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CALCULATIONS OF EXPONENT SEMIGROUPS AIDED BY A COMPUTER

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Dedicated to Professor Makoto Tsukada on the occasion of his retirement

Abstract. We study the exponent semigroups of E-3 semigroups. When the modular exponential semigroup is $\{\bar{1}, \bar{3}\}$, $\{\bar{1}, \bar{3}, \bar{5}\}$ or $\{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$, we completely determine the exponent semigroups. When the modular exponent semigroup is $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$, a partial answer is given. We require calculations aided by a computer.

1. Introduction

Let S be a semigroup and let P denote the multiplicative semigroup of positive integers. The subset of P defined by

$$E(S) = \{n \in P \mid (xy)^n = x^n y^n \text{ for all } x, y \in S\}$$

forms a subsemigroup of P and is called the *exponent semigroup* of S. This notion was introduced by Tamura [6]. Even for a general semigroup S, the structure of $E(S)$ is rather restrictive. We call S an *E-m semigroup* if $m \in E(S)$. If S is an E-2 semigroup, then $E(S)$ is equal to either P or $P \setminus \{3\}$ by Clarke, Piefer and Tamura [5].

In this paper we study the exponential semigroups of E-3 semigroups. For an E-3 semigroup S, if $k \in E(S)$, then $n \in E(S)$ for all sufficiently large n such that $n \equiv k \bmod 6$. So, it is essential to consider the *modular exponent semigroup* $\overline{E}_3(S) = \{\bar{n} \in \mathbb{Z}_6 \mid n \in E(S)\}$. The modular exponent semigroups of E-3 semigroups are determined by Kobayashi [2]; they are $\{\bar{1}, \bar{3}\}$, $\{\bar{1}, \bar{3}, \bar{5}\}$, $\{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$ and $\mathbb{Z}_6$. In this paper we completely determine the exponent semigroups of E-3 semigroups, when their modular exponent semigroups fall into one of the first three cases above. In the last case we give a partial answer. In our study we need calculations aided by a computer.

2. E-m semigroups and free E-m semigroups

For $m \geq 2$ if $m \in E(S)$, S is called an *E-m semigroup*. More generally, for $m_1, m_2, \ldots, m_k \geq 2$, if $m_1, m_2, \ldots, m_k \in E(S)$, S is called an *E-$(m_1, m_2, \ldots, m_k)$ semigroup*. We write

$$m_1, m_2, \ldots, m_k \Rightarrow n$$

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if $m_1, m_2, \ldots, m_k \in E(S)$ implies $n \in E(S)$ for any semigroup S, that is, $n \in E(S)$ for any E-$(m_1, m_2, \ldots, m_k)$ semigroup S. Define

$$E(m_1, m_2, \ldots, m_k) = \{n \in P \mid m_1, m_2, \ldots, m_k \Rightarrow n\}.$$

Specifically, set

$$\begin{aligned} E(m) = E(m,1) &= \{n \in P \mid m \Rightarrow n\} \\ &= \{n \in E(S) \mid S : \text{an arbitrary } E\text{-}m \text{ semigroup}\}. \end{aligned}$$

Consider two symbols a and b, and let F be the free semigroup generated by $\{a, b\}$. Let $\equiv_m$ be the congruence generated by $\{x^m y^m = (xy)^m \mid x, y \in F\}$, and let $F(m) = F/\equiv_m$ be the quotient semigroup of F modulo $\equiv_m$. We call $F(m)$ the *free E-m semigroup of rank* 2. More generally, for $m_1, m_2, \ldots, m_k \geq 2$, $\equiv_{m_1,\ldots,m_k}$ is the congruence generated by

$$\{x^{m_i} y^{m_i} = (xy)^{m_i} \mid i = 1, 2, \ldots k,\ x, y \in F\},$$

and $F(m_1, \ldots, m_k) = F/\equiv_{m_1,\ldots,m_k}$ is the *free E-$(m_1, m_2, \ldots, m_k)$ semigroup of rank* 2. We have

$$E(m) = E(F(m)) \quad \text{and} \quad E(m_1, \ldots, m_k) = E(F(m_1, \ldots, m_k)).$$

3. Modular exponent semigroups

The following results are known (Kobayashi [4], Tamura [7]).

