y - x\|$ for all $y \in C$. The metric projection P_C of E onto C is defined by $P_C(x) = \hat{x}$ for all $x \in E$. If E is also smooth and $x \in E$, then there exists a unique $\check{x} \in C$ such that $\phi(\check{x}, x) \leq \phi(y, x)$ for all $y \in C$. The generalized projection Q_C of E onto C is defined by $Q_C(x) = \check{x}$ for all $x \in E$; see [1, 20]. We know the following fundamental lemma:

Lemma 2.2 ([33, Corollary 6.5.5]). *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, P_C the metric projection of E onto C, x an element of E, and z an element of C. Then $P_C(x) = z$ if and only if $\langle z - y, J(x - z)\rangle \geq 0$ for all $y \in C$.*

Let E be a smooth Banach space and C a nonempty subset of E. A mapping $U \colon C \to E$ is said to be strongly relatively nonexpansive [11, 24, 32] if the following conditions are satisfied:

- $\hat{\mathcal{F}}(U) = \mathcal{F}(U) \neq \emptyset$ and $\phi(u, Ux) \leq \phi(u, x)$ for all $u \in \mathcal{F}(U)$ and $x \in C$;
- $\phi(Uz_n, z_n) \to 0$ whenever $\{z_n\}$ is a bounded sequence in C such that
$$\phi(u, z_n) - \phi(u, Uz_n) \to 0$$
for some $u \in \mathcal{F}(U)$.

See also [27, 28] on the definition of relatively nonexpansive mappings. We know the following lemma:

Theorem 2.3 ([7, Theorem 3.4]). *Let E be a uniformly smooth and uniformly convex Banach space and $S, T \colon E \to E$ strongly relatively nonexpansive mappings such that $\mathcal{F}(S) \cap \mathcal{F}(T)$ is nonempty. Then $ST \colon E \to E$ is strongly relatively nonexpansive and $\mathcal{F}(ST) = \mathcal{F}(S) \cap \mathcal{F}(T)$.*

Using [9, Example 3.1 and Theorem 4.1] and [2, Theorem 4.1], we can obtain the following weak and strong convergence theorems for strongly relatively nonexpansive mappings, respectively.

Theorem 2.4. *Let E be a smooth and uniformly convex Banach space such that J is weakly sequentially continuous, x an element of E, and $U \colon E \to E$ a strongly relatively nonexpansive mapping. Then $\{U^n x\}$ converges weakly to the strong limit of $\{Q_{\mathcal{F}(U)}(U^n x)\}$.*

Theorem 2.5. *Let E be a smooth and uniformly convex Banach space, $U \colon E \to E$ a strongly relatively nonexpansive mapping, $\{\alpha_n\}$ a sequence in $(0, 1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, v an element of E, and $\{x_n\}$ a sequence defined by $x_1 \in E$ and*

$$x_{n+1} = J^{-1}\big(\alpha_n Jv + (1 - \alpha_n)JUx_n\big) \tag{2.2}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $Q_{\mathcal{F}(U)}(v)$.

Let C be a nonempty subset of a Banach space E and $\{T_n\}$ a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} \mathcal{F}(T_n)$ is nonempty. Then $\{T_n\}$ is said to satisfy the condition (Z2) if every weak cluster point of $\{z_n\}$ belongs to F whenever $\{z_n\}$ is a bounded sequence in C such that $z_n - T_n z_n \to 0$ and $z_n - z_{n+1} \to 0$; see [8] for more details. It is also said to satisfy the condition (Z3) if $p \in F$ whenever $\{z_n\}$ is a sequence in C such that $z_n \to p$ and $z_n - T_n z_n \to 0$. Note that the condition (Z2) implies the condition (Z3). We know the following lemma:

Lemma 2.6 ([8, Lemma 2.5]). *Let C be a nonempty subset of a Banach space E, $\{S_n\}$ a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} \mathcal{F}(S_n)$ is nonempty, $\{\alpha_n\}$ a sequence of real numbers such that $\sup_n \alpha_n < 1$, $\{T_n\}$ a sequence of mappings of C into E defined by $T_n = \alpha_n I + (1 - \alpha_n)S_n$ for all $n \in \mathbb{N}$. If $\{S_n\}$ satisfies the condition (Z2), then so does $\{T_n\}$.*

We also know the following lemma; see [7, Lemma 4.1] and [6, Lemma 3.1] for related results:

Lemma 2.7 ([8, Lemma 3.1]). *Let E be a smooth and uniformly convex Banach space, both $\{M_n\}$ and $\{N_n\}$ sequences of nonempty closed convex subsets of E, x an element of E, and $\{x_n\}$ a sequence in E such that $x_n = P_{N_n}(x)$, $x_{n+1} \in N_n$, and $x_{n+1} = P_{M_n}(x)$ for all $n \in \mathbb{N}$. If $\bigcap_{n=1}^{\infty} M_n \neq \emptyset$, then $\{x_n\}$ is bounded and $x_{n+1} - x_n \to 0$. Further, if there exists a nonempty closed convex subset F of E such that $F \subset \bigcap_{n=1}^{\infty} M_n$ and every weak cluster point of $\{x_n\}$ belongs to F, then $\{x_n\}$ converges strongly to $P_F(x)$.*

Let $\{D_n\}$ be a sequence of nonempty closed convex subsets of a Banach space E. We denote by $\operatorname{s-Li}_n D_n$ the set of all $z \in E$ such that there exists a sequence $\{z_n\}$ in E satisfying $z_n \in D_n$ for all $n \in \mathbb{N}$ and $z_n \to z$. We also denote by $\operatorname{w-Ls}_n D_n$ the set of all $z \in E$ such that there exist an increasing sequence $\{n_i\}$ in $\mathbb{N}$ and a sequence $\{z_i\}$ in E satisfying $z_i \in D_{n_i}$ for all $i \in \mathbb{N}$ and $z_i \rightharpoonup z$. The sequence $\{D_n\}$ is Mosco convergent [29] to D if

$$D = \operatorname*{s-Li}_n D_n = \operatorname*{w-Ls}_n D_n.$$

It is known that if $D_n \supset D_{n+1}$ for all $n \in \mathbb{N}$, then $\{D_n\}$ is Mosco convergent to $\bigcap_{n=1}^{\infty} D_n$. See also [15,16] for more details on Mosco convergence.

