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A QUICK LOOK ON THE HYBRID PROJECTION SCHEME OF DEEPHO ET AL.

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Abstract. We give a short and simple proof of convergence theorem of the hybrid projection scheme studied by Deepho et al. [6]. In fact, we show that the main result of Deepho et al. [6] is a consequence of the convergence theorem of Takahashi et al. [12]. We also show that some assumptions in the result of Deepho et al. can be either dropped away or weakened.

1. Introduction

In the recent paper of Deepho et al. [6], the hybrid projection scheme for finding a common element of three sets: (a) the solution set of split equilibrium problem; (b) the solution set of a general system of finite variational inequalities; and (c) the set of common fixed points of a countable family of nonexpansive mappings was presented. It is our purpose to give a short and simple proof of this result. Note that some restrictions as were the case in the original results can be improved.

First, let us recall some notations of each problem above. Throughout this paper we assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces. Let C and Q be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Split Equilibrium Problem. Let $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ be two bifunctions. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The *split equilibrium problem* (see [4, 9]) is to find

$$x^* \in C \text{ such that } F_1(x^*, x) \geq 0 \text{ and } F_2(Ax^*, y) \geq 0 \text{ for all } x \in C, y \in Q.$$

The solution set of this problem is denoted by $\mathrm{SEP}(F_1, F_2; A)$.

System of Finite Variational Inequalities. Let $N \geq 1$ be a natural number. Let $B_1, B_2, \ldots, B_N : C \to \mathcal{H}_1$ be given mappings and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be given (stricltly) positive real numbers. The *system of variational inequalities* (see [13]) is to find

$$(x_1^*, x_2^*, \ldots, x_N^*) \in C \times C \times \cdots \times C$$
such that the following N inequalities hold
$$\langle \lambda_1 B_1 x_1^* + x_2^* - x_1^*, x - x_2^* \rangle \geq 0 \text{ for all } x \in C;$$
$$\langle \lambda_2 B_2 x_2^* + x_3^* - x_2^*, x - x_3^* \rangle \geq 0 \text{ for all } x \in C;$$

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$$\vdots$$

$$\langle \lambda_{N-1}B_{N-1}x_{N-1}^* + x_N^* - x_{N-1}^*, x - x_N^* \rangle \geq 0 \text{ for all } x \in C;$$

$$\langle \lambda_N B_N x_N^* + x_1^* - x_N^*, x - x_1^* \rangle \geq 0 \text{ for all } x \in C.$$

The solution set of this problem is denoted by $\mathrm{VI}(C, \lambda_1 B_1, \lambda_2 B_2, \ldots, \lambda_N B_N)$. Before moving, we give a simple note on this problem. This solution set can be regarded as a subset of C instead of a subset of $C \times C \times \cdots \times C$ as follows. By Zarantonello's characterization [14] of the projection Proj_C onto a closed convex set C, we have the following equivalent statements: $(x_1^*, x_2^*, \ldots, x_N^*) \in \mathrm{VI}(C, \lambda_1 B_1, \lambda_2 B_2, \ldots, \lambda_N B_N)$ if and only if

$$\begin{cases} x_2^* = \mathrm{Proj}_C(I - \lambda_1 B_1)x_1^* \\ x_3^* = \mathrm{Proj}_C(I - \lambda_2 B_2)x_2^* \\ \quad \vdots \\ x_N^* = \mathrm{Proj}_C(I - \lambda_{N-1} B_{N-1})x_{N-1}^* \\ x_1^* = \mathrm{Proj}_C(I - \lambda_N B_N)x_N^*. \end{cases}$$

In particular, $(x_1^*, x_2^*, \ldots, x_N^*) \in \mathrm{VI}(C, \lambda_1 B_1, \lambda_2 B_2, \ldots, \lambda_N B_N)$ can be determined by only one element $x^* \in C$ such that

$$x^* = \mathrm{Proj}_C(I - \lambda_N B_N) \cdots \mathrm{Proj}_C(I - \lambda_2 B_2) \,\mathrm{Proj}_C(I - \lambda_1 B_1)x^*.$$

From now on, we may assume that the solution set $\mathrm{VI}(C, \lambda_1 B_1, \lambda_2 B_2, \ldots, \lambda_N B_N)$ is a subset of C.

Fixed Point Problem. A *fixed point* of a mapping $S : C \to C$ is a point $x^* \in C$ such that $x^* = Sx^*$ (see [7]). The set of all fixed points of S is denoted by $\mathrm{Fix}(S)$.