Theorem 3.1. *Let $m, k \geq 1$, then we have*

(i) $m, k \ \Rightarrow\ am(m-1) + k$ *for any* $a \geq 2$.
(ii) $m, k \ \Rightarrow\ m(m-1) + k$ *if* $k \geq m$.
(iii) $m(m-1) + 1 \notin E(m)$.

Corollary 3.2 (Cherubini-Varisco [1], Kobayashi [4])**.** *Let* $m \geq 2$.

(i) $m \ \Rightarrow\ am(m-1) + m$ *for any* $a \geq 1$.
(ii) $m \ \Rightarrow\ am(m-1) + 1$ *for any* $a \geq 2$.

Viewing the above theorem we define the *modular exponent semigroup* $\overline{E}_m(S)$ of an E-m semigroup S by

$$\overline{E}_m(S) = \{\overline{n} \in \mathbb{Z}_{m(m-1)} \mid n \in E(S)\}.$$

This is a subsemigroup of the multiplicative semigroup $\mathbb{Z}_{m(m-1)}$. Due to Theorem 3.1, if $\alpha \in \overline{E}_m(S)$ then $n \in E(S)$ for all sufficiently large $n \in P$ such that $\bar{n} = \alpha$.

Note that (modular) exponent semigroups are closed under intersection. In fact, for E-m semigroups S and T we have

$$E(S) \cap E(T) = E(S \times T) \quad \text{and} \quad \overline{E}_m(S) \cap \overline{E}_m(T) = \overline{E}_m(S \times T).$$

For $n > 0$ and $m \geq 2$, set

$$M(n) = \{kn + 1, kn + n \mid k = 0, 1, \ldots\}, \ N(n) = \{kn + 1 \mid k = 0, 1, \ldots\},$$

and

$$\overline{M}_m(n) = \overline{M(n)}, \ \overline{N}_m(n) = \overline{N(n)} \pmod{m(m-1)}.$$

Theorem 3.3 (Kobayashi [4]). *For any $n > 0$ and $n_1, \dots n_s > 0$, there is an E-m semigroup S with*

$$\overline{E}_m(S) \;=\; \bigcap_{i=1}^{s} \overline{M}_m(n_i) \bigcap \overline{N}_m(n). \tag{3.1}$$

For many types of semigroups S, the modular exponent semigroups are expressed as (3.1), but we do not know if any modular exponent semigroup is so expressed (see Kobayashi [3]).

4. E-3 SEMIGROUPS

Let S be an E-3 semigroup, that is, $x^3y^3 = (xy)^3$ for all $x, y \in S$. In this section, for $m_1, m_2, \dots, m_s, n \in P$, we write

$$m_1, m_2, \dots, m_s \Rightarrow n$$

if $3, m_1, m_2, \dots, m_s \Rightarrow n$.

The following identity (4.1) is given in Kobayashi [2], and is crucial to analyze the exponent semigroups of E-3 semigroup. We give a proof for the convenience of the reader.

Lemma 4.1. *Let S be an E-3 semigroup. Then we have*

$$(xy)^3x^{2n+1} = x^{2n}(xy)^3x \tag{4.1}$$

for any $x, y \in S$ and $n \geq 0$.

Proof. We proceed by induction on n. When $n = 0$, the identity is trivial. Let $n > 0$ and assume that

$$(xy)^3x^{2(n-1)+1} = x^{2(n-1)}(xy)^3x$$

holds. Then, we have

$$\begin{aligned} (xy)^3x^{2n+1} &= x^{2(n-1)}(xy)^3x^3 = x^{2(n-1)}(xyx)^3 \\ &= x^{2n+1}(yx)^3 = x^{2n}(xy)^3x. \end{aligned}$$

□

Corollary 4.2. *For any $n, a \geq 1$,*

$$n, n+3, n+2a \;\Rightarrow\; n+3+2a. \tag{4.2}$$

Proof. Let $x, y \in S$, and assume $n, n+3, n+2a \in E(S)$. Then, using (4.1), we have

$$\begin{aligned} (xy)^{n+3+2a} &= (xy)^3(xy)^{n+2a} = (xy)^3x^{n+2a}y^{n+2a} \\ &= x^{2a}(xy)^3x^ny^{n+2a} = x^{2a}(xy)^{n+3}y^{2a} \\ &= x^{n+3+2a}y^{n+3+2a}. \end{aligned}$$