In 1984, Tsukada [35] obtained the following celebrated theorem on the relation between the Mosco convergence and the pointwise convergence of metric projections in Banach spaces:

Theorem 2.8 ([35, Theorem 3.2]). *Let E be a strictly convex and reflexive Banach space with the Kadec–Klee property and $\{D_n\}$ a sequence of nonempty closed convex subsets of E. If $\{D_n\}$ is Mosco convergent to a nonempty subset D of E, then $\{P_{D_n}(x)\}$ converges strongly to $P_D(x)$ for all $x \in E$.*

3. Fundamental properties of cutter mappings

In this section, we show some fundamental properties of cutter mappings of type (P) in Banach spaces and obtain weak and strong convergence theorems for a finite family of such mappings.

Using some ideas in [5, Lemma 3.2 and Corollary 4.2] and [8, Lemma 2.2], we show the following two lemmas for cutter mappings of type (P), respectively:

Lemma 3.1. *Let E be a uniformly smooth and 2-uniformly convex Banach space, $T\colon E \to E$ a cutter mapping of type (P) such that $\hat{\mathcal{F}}(T) = \mathcal{F}(T)$, β a real number such that $0 < \beta < 2/(\mu_E)^2$, and S the mapping defined by*

$$S = J^{-1}\big(J - \beta J(I - T)\big).$$

Then $\mathcal{F}(S) = \mathcal{F}(T)$,

$$\phi(u, Sx) + \frac{1}{2}\left(\frac{2}{(\mu_E)^2} - \beta\right)\|Sx - x\|^2 \leq \phi(u, x) \tag{3.1}$$

for all $u \in \mathcal{F}(S)$ and $x \in E$, and $S\colon E \to E$ is strongly relatively nonexpansive.

Proof. By the definition of S, we can easily show that $\mathcal{F}(S) = \mathcal{F}(T)$. This implies that $\mathcal{F}(S)$ is nonempty.

We first show (3.1). Let $u \in \mathcal{F}(S)$ and $x \in E$ be given. Since T is a cutter mapping of type (P), we have

$$\begin{aligned}
&\phi(u, Sx) + \phi(Sx, x) - \phi(u, x) \\
&= 2 \langle u - Sx, Jx - JSx \rangle \\
&= 2\beta \langle u - Sx, J(x - Tx) \rangle \\
&= 2\beta\big(\langle u - Tx, J(x - Tx) \rangle + \langle Tx - Sx, J(x - Tx) \rangle\big) \\
&\leq 2\beta\big(- \|x - Tx\|^2 + \langle x - Sx, J(x - Tx) \rangle\big) \\
&\leq 2\beta\big(- \|x - Tx\|^2 + \|x - Sx\| \|x - Tx\|\big) \leq \frac{\beta}{2} \|x - Sx\|^2 .
\end{aligned} \tag{3.2}$$

By Lemma 2.1, we have

$$\frac{1}{(\mu_E)^2} \|x - Sx\|^2 \leq \phi(Sx, x). \tag{3.3}$$

By (3.2) and (3.3), we see that (3.1) holds, and thus $\phi(u, Sx) \leq \phi(u, x)$ for all $u \in \mathcal{F}(S)$ and $x \in E$.

We next show that $\hat{\mathcal{F}}(S) = \mathcal{F}(S)$. Let p be an element of $\hat{\mathcal{F}}(S)$. Then we have a sequence $\{x_n\}$ in E such that $x_n \rightharpoonup p$ and $x_n - Sx_n \to 0$. Since E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E. This implies that

$$\beta \|x_n - Tx_n\| = \|Jx_n - JSx_n\| \to 0$$

and hence $p \in \hat{\mathcal{F}}(T)$. Since $\hat{\mathcal{F}}(T) = \mathcal{F}(T)$ and $\mathcal{F}(S) = \mathcal{F}(T)$, we know that $p \in \mathcal{F}(S)$. Thus we see that $\hat{\mathcal{F}}(S) = \mathcal{F}(S)$.

Let $\{z_n\}$ be a bounded sequence in E such that

$$\phi(v, z_n) - \phi(v, Sz_n) \to 0 \tag{3.4}$$

for some $v \in \mathcal{F}(S)$. By (3.1) and (3.4), we have

$$\frac{1}{2}\left(\frac{2}{(\mu_E)^2} - \beta\right) \|Sz_n - z_n\|^2 \leq \phi(v, z_n) - \phi(v, Sz_n) \to 0.$$

This implies that $\|Sz_n - z_n\| \to 0$ and hence $\phi(Sz_n, z_n) \to 0$. □

Lemma 3.2. *Let E be a smooth Banach space, C a nonempty subset of E, and $T\colon C \to E$ a cutter mapping of type (P). Then the following hold:*