In each problem above, the following assumptions are assumed in [6]:

Assumption on bifunctions: For a given bifunction $f : C \times C \to \mathbb{R}$, we assume the following conditions:

(A1) $f(x, x) \geq 0$ for all $x \in C$;
(A2) $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) $\limsup_{t \to 0^+} f((1-t)x + tz, y) \leq f(x, y)$ for all $x, y, z \in C$;
(A4) The function $y \mapsto f(x, y)$ is convex and lower semicontinuous for all $x \in C$;
(A5) For each $r > 0$ and each $z \in C$ there exist a compact convex subset $K \subset \mathcal{H}_1$ and an element $x \in C \cap K$ such that $f(y, x) + \frac{1}{r}\langle y - x, x - z \rangle < 0$ for all $y \in C \setminus K$.

Assumption for variational inequalities: For a given mapping $B : C \to \mathcal{H}_1$, we assume that B is *β-inverse strongly monotone* where $\beta > 0$, that is, $\langle x - y, Bx - By \rangle \geq \beta \|x - y\|^2$ for all $x, y \in C$.

Assumption on mappings: For a mapping $S : C \to C$, we assume that S is *nonexpansive*, that is, $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$.

It follows from [5] that if a bifunction $f : C \times C \to \mathbb{R}$ satisfies conditions (A1)–(A4), then for each $r > 0$ and for each $x \in \mathcal{H}_1$ there exists a unique element $z \in C$

such that

$$f(z,y)+\frac{1}{r}\langle x-z, z-y\rangle \geq 0 \text{ for all } y\in C.$$

In this situation, this element z is denoted by $T_r^f(x)$. Moreover, it is also known that

- T_r^f is *firmly nonexpansive*, that is, $\|T_r^f(x)-T_r^f(y)\|^2 \leq \langle T_r^f(x)-T_r^f(y), x-y\rangle$ for all $x,y\in\mathcal{H}_1$;
- $z\in \mathrm{Fix}(T_r^f) \iff f(z,y)\geq 0$ for all $y\in C$.

It was proved in [11, Lemma 2.3] that

$$\|T_r^f(x)-T_s^f(x)\|^2 \leq \frac{r-s}{r}\langle T_r^f(x)-T_s^f(x), T_r^f(x)-x\rangle$$

for all $r,s>0$ and $x\in\mathcal{H}_1$.

Fact 1. *If $\{r_n\}_{n=1}^\infty$ is a sequence in $(0,\infty)$ such that $\liminf_n r_n>0$ and $\{z_n\}_{n=1}^\infty$ is a bounded sequence in $\mathcal{H}_1$ such that $\lim_n\|z_n-T_{r_n}^f z_n\|=0$, then*

$$\lim_n \|z_n - T_1^f(z_n)\| = 0.$$

To see this, we note that

$$\begin{aligned}\|T_{r_n}^f(z_n)-T_1^f(z_n)\|^2 &\leq \frac{r_n-1}{r_n}\langle T_{r_n}^f(z_n)-T_1^f(z_n), T_{r_n}^f(z_n)-z_n\rangle\\ &\leq \left|1-\frac{1}{r_n}\right| \|T_{r_n}^f(z_n)-T_1^f(z_n)\|\|T_{r_n}^f(z_n)-z_n\|.\end{aligned}$$

Hence

$$\|T_{r_n}^f(z_n)-T_1^f(z_n)\| \leq \left|1-\frac{1}{r_n}\right| \|T_{r_n}^f(z_n)-z_n\|.$$

It follows from $\liminf_{n\to\infty} r_n>0$ and $\lim_n\|z_n-T_{r_n}^f(z_n)\|=0$ that $\lim_n\|T_{r_n}^f(z_n)-T_1^f(z_n)\|=0$ and hence $\lim_n\|z_n-T_1^f(z_n)\|=0$.

Convergence result of Deepho et al. The following is (the corrected version of) the main result of [6].

Theorem DMSK. *Let C and Q be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let $A:\mathcal{H}_1\to H_2$ be a bounded linear operator with its adjoint operator A^*. Let us assume the following conditions:*

- *$F_1: C\times C\to\mathbb{R}$ and $F_2: Q\times Q\to\mathbb{R}$ is bifunctions satisfying (A1)–(A4).*
- *$B_i: C\to\mathcal{H}_1$ is a β_i-inverse strongly monotone mapping for all $i=1,2,\ldots,N$ and let $\lambda_i\in(0,2\beta_i)$ for all $i=1,2,\ldots,N$.*
- *$S: C\to C$ is a nonexpansive mapping.*

Suppose that

$$\mathbf{S} := \mathrm{SEP}(F_1,F_2;A)\cap \mathrm{VI}(C,\lambda_1B_1,\lambda_2B_2,\ldots,\lambda_NB_N)\cap\mathrm{Fix}(S)\neq\varnothing.$$

We also assume the following conditions:

- *$\{r_n\}_{n=1}^\infty$ is a sequence in $(0,\infty)$ such that $\liminf_n r_n>0$.*
- *γ is a real number such that $\gamma\in(0,2/L)$ where $L:=\|A\|^2$.*

- $\{\alpha_n\}_{n=1}^{\infty}$ *is a sequence in* $[0,1]$ *such that* $\alpha_n \le \alpha$ *for some* $\alpha \in (0,1)$.