□

Corollary 4.3. *For $n \geq 1$ we have*

$$n, n+2, n+3 \;\Rightarrow\; n+5, \tag{4.3}$$

and

$$n, n+3, n+4 \;\Rightarrow\; n+7. \tag{4.4}$$

Proof. Letting $a = 1$ and 2 in (4.2), we obtain (4.3) and (4.4), respectively. $\square$

Lemma 4.4. *For $n \geq 1$ we have*

$$3n + 1 \;\Rightarrow\; 3n + 3. \tag{4.5}$$

Proof. Let S be an E-$(3, 3n + 1)$ semigroup. For $x, y \in S$ we have

$$\begin{aligned} x^{3n+3}y^{3n+3} &= x^2(xy)^{3n+1}y^2 = x^3(yx)^{3n}y^3 = (x(yx)^n y)^3 \\ &= (xy)^{3n+3}. \end{aligned}$$

$\square$

Lemma 4.5. *For $n \geq 2$ we have*

$$n - 1,\, n + 1,\, 2n \;\Rightarrow 2n + 2. \tag{4.6}$$

Proof. Let S be an E-3 semigroup with $n - 1, n + 1, 2n \in E(S)$. For $x, y \in S$ we have

$$\begin{aligned} x^{2n+2}y^{2n+2} &= x^2(xy)^{2n}y^2 = x^3(yx)^{2n-1}y^3 \\ &= (xyx)^3(yx)^{2n-4}y^3 = (xyx)^2(xyxy)^{n-1}y^2 \\ &= (xyx)^2(xyx)^{n-1}y^{n-1}y^2 = (xyx)^{n+1}y^{n+1} \\ &= (xy)^{2n+2}. \end{aligned}$$

$\square$

By Corollary 3.2 and Theorem 3.1 we see

$$6n + 1 \ (n \geq 2),\ 6n + 3 \ (n \geq 0) \in E(S), \tag{4.7}$$

and

$$7 \notin E(3). \tag{4.8}$$

Moreover, by Kobayashi [2, Theorem 1] we have

$$6n + a \;\Rightarrow\; 6(n + 1) + a \quad (n \geq 0, a \geq 2). \tag{4.9}$$

Corollary 4.6. *For any $n \geq 0$ we have*

$$6n + 4 \;\Rightarrow\; 6(n + 1), \tag{4.10}$$

$$6(n + 1) \;\Rightarrow\; 6(n + 1) + 4, \tag{4.11}$$

$$6n + 5,\ 6(n + 1) + 2 \;\Rightarrow\; 6(n + 1) + 4, \tag{4.12}$$

$$6n + 5,\ 6(n + 1) \;\Rightarrow\; 6(n + 1) + 2. \tag{4.13}$$

Proof. Replacing n by $2n + 1$ in (4.5) in Lemma 4.4, we obtain (4.10).

Because $6n+3, 6(n+1)+1 \in E(S)$ for $n \geq 1$ by (4.7), we see $6(n+1)+4 \in E(S)$ if $6(n + 1) \in E(S)$ for $n \geq 1$ by Corollary 4.3 (replace n by $6n + 3$ in (4.4)). By a calculation using a computer (see (7.1) in Section 7), we find (4.11) also holds even for $n = 0$.

Replacing n by $6n + 5$ in (4.3), we get (4.12) if $n \geq 1$, because $6n + 7 \in E(S)$ by (4.7). It also holds for $n = 0$ by Lemma 4.5 (let $n = 4$ in (4.6)).

Lastly, we get (4.13) replacing n by $6n + 3$ in (4.3). $\square$

The following tells that if we have 5 consecutive exponents, then the all numbers after them are exponents.

Corollary 4.7. *For any $n \geq 1$ and $k \geq n+5$,*

$$n, n+1, n+2, n+3, n+4 \;\Rightarrow\; k. \tag{4.14}$$

Proof. Apply (4.3) repeatedly. □

Lemma 4.8. *For any $n \geq 0$,*

$$3n+3, 3n+4, 3n+5 \;\Rightarrow\; 3n+7. \tag{4.15}$$

Proof. Let $x, y \in$ and suppose $3n+3, 3n+4, 3n+5 \in E(S)$, then we have

$$\begin{aligned} x^{3n+7}y^{3n+7} &= x^2(xy)^{3n+5}y^2 = x^3(yx)^{3n+4}y^3 \\ &= (xyx)^3(yx)^{3n+1}y^3 = (xyx)^2(xy)^{3n+3}y^2 \\ &= (xyx)^2x^{3n+3}y^{3n+5} = (xyx)xyx^{3n+4}y^{3n+4}y \\ &= xyx(xy)^{3n+5}y = xyx^{3n+6}y^{3n+6} \\ &= xyx^2(xy)^{3n+4}y^2 = xyx^3(yx)^{3n+3}y^3 \\ &= xy(x(yx)^{n+1}y)^3 = (xy)^{3n+7}. \end{aligned}$$