(i) *If C is closed and convex, then $\mathcal{F}(T)$ is closed and convex;*
(ii) *if $\lambda \in [0, 1)$, then the mapping $U\colon C \to E$ defined by $U = \lambda I + (1 - \lambda)T$ is a cutter mapping of type (P).*

Proof. We first show (i). Let $u, v \in \mathcal{F}(T)$ and $\lambda \in [0, 1]$ be given and set $z = \lambda u + (1 - \lambda)v$. Then the convexity of C implies that $z \in C$. Since T is a cutter mapping of type (P), we have

$$\begin{aligned}
\|z - Tz\|^2 &= \langle z - Tz, J(z - Tz) \rangle \\
&= \lambda \langle u - Tz, J(z - Tz) \rangle + (1 - \lambda) \langle v - Tz, J(z - Tz) \rangle \leq 0
\end{aligned}$$

and hence $z \in \mathcal{F}(T)$. Thus $\mathcal{F}(T)$ is convex. Let $\{u_n\}$ be a sequence in $\mathcal{F}(T)$ which is strongly convergent to w. Then the closedness of C implies that $w \in C$. Since T is a cutter mapping of type (P), we have

$$\begin{aligned}\|w - Tw\|^2 &= \langle w - Tw, J(w - Tw)\rangle \\ &= \langle w - u_n, J(w - Tw)\rangle + \langle u_n - Tw, J(w - Tw)\rangle \\ &\le \langle w - u_n, J(w - Tw)\rangle \to 0\end{aligned}$$

and hence $w \in \mathcal{F}(T)$. Thus $\mathcal{F}(T)$ is closed.

We next show (ii). It obviously holds that $\mathcal{F}(T) = \mathcal{F}(U)$ and hence $\mathcal{F}(U)$ is nonempty. Let $x \in C$ and $u \in \mathcal{F}(U)$ be given. Since $I - U = (1 - \lambda)(I - T)$ and $u \in \mathcal{F}(T)$, we have

$$\begin{aligned}&\langle Ux - u, J(x - Ux)\rangle \\ &= (1 - \lambda)\lambda \|x - Tx\|^2 + (1 - \lambda) \langle Tx - u, J(x - Tx)\rangle \ge 0.\end{aligned}$$

Therefore, U is a cutter mapping of type (P). □

By Theorems 2.3, 2.4, and Lemma 3.1, we obtain the following weak convergence theorem:

Theorem 3.3. *Let E be a uniformly smooth and 2-uniformly convex Banach space such that J is weakly sequentially continuous, $\{T_i\}_{i=1}^m$ a finite family of cutter mappings of type (P) of E into itself such that $\hat{\mathcal{F}}(T_i) = \mathcal{F}(T_i)$ for all $i \in \{1, 2, \dots, m\}$ and $F = \bigcap_{i=1}^m \mathcal{F}(T_i)$ is nonempty, β a real number such that $0 < \beta < 2/(\mu_E)^2$, S_i the mapping defined by $S_i = J^{-1}\big(J - \beta J(I - T_i)\big)$ for all $i \in \{1, 2, \dots, m\}$, and $\{x_n\}$ a sequence defined by $x_1 \in E$ and*

$$x_{n+1} = S_1 S_2 \cdots S_m x_n$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to the strong limit of $\{Q_F(x_n)\}$.

By Theorems 2.3, 2.5, and Lemma 3.1, we also obtain the following strong convergence theorem:

Theorem 3.4. *Let E, $\{T_i\}_{i=1}^m$, F, β, and $\{S_i\}_{i=1}^m$ be the same as in Theorem 3.3, $\{\alpha_n\}$ a sequence in $(0, 1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$, v an element of E, and $\{x_n\}$ a sequence defined by $x_1 \in E$ and*

$$x_{n+1} = J^{-1}\big(\alpha_n Jv + (1 - \alpha_n) J S_1 S_2 \cdots S_m x_n\big)$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $Q_F(v)$.

4. Convergence theorems for a sequence of cutter mappings

In this section, we obtain two strong convergence theorems for a sequence of cutter mappings of type (P) in Banach spaces.

The following theorem is a generalization of [8, Theorem 3.2]:

Theorem 4.1. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, and $\{T_n\}$ a sequence of cutter mappings of* *type (P) of C into E such that $F = \bigcap_{n=1}^{\infty} \mathcal{F}(T_n)$ is nonempty, x an element of E, and $\{x_n\}$ a sequence defined by $x_1 = P_C(x)$ and*

$$\begin{cases} C_n = \big\{ z \in C : \langle T_n x_n - z, J(x_n - T_n x_n) \rangle \geq 0 \big\}; \\ D_n = \big\{ z \in C : \langle x_n - z, J(x - x_n) \rangle \geq 0 \big\}; \\ x_{n+1} = P_{C_n \cap D_n}(x) \end{cases}$$

for all $n \in \mathbb{N}$. Then the following hold:

(i) *$F \subset C_n \cap D_n$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is well defined;*
(ii) *if E is uniformly convex and $\{T_n\}$ satisfies the condition (Z2), then $\{x_n\}$ converges strongly to $P_F(x)$.*

Proof. We first show (i). It is obvious that C_n and D_n are closed and convex for all $n \in \mathbb{N}$. Since T_n is a cutter mapping of type (P), it follows that $F \subset C_n$ for all $n \in \mathbb{N}$. We next show that $F \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. Noting that $D_1 = C$, we have

$$F \subset C_1 = C_1 \cap D_1.$$

If $F \subset C_1 \cap D_1, C_2 \cap D_2, \dots, C_{n-1} \cap D_{n-1}$ for some $n \geq 2$, then $C_k \cap D_k$ is nonempty for all $k \in \{1, 2, \dots, n-1\}$ and hence $\{x_2, \dots, x_n\}$, C_n, and D_n are well defined. By Lemma 2.2 and the definition of $\{x_2, \dots, x_n\}$, we know that $C_{k-1} \cap D_{k-1} \subset D_k$ for all $k \in \{2, \dots, n\}$. Consequently, we have

$$\begin{aligned} C_n \cap D_n &\supset C_n \cap (C_{n-1} \cap D_{n-1}) \\ &\supset C_n \cap C_{n-1} \cap \cdots \cap C_1 \cap D_1 = \bigcap_{i=1}^{n} C_i \supset F \end{aligned}$$

and hence $F \subset C_n \cap D_n$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is well defined.