Let $x_0 \in \mathcal{H}_1$ *be arbitrarily chosen and* $C_1 := C$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* $\mathcal{H}_1$ *and* $\{C_n\}_{n=1}^{\infty}$ *be a sequence of closed convex subsets of* C *defined by* $x_1 := \operatorname{Proj}_{C_1} x_0$ *and*

$$\begin{cases} u_n := T_{r_n}^{F_1}(I + \gamma A^*(T_{r_n}^{F_2} - I)A)(x_n); \\ y_n := \operatorname{Proj}_C(I - \lambda_N B_N) \cdots \operatorname{Proj}_C(I - \lambda_2 B_2) \operatorname{Proj}_C(I - \lambda_1 B_1) u_n \\ z_n := \alpha_n y_n + (1 - \alpha_n)\left(\frac{1}{n+1}\left(y_n + S y_n + S^2 y_n + \cdots + S^n y_n\right)\right) \\ C_{n+1} := \{z \in C_n : \|z_n - z\| \le \|x_n - z\|\} \\ x_{n+1} := \operatorname{Proj}_{C_{n+1}} x_0 \text{ for all } n \ge 1. \end{cases}$$

Then the sequence $\{x_n\}_{n=1}^{\infty}$ *converges strongly to* $x^* \in \mathbf{S}$ *which is nearest to* x_0, *that is,* $x^* = \operatorname{Proj}_{\mathbf{S}} x_0$.

Remark 1. Let us comment on the original statements of Theorem DMSK.

(1) The original paper assumes that $\{S^i\}_{i=1}^{n}$ is a sequence of nonexpansive mappings. But in the last line of Step 4, they make use of Lemma 2.4 which is for a single nonexpansive mapping.
(2) Lemma 2.2 is not complete. It contains only assumptions.
(3) The assumption (A5) of Lemma 2.3 in [6] is superfluous.

The corrected version of their result is therefore stated as above. Moreover, the result above is stated for $\gamma \in (0, 2/L)$ and $\alpha_n \in [0, \alpha]$ while the original work assumes that $\gamma \in (0, 1/L)$ and $\alpha_n \in (0, \alpha]$. As noted in [4], permitting γ to take on larger values accelerates the convergence of the iterative sequence.

2. A short and simple proof of Theorem DMSK

We note that the scheme of Deepho et al. [6] is a combination of the following known methods:

- Baillon's nonlinear ergodic theorem (see [2]).
- Shrinking projection method of Takahashi et al. (see [12]).

Many pieces of the proof given there are taken from the corresponding previous known results. First we note that the shrinking projection method given below (Theorem TKK) is the stem of Theorem DMSK. We then prove that Theorem DMSK is a direct consequence of Theorem TTK.

Theorem TTK ([12, Theorem 3.3 with $\alpha_n \equiv 0$])**.** *Let* C *be a nonempty closed convex subset of a Hilbert space* $\mathcal{H}$. *Let* $\{T_n : C \to C\}_{n=1}^{\infty}$ *be a sequence of nonexpansive mappings and* $T : C \to C$ *be a nonexpansive mapping such that*

$$\bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) = \operatorname{Fix}(T) \neq \varnothing.$$

Suppose that $\{T_n\}$ *satisfies the NST-condition with* T. *Let* $u_0 \in \mathcal{H}$ *be arbitrarily chosen and* $C_1 := C$, *and* $u_1 := \operatorname{Proj}_{C_1} u_0$, *define a sequence* $\{u_n\}_{n=1}^{\infty}$ *of* C *as* *follows:*

$$\begin{cases} C_{n+1} := \{z \in C_n : \|T_n u_n - z\| \leq \|u_n - z\|\}; \\ u_{n+1} := \operatorname{Proj}_{C_{n+1}} u_0 \text{ for all } n \geq 1. \end{cases}$$

Then, $\{u_n\}_{n=1}^{\infty}$ *converges strongly to* $x^* = \operatorname{Proj}_F u_0$.