□

The modular exponent semigroups of E-3 semigroups are completely determined by Kobayashi [2, Theorem 2] as follows.

Theorem 4.9. *$\overline{E}_3(S)$ is equal to one of the subsemigroups*

(i) $\{\bar{1}, \bar{3}\}$, (ii) $\{\bar{1}, \bar{3}, \bar{5}\}$, (iii) $\{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$ *and* (iv) $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

of $\mathbb{Z}_6$, and they are all possible.

5. Determination of exponents of E-3 semigroups, I

We try to determine the exponent semigroups of E-3 semigroups exactly. Let S be an E-3 semigroup.

First we treat the case where $\overline{E}_3(S) = \{\bar{1}, \bar{3}\}$. Set

$$P_1 = \{6a+1, 6a+3 \mid a = 0, 1, 2, \dots\},$$

and

$$P_1' = P_1 \setminus \{7\}.$$

Proposition 5.1. *If $\overline{E}_3(S) = \{\bar{1}, \bar{3}\}$, then either $E(S) = P_1$ or $E(S) = P_1'$.*

Proof. Suppose that $\overline{E}(S) = \{\bar{1}, \bar{3}\}$, then $E(S) \subset P_1$. On the other hand, $P_1' \subset E(S)$ by (4.7). Hence, P_1 and P_1' are the only possibilities. Because $7 \notin E(3)$ by (4.8), we see $E(3) = P_1'$. P_1 is realized as $E(3, 7) = P_1$. □

Next we treat the case where $\overline{E}_3(S) = \{\bar{1}, \bar{3}, \bar{5}\}$. For $r \geq 0$ set

$$P(r) = \{6a+1, 6a+3, 6(r+a)+5 \mid a = 0, 1, 2, \dots\},$$

and

$$P'(r) = P(r) \setminus \{7\}.$$

Proposition 5.2. *If* $\overline{E}_3(S) = \{\bar{1}, \bar{3}, \bar{5}\}$*, then* $E(S) = P(r)$ *or* $E(S) = P'(r)$ *for some* $r \geq 0$*. Actually,*

$$E(3, 6r+5) = P'(r) \ \text{and} \ E(3, 7, 6r+5) = P(r).$$

Proof. Let $n = 6r + 5 = \min\{n \in E(S) | n \equiv 5 \,(\mathrm{mod}\, 6)\}$, then $E(S) \subset P(r)$. We see $P'(r) \subset E(S)$ by (4.7) and (4.9). So, if $7 \in E(S)$, then $E(S) = P(r)$. By computer we find $7 \notin E(3, 5)$ (see Section 7). This implies $E(3, 6r + 5) = P'(r)$ for any r. $\square$

Now, we treat the case where $\overline{E}_3(S) = \{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$. For $r \geq 0$ set

$$\begin{aligned} Q(r) &= \{6a+1, 6a+3, 6(r+a)+4, 6(r+a)+6 \,|\, a = 0, 1, 2, \dots\}, \\ R_(r) &= \{6a+1, 6a+3, 6(r+a)+6, 6(r+a)+10 \,|\, a = 0, 1, 2, \dots\}, \\ Q'(r) &= Q(r) \setminus \{7\}, \\ R'(r) &= R(r) \setminus \{7\}. \end{aligned}$$

Proposition 5.3. *If* $\overline{E}_3(S) = \{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$*, then* $E(S)$ *is equal to* $Q(r)$*,* $Q'(r)$*,* $R(r)$ *or* $R'(r)$ *for some* $r \geq 0$*. Actually,*

$$E(3, 6r+4) = Q'(r), \ E(3, 6r+6) = R'(r),$$

$$E(3, 7, 6r+4) = Q(r), \ E(3, 7, 6r+6) = R(r).$$

Proof. If $6r + 4 \in E(S)$, then $6r + 6 \in E(S)$ for $r \geq 0$ by (4.10). So, if $6r + 4$ is the smallest $n \in E(S)$ such that $n \equiv 4 \pmod 6$, then $Q'(r) \subset E(S)$ by (4.9). By computer, $7 \notin E(3, 4)$. So, because $E(S) \subset Q(r)$, we have two possibilities $E(3, 6r + 4) = Q'(r)$ and $E(3, 7, 6r + 4) = Q(r)$.