We next show (ii). Set $M_n = C_n \cap D_n$ and $N_n = D_n$ for all $n \in \mathbb{N}$. Then it is obvious that both $\{M_n\}$ and $\{N_n\}$ are sequences of nonempty closed convex subsets of E and $F \subset \bigcap_{n=1}^{\infty} M_n$. It is also obvious that $x_{n+1} = P_{M_n}(x)$, $x_n = P_{N_n}(x)$, and $x_{n+1} \in N_n$ for all $n \in \mathbb{N}$. Hence it follows from Lemma 2.7 that $\{x_n\}$ is bounded and $x_{n+1} - x_n \to 0$. Since $x_{n+1} \in C_n$, we have

$$\begin{aligned} \|x_n - T_n x_n\|^2 &= \langle x_n - x_{n+1}, J(x_n - T_n x_n) \rangle + \langle x_{n+1} - T_n x_n, J(x_n - T_n x_n) \rangle \\ &\leq \langle x_n - x_{n+1}, J(x_n - T_n x_n) \rangle \\ &\leq \|x_n - x_{n+1}\| \, \|x_n - T_n x_n\| \end{aligned}$$

and hence $\|x_n - T_n x_n\| \leq \|x_n - x_{n+1}\| \to 0$. Since $\{T_n\}$ satisfies the condition (Z2), we know that every weak cluster point of $\{x_n\}$ belongs to F. It also follows from Lemma 3.2 that F is closed and convex. Therefore Lemma 2.7 implies that $\{x_n\}$ converges strongly to $P_F(x)$. □

We next show another convergence theorem by the shrinking projection method introduced in [34]. Before obtaining it, we show the following:

Lemma 4.2. *Let E be a smooth Banach space, C a nonempty subset of E, and $\{T_n\}$ a sequence of mappings of C into E such that $F = \bigcap_{n=1}^{\infty} \mathcal{F}(T_n)$ is nonempty* *and $\{T_n\}$ satisfies the condition (Z3). If $\{x_n\}$ is a sequence in C which is strongly convergent to an element u of the set*

$$\bigcap_{n=1}^{\infty}\{z\in C:\langle T_nx_n-z,J(x_n-T_nx_n)\rangle\geq 0\},$$

then $u\in F$.

Proof. By assumption, we have

$$\begin{aligned}\|x_n-T_nx_n\|^2&=\langle x_n-u,J(x_n-T_nx_n)\rangle+\langle u-T_nx_n,J(x_n-T_nx_n)\rangle\\&\leq\langle x_n-u,J(x_n-T_nx_n)\rangle\\&\leq\|x_n-u\|\,\|x_n-T_nx_n\|\end{aligned}$$

and hence $\|x_n-T_nx_n\|\leq\|x_n-u\|$ for all $n\in\mathbb{N}$. Since $x_n\to u$, we know that $x_n-T_nx_n\to 0$. Since $\{T_n\}$ satisfies the condition (Z3), we conclude that $u\in F$. □

Using some techniques in [21, 23], we obtain the following theorem, which is a generalization of [8, Theorem 3.5]:

Theorem 4.3. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, $\{T_n\}$ a sequence of cutter mappings of type (P) of C into E such that $F=\bigcap_{n=1}^{\infty}\mathcal{F}(T_n)$ is nonempty, x an element of E, and $\{x_n\}$ a sequence defined by $x_1\in C$, $C_1=C$, and*

$$\begin{cases}C_{n+1}=\{z\in C:\langle T_nx_n-z,J(x_n-T_nx_n)\rangle\geq 0\}\cap C_n;\\x_{n+1}=P_{C_{n+1}}(x)\end{cases}$$

for all $n\in\mathbb{N}$. Then the following hold:

(i) *$F\subset C_n$ for all $n\in\mathbb{N}$ and $\{x_n\}$ is well defined;*
(ii) *if E has the Kadec–Klee property and $\{T_n\}$ satisfies the condition (Z3), then $\{x_n\}$ converges strongly to $P_F(x)$.*

Proof. We first show (i). Since each T_n is a cutter mapping of type (P), we have

$$\langle T_ny-z,J(y-T_ny)\rangle\geq 0$$

for all $y\in C$, $z\in F$, and $n\in\mathbb{N}$ and hence $F\subset C_n$ for all $n\in\mathbb{N}$. This implies that each C_n is a nonempty closed convex subset of E and hence $\{x_n\}$ is well defined.