Recall that $\{T_n\}_{n=1}^{\infty}$ satisfies the *NST-condition with* T if for each bounded sequence $\{z_n\}_{n=1}^{\infty}$ in C, the following implication holds:

$$\lim_n \|z_n - T_n z_n\| = 0 \implies \lim_n \|z_n - T z_n\| = 0.$$

The following concept was introduced by Aoyama et al. [1].

Definition. Let $\mathcal{H}$ be a Hilbert space and let C be a nonempty closed convex subset of $\mathcal{H}$. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of nonexpansive mappings of C into itself. We say that $\{T_n\}_{n=1}^{\infty}$ is a *strongly nonexpansive sequence* if

$$\lim_n \|(x_n - y_n) - (T_n x_n - T_n y_n)\| = 0$$

whenever $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences in C such that $\{x_n - y_n\}_{n=1}^{\infty}$ is bounded and $\lim_n(\|x_n - y_n\| - \|T_n x_n - T_n y_n\|) = 0$.

In particular, if $T_n \equiv T$, then the concept above is deduced to the one introduced by Bruck and Reich [3]. More precisely, $T : C \to C$ is a *strongly nonexpansive mapping* if T is nonexpansive and

$$\lim_n \|(x_n - y_n) - (T x_n - T y_n)\| = 0$$

whenever $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are sequences in C such that $\{x_n - y_n\}_{n=1}^{\infty}$ is bounded and $\lim_n(\|x_n - y_n\| - \|T x_n - T y_n\|) = 0$.

Fact 2 ([1])**.** *Suppose that* $\{S_n : C \to C\}_{n=1}^{\infty}$ *and* $\{T_n : C \to C\}_{n=1}^{\infty}$ *are two sequences of nonexpansive mappings. The following statements are true.*

(1) *Suppose that* $\bigcap_{n=1}^{\infty} \operatorname{Fix}(S_n) \cap \bigcap_{n=1}^{\infty} \operatorname{Fix}(T_n) \neq \varnothing$ *and that either* $\{S_n\}_{n=1}^{\infty}$ *or* $\{T_n\}_{n=1}^{\infty}$ *is a strongly nonexpansive sequence. Let* $\{z_n\}_{n=1}^{\infty}$ *be a bounded sequence in* C*. Then*

$$\lim_n \|z_n - (S_n \circ T_n) z_n\| = 0 \iff \lim_n \|z_n - S_n z_n\| = \lim_n \|z_n - T_n z_n\| = 0.$$

(2) *Suppose that both* $\{S_n\}_{n=1}^{\infty}$ *and* $\{T_n\}_{n=1}^{\infty}$ *are strongly nonexpansive sequences. Then* $\{S_n \circ T_n\}_{n=1}^{\infty}$ *is a strongly nonexpansive sequence.*

Fact 3 ([1])**.** *If a bifunction* $f : C \times C \to \mathbb{R}$ *satisfies conditions (A1)–(A4) and* $\{r_n\}_{n=1}^{\infty}$ *is a sequence in* $(0, \infty)$*, then* $\{T_{r_n}^f\}_{n=1}^{\infty}$ *is a strongly nonexpansive sequence.*

Fact 4 ([8])**.** *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be Hilbert spaces. Let* $A : \mathcal{H}_1 \to \mathcal{H}_2$ *be a bounded linear operator with its adjoint operator* A^**. Assume that* $\gamma \in (0, 2/L)$ *where* $L := \|A\|^2$*. The following results are true.*

(1) *If* $T : \mathcal{H}_2 \to \mathcal{H}_2$ *is a firmly nonexpansive operator, then the mapping* $W : \mathcal{H}_1 \to \mathcal{H}_1$ *defined by*

$$W := I + \gamma A^*(T - I)A$$

is a strongly nonexpansive mapping.

(2) *Suppose that* $F_2 : Q \times Q \to \mathbb{R}$ *satisfies conditions (A1)–(A4) and* $\{r_n\}_{n=1}^{\infty}$ *is a sequence in* $(0, \infty)$. *For each* n, *we define*

$$W_n := I + \gamma A^*(T_{r_n}^{F_2} - I)A.$$

Then $\{W_n\}_{n=1}^{\infty}$ *is a strongly nonexpansive sequence.*

Fact 5 ([1]). *If* $B : C \to \mathcal{H}_1$ *is* β*-inverse strongly monotone and* $\lambda \in (0, 2\beta)$, *then* $\mathrm{Proj}(I - \lambda B) : C \to C$ *is a strongly nonexpansive mapping. In particular, let* $B_i : C \to \mathcal{H}_1$ *be a* β_i*-inverse strongly monotone mapping for all* $i = 1, 2, \ldots, N$ *and let* $\lambda_i \in (0, 2\beta_i)$ *for all* $i = 1, 2, \ldots, N$. *It follows then that*

$$\mathrm{Proj}_C(I - \lambda_N B_N) \cdots \mathrm{Proj}_C(I - \lambda_2 B_2)\,\mathrm{Proj}_C(I - \lambda_1 B_1)$$

is a strongly nonexpansive mapping.