On the other hand, if $6r + 6 \in E(S)$, then $6r + 10 \in E(S)$ for $r \geq 1$ by (4.11). Hence, if $6r + 6$ is the smallest $n \in E(S)$ such that $n \equiv 0 \pmod 6$, then $R'(r) \subset E(S) \subset R(r)$, and we have $E(7, 6r + 6) = R'(r)$ and $E(3, 7, 6r + 6) = R(r)$. $\square$

In the next section we treat the case where $\overline{E}_3(S) = \mathbb{Z}_6$.

6. Determination of exponents of E-3 semigroups, II

Let S be an E-3 semigroup such that $\overline{E}_3(S) = \mathbb{Z}_6$. Due to Theorem 4.9, $\overline{E}_3(S) = \mathbb{Z}_6$ if and only if $\bar{2} \in \overline{E}_3(S)$. For $k \geq 1$ set

$$P_{\geq k} = \{n \in P \,|\, n \geq k\} \ \text{ and } \ P_{\leq k} = \{n \in P \,|\, n \leq k\}.$$

(a) First of all, as we stated in Introduction, if $2 \in E(S)$, then

$$E(S) = E(2, 3) = P.$$

(b) Now suppose that $2 \notin E(S)$ and $8 \in E(S)$. Let $E = E(3, 8)$. We see $2, 4, 5, 6, 7 \notin E$, $9 = 3 \cdot 3 \in E$, and by computer $10 \notin E$ and $11, 12 \in E$. Moreover, $13 = 2 \cdot 6 + 1, 14 = 6 + 8 \in E$ and $15 = 2 \cdot 6 + 3 \in E$ by (4.7) and (4.9). Hence, $m \in E$ for all $m \geq 11$ by (4.14). Moreover, $10 \notin E(3, 7, 8)$ by computer. Thus,

$$E(3, 8) = \{1, 3, 8, 9\} \cup P_{\geq 11}, \tag{6.1}$$

$$E(3, 7, 8) = E(3, 8) \cup \{7\}, \tag{6.2}$$

$$E(3, 8, 10) = E(3, 8) \cup \{10\} \tag{6.3}$$

and

$$E(3,7,8,10)=E(3,8)\cup\{7,10\}. \tag{6.4}$$

(b1) Let $E=E(3,4,8)$. Then $E(3,8)\subset E$. We see $6\in E$ by (4.10) and $10\in E$ by (4.11), and $5,7\notin E(S)$ because $5,7\notin E(3,4)$ as we saw in the proof of Proposition 5.3. Hence,

$$E(3,4,8)=\{1,3,4,6\}\cup P_{\geq 8} \tag{6.5}$$

and

$$E(3,4,7,8)=E(3,4,8)\cup\{7\}. \tag{6.6}$$

(b2) Let $E=E(3,5,8)$. Then, $6\notin E$ because $6<8$ and $6\not\equiv 5 \pmod 6$. Moreover, $10\in E$ by (4.12) and $7\notin E$ by computer. Hence,

$$E(3,5,8)=\{1,3,5\}\cup P_{\geq 8} \tag{6.7}$$

and

$$E(3,5,7,8)=E(3,5,8)\cup\{7\}. \tag{6.8}$$

(b3) Let $E=E(3,6,8)$. We see $7\notin E$ by computer and $10\in E$ by (4.11). Thus,

$$E(3,6,8)=\{1,3,6\}\cup P_{\geq 8} \tag{6.9}$$

and

$$E(3,6,7,8)=E(3,6,8)\cup\{7\}. \tag{6.10}$$

(b4) Let $E=E(3,4,5,8)$. We see $7\in E$ by (4.15) in Lemma 4.8, $4\Rightarrow 6$ by (4.10) and $5,6\Rightarrow 8$ by (4.13). Hence, by (b1) and (b2) above we have