We next show (ii). Set $F_0=\bigcap_{n=1}^{\infty}C_n$. Since $F\subset F_0$, we have

$$\|P_{F_0}(x)-x\|=\inf_{y\in F_0}\|y-x\|\leq\inf_{y\in F}\|y-x\|.\tag{4.1}$$

On the other hand, since $C_n\supset C_{n+1}$ for all $n\in\mathbb{N}$, the sequence $\{C_n\}$ is Mosco convergent to F_0. Since F_0 is nonempty, Theorem 2.8 ensures that $\{x_n\}$ converges strongly to $P_{F_0}(x)$. Since $\{T_n\}$ satisfies the condition (Z3) and

$$x_n\to P_{F_0}(x)\in\bigcap_{n=1}^{\infty}\{z\in C:\langle T_nx_n-z,J(x_n-T_nx_n)\rangle\geq 0\},$$

Lemma 4.2 implies that

$$P_{F_0}(x)\in F.\tag{4.2}$$

By (4.1) and (4.2), we conclude that $P_{F_0}(x) = P_F(x)$. Therefore, the sequence $\{x_n\}$ converges strongly to $P_F(x)$. □

We next apply Theorems 4.1 and 4.3 to the fixed point problem for NST mappings in Banach spaces.

Let E be a smooth Banach space, C a nonempty subset of E, and α a positive real number. A mapping $S\colon C \to E$ is said to be an α-NST-mapping if

- $\mathcal{F}(S)$ is nonempty;
- $\langle x - u, J(x - Sx)\rangle \geq \alpha \|x - Sx\|^2$ for all $x \in C$ and $u \in \mathcal{F}(S)$;

see [21,30] for more details. Every cutter mapping of type (P) of C into E is clearly a 1-NST mapping. We show the following lemmas:

Lemma 4.4. *Let E be a smooth Banach space, C a nonempty subset of E, α a positive real number, $S\colon C \to E$ an α-NST mapping, and $T\colon C \to E$ a mapping defined by $T = (1-\alpha)I + \alpha S$. Then $\mathcal{F}(T) = \mathcal{F}(S)$ and T is a cutter mapping of type (P).*

Proof. The first part is clear. We show the second part. Let $x \in C$ and $u \in \mathcal{F}(T) = \mathcal{F}(S)$ be given. Then we have

$$\begin{aligned}\langle Tx - u, J(x - Tx)\rangle &= -\|x - Tx\|^2 + \langle x - u, J(x - Tx)\rangle \\ &= \alpha\left(-\alpha \|x - Sx\|^2 + \langle x - u, J(x - Sx)\rangle\right) \geq 0.\end{aligned}$$

This completes the proof. □

Lemma 4.5. *Let E be a smooth Banach space, C a nonempty subset of E, $\{\alpha_n\}$ a sequence of positive real numbers such that $\inf_n \alpha_n > 0$, $\{S_n\}$ a sequence of mappings of C into E such that $F = \bigcap_{n=1}^\infty \mathcal{F}(S_n)$ is nonempty, and $\{T_n\}$ a sequence of mappings of C into E defined by $T_n = (1-\alpha_n)I + \alpha_n S_n$ for each $n \in \mathbb{N}$. If $\{S_n\}$ satisfies the condition (Z3), then so does $\{T_n\}$.*

Proof. It is clear that $\bigcap_{n=1}^\infty \mathcal{F}(T_n) = F \neq \emptyset$. Let $\{z_n\}$ be a sequence in C such that $z_n - T_n z_n \to 0$. Then it is clear that

$$\|z_n - S_n z_n\| = \frac{1}{\alpha_n}\|z_n - T_n z_n\| \leq \frac{1}{\inf_n \alpha_n}\|z_n - T_n z_n\| \to 0.$$

This implies the conclusion. □

Using Lemmas 2.6, 4.4, and Theorem 4.1, we obtain the following theorem, which is a slight generalization of [30, (i) of Theorem 3.1] and [30, Theorem 3.2]:

Theorem 4.6. *Let E be a smooth, strictly convex, and reflexive Banach space, C a nonempty closed convex subset of E, $\{\alpha_n\}$ a sequence of positive real numbers, and $S_n\colon C \to E$ an α_n-NST mapping for all $n \in \mathbb{N}$. Suppose that $F = \bigcap_{n=1}^\infty \mathcal{F}(S_n)$ is nonempty and $\inf_n \alpha_n > 0$. Let x be an element of E and $\{x_n\}$ a sequence defined by $x_1 = P_C(x)$ and*

$$\begin{cases} C_n = \left\{z \in C : \langle x_n - z, J(x_n - S_n x_n)\rangle \geq \alpha_n \|x_n - S_n x_n\|^2\right\}; \\ D_n = \left\{z \in C : \langle x_n - z, J(x - x_n)\rangle \geq 0\right\}; \\ x_{n+1} = P_{C_n \cap D_n}(x) \end{cases}$$

for all $n \in \mathbb{N}$. Then the following hold:

(i) *$F \subset C_n \cap D_n$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is well defined;*
(ii) *if E is uniformly convex and $\{S_n\}$ satisfies the condition (Z2), then $\{x_n\}$ converges strongly to $P_F(x)$.*

Using Lemmas 4.4, 4.5, and Theorem 4.3, we obtain the following theorem, which is a slight generalization of [21, Theorem 3.1]:

Theorem 4.7. *Let E and C be the same as in Theorem 4.3, $\{\alpha_n\}$ a sequence of positive real numbers, and $S_n\colon C \to E$ an α_n-NST mapping for all $n \in \mathbb{N}$. Suppose that $F = \bigcap_{n=1}^{\infty} \mathcal{F}(S_n)$ is nonempty and $\inf_n \alpha_n > 0$. Let x be an element of E and $\{x_n\}$ a sequence defined by $x_1 \in C$, $C_1 = C$, and*

$$\begin{cases} C_{n+1} = \left\{ z \in C : \langle x_n - z, J(x_n - S_n x_n)\rangle \geq \alpha_n \left\| x_n - S_n x_n \right\|^2 \right\} \cap C_n; \\ x_{n+1} = P_{C_{n+1}}(x) \end{cases}$$

for all $n \in \mathbb{N}$. Then the following hold:
(i) *$F \subset C_n$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is well defined;*
(ii) *if E has the Kadec–Klee property and $\{S_n\}$ satisfies the condition (Z3), then $\{x_n\}$ converges strongly to $P_F(x)$.*

5. Subgradient projections in Banach spaces

In this section, we study some fundamental properties of subgradient projections associated with metric projections in Banach spaces.