$$E(3,4,5,8)=E(3,4,5)=P\setminus\{2\}. \tag{6.11}$$

(b5) Let $E=E(3,5,6,8)$. We find $7\notin E(3,5,6)$ by computer and $5,6\Rightarrow 8$ by (4.13). By (b2) and (b3) we have

$$E(3,5,6,8)=E(3,5,6)=\{1,3,5,6\}\cup P_{\geq 8} \tag{6.12}$$

and

$$E(3,5,6,7,8)=E(3,5,6,7)=E(3,5,6,8)\cup\{7\}. \tag{6.13}$$

(c) Next we suppose that $8\notin E(S)$ and $14\in E(S)$. Let $E=E(3,14)$. By computer we find $16\notin E$ and $17,18\in E$ (see Section 7). We see $19,21\in E$ by (4.7) and $20\in E$ by (4.9). Hence, $k\in E(S)$ for all $k\geq 17$ by (4.14). Let

$$E'(S)=E(S)\cap P_{\leq 13} \quad\text{and}\quad \overline{E}'(S)=\{\overline{k}\in\mathbb{Z}_6\,|\,k\in E'(S)\}.$$

(c1) Suppose $\overline{E}'(S)=\{\bar{1},\bar{3}\}$. By computer $16\notin E(3,7,14)$. So, by Proposition 5.1, with

$$P_0=\{1,3,9,13,14,15\}\cup P_{\geq 17},$$

we have 4 possibilities

$$\begin{gathered}E(3,14)=P_0, E(3,7,14)=P_0\cup\{7\}, E(3,14,16)=P_0\cup\{16\}\ \text{ and}\\ E(3,7,14,16)=P_0\cup\{7,16\}.\end{gathered} \tag{6.14}$$

(c2) Suppose $\overline{E}'(S) = \{\bar{1}, \bar{3}, \bar{5}\}$. By (4.12) we see $16 \in E(3, 11, 14)$. By Proposition 5.2, we have 4 possibilities

$$E(3,5,14) = \{1,3,5,9,11\} \cup P_{\geq 13},\ E(3,5,7,14) = E(3,5,14) \cup \{7\},$$
$$E(3,11,14) = \{1,3,9,11\} \cup P_{\geq 13},\ E(3,7,11,14) = E(3,11,14) \cup \{7\}. \tag{6.15}$$

(c3) Suppose $\overline{E}'(S) \subset \{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$ and $\overline{E}'(S) \neq \{\bar{1}, \bar{3}\}$. By (4.10) and (4.11), we have $4 \Rightarrow 6 \Rightarrow 10 \Rightarrow 12$. We see $16 \in E(3, 12, 14)$ by (4.15) and $7 \notin E(3, 4)$ as we stated in the proof of Proposition 5.3. So we have 8 possibilities

$$\begin{aligned} E(3,4,14) &= \{1,3,4,6,9,10\} \cup P_{\geq 12}, & E(3,6,14) &= \{1,3,6,9,10\} \cup P_{\geq 12},\\ E(3,10,14) &= \{1,3,9,10\} \cup P_{\geq 12}, & E(3,12,14) &= \{1,3,9\} \cup P_{\geq 12},\\ E(3,4,7,14) &= E(3,4,14) \cup \{7\}, & E(3,6,7,14) &= E(3,6,14) \cup \{7\},\\ E(3,7,10,14) &= E(3,10,14) \cup \{7\}, & E(3,7,12,14) &= E(3,12,14) \cup \{7\}. \end{aligned} \tag{6.16}$$

(c4) Suppose that $\overline{E}'(S)$ is not in $\{\bar{1}, \bar{3}, \bar{5}\}$ nor in $\{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$, that is, $E'(S)$ contains two elements a, b with $a \equiv 4 \pmod 6$ and $b \equiv 5 \pmod 6$. Because $4 \Rightarrow 6 \Rightarrow 10 \Rightarrow 12$ and $5 \Rightarrow 11$, we see $n \in E(S)$ for all $n \geq 11$. Because $4, 5 \Rightarrow 8$ and $5, 6 \Rightarrow 8$ by (4.10) and (4.13), we can exclude the cases where $\{a, b\} = \{4, 5\}$ and $\{a, b\} = \{6, 5\}$. We already see above that $7 \notin E(3, 4) \cup E(3, 5)$. Moreover, $11, 12 \Rightarrow 14$ by (4.13). Hence, $E(3, 4, 11, 14) = E(3, 4, 11), E(3, 5, 10, 14) = E(3, 5, 10)$ etc. Taking account of these facts, we have 12 possibilities

$$\begin{aligned} E(3,4,11) &= \{1,3,4,6\} \cup P_{\geq 9}, & E(3,4,7,11) &= E(3,4,11) \cup \{7\},\\ E(3,6,11) &= \{1,3,6\} \cup P_{\geq 9}, & E(3,6,7,11) &= E(3,6,11) \cup \{7\},\\ E(3,5,10) &= \{1,3,5\} \cup P_{\geq 9}, & E(3,5,7,10) &= E(3,5,10) \cup \{7\},\\ E(3,10,11) &= \{1,3\} \cup P_{\geq 9}, & E(3,7,10,11) &= E(3,10,11) \cup \{7\},\\ E(3,5,12) &= \{1,3,5,9\} \cup P_{\geq 11}, & E(3,5,7,12) &= E(3,5,12) \cup \{7\},\\ E(3,11,12) &= \{1,3,9\} \cup P_{\geq 11}, & E(3,7,11,12) &= E(3,11,12) \cup \{7\}. \end{aligned} \tag{6.17}$$

Proposition 6.1. *Suppose that $\overline{E}_3(S) = \mathbb{Z}_6$, and let a be the minimal number in $E(S)$ such that $a \equiv 2 \pmod 6$.*

(a) *If $a = 2$, then $E(S) = P$.*
(b) *If $a = 8$, then we have 13 possibilities* (6.1) – (6.13).
(c) *If $a = 14$, then we have 28 possibilities* (6.14) – (6.17).

7. Discussions

In this paper we studied the exponent semigroups of E-3 semigroups. When the modular exponential semigroup is $\{\bar{1}, \bar{3}\}$, $\{\bar{1}, \bar{3}, \bar{5}\}$ or $\{\bar{0}, \bar{1}, \bar{3}, \bar{4}\}$, we completely determine them. When the modular exponent semigroup is $\{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$, a partial answer is given.

In our calculations we used a computer. For two words f, g over $\{a, b\}$, $f = g$ in the free E-m semigroup $F(m)$ generated by $\{a, b\}$, if there is a sequence $f = f_0, f_1, \ldots, f_n = g$ such that

$$f_{i-1} = u(xy)^m v, f_i = ux^m y^m v \text{ or } f_{i-1} = ux^m y^m v, f_i = u(xy)^m v$$

for some $x, y, u, v \in F = \{a, b\}^+$. This is a kind of *word problems*, which are in general undecidable. However, because the lengths of the words in the left and the right sides of $(xy)^m = x^m y^m$ are equal, for a given word f there are only a finite number of words g such that $f = g$ in $F(m)$. So, the derivation sequences starting from f can be calculated by a computer utilizing the depth first search. Here, to avoid infinite loops, the computer needs to remember all words already visited.

For example, a computer calculation gives the following derivation sequence

$$\begin{aligned} a^{10}b^{10} &\to a^4(ab)^6b^4 \\ &\to a(a^2b)^3(ab)^3b^4 \\ &\to a(a^2b)^3a^3b^7 = a^3(ba^2)^3ab^7 \\ &\to (aba^2)^3ab^7 = a(ba^3)^3b^7 \\ &\to a(ba^3b^2)^3b = ab(a^3b^3)^3 \\ &\to (ab)^{10}. \end{aligned} \tag{7.1}$$

This shows $6 \Rightarrow 10$ for E-3 semigroups. On the other hand, by searching all possible derivation sequences from $(ab)^7$ to a^7b^7 in $F(3, 5)$ by computer, we find that $7 \notin E(3, 5)$.

As stated in (c) in the previous section, we find that $18 \in E(3, 14)$ by a computer calculation. Our search program gives a derivation sequence from $a^{18}b^{18}$ to $(ab)^{18}$ of length 12037856. It seems very hard to get this result by hand calculation.

Suggested by these computer calculations we conjecture that for any $n \geq 1$

$$6n + 4 \notin E(3, 6n + 2) \quad \text{and} \quad 6n + 5, 6n + 6 \in E(3, 6n + 2). \tag{7.2}$$

If (7.2) is true, we would be able to completely describe the exponent semigroups of E-3 semigroups.

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