The notion of subgradient projections in Hilbert spaces was studied in [4, Section 4], [12, Section 7], and [13, Proposition 2.3]. It is known that this projection is not generally nonexpansive even in Hilbert spaces; see, for example, [4, Section 4]. This implies that the subgradient projection is not generally of type (P).

Let E be a smooth, strictly convex, and reflexive Banach space and $g\colon E \to \mathbb{R}$ a continuous and convex function such that the set C given by

$$C = \{x \in E : g(x) \leq 0\} \tag{5.1}$$

is nonempty, and $h\colon E \to E^*$ a mapping such that $h(x) \in \partial g(x)$ for all $x \in E$. Since g is continuous and convex, the subdifferential $\partial g(x)$ given by

$$\partial g(x) = \{x^* \in E^* : (g - x^*)(x) = \inf (g - x^*)(E)\}$$

is nonempty for all $x \in E$. It is known that

$$g(u) = \inf g(E) \iff 0 \in \partial g(u).$$

This implies that $h(x) \neq 0$ for all $x \in E \setminus C$.

The subgradient projection $P_{g,h}\colon E \to E$ with respect to g and h is defined by

$$P_{g,h}(x) = P_{L(x)}(x)$$

for all $x \in E$, where $P_{L(x)}$ denotes the metric projection of E onto the set $L(x)$ given by

$$L(x) = \{y \in E : g(x) + \langle y - x, h(x)\rangle \leq 0\}$$

for all $x \in E$. It is obvious that $C \subset L(x)$ for all $x \in E$ and

$$C = \{u \in E : u \in L(u)\}.$$

It is well known that if M is a closed half space given by

$$M = \{y \in E : \langle y, x^* \rangle \le \beta\}$$

for some $x^* \in E^* \setminus \{0\}$ and $\beta \in \mathbb{R}$, then the metric projection P_M of E onto M is given by

$$P_M(x) = x - \frac{[\langle x, x^* \rangle - \beta]_+}{\|x^*\|^2} J^{-1} x^*$$

for all $x \in E$, where $[t]_+ = \max\{t, 0\}$ for all $t \in \mathbb{R}$. Hence we obtain the following explicit formula for $P_{g,h}$:

$$P_{g,h}(x) = \begin{cases} x - \dfrac{g(x)}{\|h(x)\|^2} J^{-1} h(x) & (x \in E \setminus C); \\ x & (x \in C). \end{cases} \tag{5.2}$$

The concept of subgradient projections is a generalization of that of metric projections. In fact, if P_D is the metric projection of E onto a nonempty closed convex subset D of E, then P_D coincides with the subgradient projection $P_{g,h}$ with respect to $g\colon E \to \mathbb{R}$ and $h\colon E \to E^*$ defined by $g(x) = \inf_{y \in D} \|y - x\|$ for all $x \in E$ and

$$h(x) = \begin{cases} J\big(x - P_D(x)\big) / \|x - P_D(x)\| & (x \in E \setminus D); \\ 0 & (x \in D), \end{cases}$$

respectively.

We next show the following properties of subgradient projections:

Lemma 5.1. *Let E be a smooth, strictly convex, and reflexive Banach space, $g\colon E \to \mathbb{R}$ a continuous and convex function such that the set C given by* (5.1) *is nonempty, $h\colon E \to E^*$ a mapping such that $h(x) \in \partial g(x)$ for all $x \in E$, and $P_{g,h}$ the subgradient projection with respect to g and h. Then the following hold:*

(i) *$\mathcal{F}(P_{g,h}) = C$;*
(ii) *$P_{g,h}$ is a cutter mapping of type (P);*
(iii) *if $g(V)$ is bounded for each bounded subset V of E, then $\hat{\mathcal{F}}(P_{g,h}) = \mathcal{F}(P_{g,h})$.*

Proof. We first show (i). If $u \in C$, then we have $u \in L(u)$ and hence

$$P_{g,h}(u) = P_{L(u)}(u) = u.$$

Thus $C \subset F(P_{g,h})$. Conversely, if $u \in E \setminus C$, then it follows from (5.2) that

$$\|u - P_{g,h} u\| = \left\| u - \left(u - \frac{g(u)}{\|h(u)\|^2} J^{-1} h(u) \right) \right\| = \frac{g(u)}{\|h(u)\|} > 0.$$

Hence $u \in E \setminus \mathcal{F}(P_{g,h})$. Thus $C \supset \mathcal{F}(P_{g,h})$.

We next show (ii). Let $y \in \mathcal{F}(P_{g,h}) = C$ and $x \in E$ be given. If $x \in C$, then $P_{g,h}(x) = x$ and hence $\big\langle y - P_{g,h}(x), J\big(x - P_{g,h}(x)\big) \big\rangle = 0$. If $x \notin C$, then it follows from (5.2) that

$$\begin{aligned}
&\big\langle y - P_{g,h}(x), J\big(x - P_{g,h}(x)\big)\big\rangle \\
&= \left\langle y - \left(x - \frac{g(x)}{\|h(x)\|^2} J^{-1}h(x)\right), J\left(\frac{g(x)}{\|h(x)\|^2} J^{-1}h(x)\right)\right\rangle \\
&= \frac{g(x)}{\|h(x)\|^2}\left(\langle y - x, h(x)\rangle + \frac{g(x)}{\|h(x)\|^2}\left\langle J^{-1}h(x), h(x)\right\rangle\right) \\
&= \frac{g(x)}{\|h(x)\|^2}\big(g(x) + \langle y - x, h(x)\rangle\big).
\end{aligned}$$

Since $g(x) > 0$, $h(x) \in \partial g(x)$, and $g(y) \le 0$, we have

$$\frac{g(x)}{\|h(x)\|^2}\big(g(x) + \langle y - x, h(x)\rangle\big) \le \frac{g(x)}{\|h(x)\|^2} g(y) \le 0.$$

Hence we obtain

$$\big\langle y - P_{g,h}(x), J\big(x - P_{g,h}(x)\big)\big\rangle \le 0.$$

for all $y \in \mathcal{F}(P_{g,h})$ and $x \in E$. Thus $P_{g,h}$ is a cutter mapping of type (P).

We finally show (iii). Suppose that $g(V)$ is bounded for each bounded subset V of E. Then it follows from [19, Proposition 1.1.11] that the set

$$\partial g(V) = \bigcup_{x \in V} \partial g(x)$$

is bounded. This implies that $h(V)$ is bounded for each bounded subset V of E. Since the inclusion $\hat{\mathcal{F}}(P_{g,h}) \supset \mathcal{F}(P_{g,h})$ is obvious, we show the converse inclusion. Let $u \in \hat{\mathcal{F}}(P_{g,h})$ be given. Then there exists a sequence $\{z_n\}$ in E such that $z_n \rightharpoonup u$ and $z_n - P_{g,h}(z_n) \to 0$. We first consider the case where there exists $m \in \mathbb{N}$ such that $z_n \in C$ whenever $n \in \mathbb{N}$ and $n \ge m$. Then the weak closedness of C implies that $u \in C = \mathcal{F}(P_{g,h})$. We next consider the case where for each $m \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $n \ge m$ and $z_n \notin C$. In this case, we have a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \notin C$ for all $i \in \mathbb{N}$. This gives us that

$$\|z_{n_i} - P_{g,h}(z_{n_i})\| = \left\| \frac{g(z_{n_i})}{\|h(z_{n_i})\|^2} J^{-1}h(z_{n_i}) \right\| = \frac{g(z_{n_i})}{\|h(z_{n_i})\|}$$

for all $i \in \mathbb{N}$. Since $\{z_n\}$ is bounded, we know that $\{h(z_n)\}$ is bounded. Thus the weak lower semicontinuity of g implies that

$$g(u) \le \liminf_{i \to \infty} g(z_{n_i}) = \liminf_{i \to \infty} \|h(z_{n_i})\| \, \|z_{n_i} - P_{g,h}(z_{n_i})\| = 0$$

and hence $u \in C = \mathcal{F}(P_{g,h})$. □

Motivated by Bauschke and Combettes [14, Theorem 4.7], we prove the following lemma:

Lemma 5.2. *Let $\mathcal{I}$ be the finite set $\{1, 2, \dots, m\}$, E a smooth, strictly convex, and reflexive Banach space, g_n a continuous and convex function of E into $\mathbb{R}$ such that*

$$\{x \in E : g_n(x) \le 0\}$$

is nonempty and $g_n(V)$ is bounded for each bounded subset V of E for all $n \in \mathcal{I}$, h_n a mapping of E into E^* such that $h_n(x) \in \partial g_n(x)$ for all $n \in \mathcal{I}$ and $x \in E$, P_n the subgradient projection with respect to g_n and h_n for all $n \in \mathcal{I}$, and $r\colon \mathbb{N} \to \mathcal{I}$ a mapping such that for each $k \in \mathcal{I}$, there exists $p_k \in \mathbb{N}$ such that

$$k \in \big\{r(n), r(n+1), \dots, r(n+p_k-1)\big\} \tag{5.3}$$

for all $n \in \mathbb{N}$. Then $\{P_{r(n)}\}$ satisfies the condition (Z2).

Proof. Let $\{z_n\}$ be a bounded sequence in E such that both $z_n - P_{r(n)}(z_n) \to 0$ and $z_n - z_{n+1} \to 0$ hold and $\{z_{n_i}\}$ a subsequence of $\{z_n\}$ which is weakly convergent to u. Fix any $k \in \mathcal{I}$. Then there exists $p_k \in \mathbb{N}$ such that (5.3) holds for all $n \in \mathbb{N}$. Without loss of generality, we may suppose that $p_k \geq 2$ and

$$n_i + p_k - 1 < n_{i+1}$$

for all $i \in \mathbb{N}$. It then follows from (5.3) that for each $i \in \mathbb{N}$, there exists

$$m_i \in \{n_i, n_i+1, \dots, n_i+p_k-1\}$$

such that $r(m_i) = k$. Since $z_n - z_{n+1} \to 0$ and $n_i \leq m_i \leq n_i + p_k - 1$, we have

$$\begin{aligned} &\|z_{n_i} - z_{m_i}\| \\ &\quad\leq \sum_{j=n_i}^{n_i+p_k-2} \|z_j - z_{j+1}\| \\ &\quad\leq (p_k - 1)\max\big\{\|z_j - z_{j+1}\| : j \in \{n_i, \dots, n_i+p_k-2\}\big\} \to 0 \end{aligned}$$

as $i \to \infty$. It also follows from $z_{n_i} - z_{m_i} \to 0$ and $z_n - P_{r(n)}(z_n) \to 0$ that

$$z_{m_i} \rightharpoonup u \quad \text{and} \quad \|z_{m_i} - P_k z_{m_i}\| = \big\|z_{m_i} - P_{r(m_i)}(z_{m_i})\big\| \to 0$$

as $i \to \infty$, and hence $u \in \hat{\mathcal{F}}(P_k)$. By Lemma 5.1, we obtain $u \in \mathcal{F}(P_k)$. Therefore, we conclude that $u \in \bigcap_{k\in\mathcal{I}} \mathcal{F}(P_k) = \bigcap_{n=1}^{\infty} \mathcal{F}(P_{r(n)})$. □

6. Convergence theorems with subgradient projections

In this section, we obtain some strong convergence theorems with subgradient projections in Banach spaces.

Throughout this section, we suppose the following:

- $\mathcal{I}$ is the finite set $\{1, 2, \dots, m\}$;
- E is a smooth, strictly convex, and reflexive Banach space;
- $g_i\colon E \to \mathbb{R}$ is continuous and convex for all $i \in \mathcal{I}$ such that

  $$F = \bigcap_{i\in\mathcal{I}} \{x \in E : g_i(x) \leq 0\} \neq \emptyset$$

  and $g_i(V)$ is bounded for each bounded subset V of E for all $i \in \mathcal{I}$;
- h_i is a mapping of E into E^* such that $h_i(x) \in \partial g_i(x)$ for all $i \in \mathcal{I}$ and $x \in E$;
- P_i is the subgradient projection with respect to g_i and h_i for all $i \in \mathcal{I}$.

As a direct consequence of Theorem 3.3 and Lemma 5.1, we obtain the following:

Theorem 6.1. *Suppose that E is uniformly smooth and 2-uniformly convex and that J is weakly sequentially continuous. Let β be a real number such that $0 < \beta < 2/(\mu_E)^2$, S_i the mapping defined by*

$$S_i = J^{-1}\big(J - \beta J(I - P_i)\big)$$

for all $i \in \mathcal{I}$, and $\{x_n\}$ a sequence defined by $x_1 \in E$ and

$$x_{n+1} = S_1 S_2 \cdots S_m x_n$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to the strong limit of $\{Q_F(x_n)\}$.

As a direct consequence of Theorem 3.4 and Lemma 5.1, we also obtain the following:

Theorem 6.2. *Suppose that E is uniformly smooth and 2-uniformly convex. Let β and $\{S_i\}_{i\in\mathcal{I}}$ be the same as in Theorem 6.1, $\{\alpha_n\}$ a sequence in $(0,1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, v an element of E, and $\{x_n\}$ a sequence defined by $x_1 \in E$ and*

$$x_{n+1} = J^{-1}\big(\alpha_n J v + (1-\alpha_n) J S_1 S_2 \cdots S_m x_n\big) \tag{6.1}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $Q_F(v)$.

Using Theorem 4.1, Lemmas 5.1, and 5.2, we obtain the following:

Theorem 6.3. *Suppose that E is uniformly convex. Let $\{\alpha_n\}$ be a sequence in $[0,1)$ such that $\sup_n \alpha_n < 1$, r the same as in Lemma 5.2, x an element of E, and $\{x_n\}$ a sequence defined by $x_1 = x$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) P_{r(n)}(x_n); \\ C_n = \big\{z \in E : \langle y_n - z, J(x_n - y_n)\rangle \geq 0\big\}; \\ D_n = \big\{z \in E : \langle x_n - z, J(x - x_n)\rangle \geq 0\big\}; \\ x_{n+1} = P_{C_n \cap D_n}(x) \end{cases}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_F(x)$.

Proof. Set $T_n = \alpha_n I + (1-\alpha_n) P_{r(n)}$ for all $n \in \mathbb{N}$. Then Lemma 5.1 implies that

$$\mathcal{F}(T_n) = \mathcal{F}(P_{r(n)}) = C_{r(n)}$$

for all $n \in \mathbb{N}$ and thus

$$\bigcap_{n=1}^{\infty} \mathcal{F}(T_n) = \bigcap_{n=1}^{\infty} C_{r(n)} = \bigcap_{n=1}^{m} C_n \neq \emptyset.$$

Lemma 5.1 also implies that each $P_{r(n)}$ is a cutter mapping of type (P). By Lemmas 3.2, 5.2, and 2.6, we also know that each T_n is a cutter mapping of type (P) and $\{T_n\}$ satisfies the condition (Z2). Therefore, Theorem 4.1 implies the conclusion. □

Using Theorem 4.3, Lemmas 5.1, and 5.2, we can similarly obtain the following theorem, which is a generalization of [31, Theorem 3.3 with (3.1a)].

Theorem 6.4. *Suppose that E has the Kadec–Klee property. Let $\{\alpha_n\}$ and r be the same as in Theorem 6.3, x an element of E, and $\{x_n\}$ a sequence defined by $x_1 \in E$, $C_1 = E$, and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) P_{r(n)}(x_n); \\ C_{n+1} = \big\{ z \in E : \langle y_n - z, J(x_n - y_n) \rangle \geq 0 \big\} \cap C_n; \\ x_{n+1} = P_{C_{n+1}}(x) \end{cases}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_F(x)$.

Remark 6.5. [31, Theorem 3.3] is a direct consequence of [8, Theorem 5.2] and [23, Theorem 3.2].

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