Fact 6. [10, Lemma 3.10] *Let* C *be a closed convex subset of* $\mathcal{H}_1$ *and* $S : C \to C$ *be a nonexpansive mapping with* $\mathrm{Fix}(S) \neq \varnothing$. *If* $S_n := \frac{1}{n}\sum_{i=0}^{n-1} S^i$ *for all* $n \geq 1$, *then* $\mathrm{Fix}(S_n) = \mathrm{Fix}(S)$ *for all* $n \geq 2$ *and* $\{S_n\}_{n=1}^{\infty}$ *satisfies the NST-condition with* S.

We use the preceding known facts to give the following proof of Theorem DMSK.

A short and simple proof of Theorem DMSK via Theorem TTK. For convenience, we set

$$T_n := S_n \circ V \circ U_n,$$

where

$$\begin{aligned} U_n &:= T_{r_n}^{F_1}(I + \gamma A^*(T_{r_n}^{F_2} - I)A);\\ V &:= \mathrm{Proj}_C(I - \lambda_N B_N) \cdots \mathrm{Proj}_C(I - \lambda_2 B_2)\,\mathrm{Proj}_C(I - \lambda_1 B_1);\\ S_n &:= \alpha_n I + (1 - \alpha_n)\left(\frac{1}{n+1}(I + S + S^2 + \cdots + S^n)\right). \end{aligned}$$

We also set

$$T := S \circ V \circ U,$$

where

$$U := T_1^{F_1}(I + A^*\gamma(T_1^{F_2} - I)A).$$

First, we observe that each T_n and T are nonexpansive mappings. Moreover,

$$\mathbf{S} = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n) = \mathrm{Fix}(T) \neq \varnothing.$$

Secondly, the iterative sequence generated by the scheme in Theorem DMSK is exactly the same as the one in Theorem TTK.

Finally, we prove that $\{T_n\}_{n=1}^{\infty}$ satisfies the NST-condition with T. To see this, let $\{z_n\}$ be a bounded sequence in C such that $\lim_n \|z_n - T_n z_n\| = 0$. It follows from Facts 2(2), 4 and 5 that $\{V \circ U_n\}_{n=1}^{\infty}$ is a strongly nonexpansive sequence. Hence, by Fact 2(1), we have

$$\lim_n \|z_n - S_n z_n\| = \lim_n \|z_n - (V \circ U_n) z_n\| = 0.$$

Note that

$$\lim_n \left\| z_n - \frac{1}{n+1}(I + S + \cdots + S^n) z_n \right\| = \lim_n \frac{1}{1-\alpha_n} \|z_n - S_n z_n\| = 0.$$

Using Fact 6 gives

$$(\spadesuit) \qquad \lim_n \|z_n - S z_n\| = 0.$$

Moreover, using Fact 2(1) for $\lim_n \|z_n - (V \circ U_n) z_n\| = 0$ again gives

$$(\heartsuit) \qquad \lim_n \|z_n - V z_n\| = \lim_n \|z_n - U_n z_n\| = 0.$$

It follows from the last expression above and Fact 4 that

$$\lim_n \|z_n - T_{r_n}^{F_1} z_n\| = \lim \|z_n - (I + \gamma A^*(T_{r_n}^{F_2} - I)A) z_n\| = 0.$$

In particular, we have $\lim_n \|z_n - T_{r_n}^{F_1} z_n\| = \lim_n \|(T_{r_n}^{F_2} - I) A z_n\| = 0$. Since A is a bounded linear operator, $\{A z_n\}$ is a bounded sequence. It follows then from Fact 1 that $\lim_n \|z_n - T_1^{F_1} z_n\| = \lim_n \|(T_1^{F_2} - I) A z_n\| = 0$. In particular, $\lim_n \|z_n - (I + \gamma A^*(T_1^{F_2} - I)A) z_n\| = 0$. Hence

$$(\diamondsuit) \qquad \lim_n \|z_n - U z_n\| = 0.$$

Using Fact 2(1) for the expressions ($\spadesuit$), ($\heartsuit$), and ($\diamondsuit$) gives

$$\lim_n \|z_n - T z_n\| = 0.$$

This completes the proof. □

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References

[1] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *On a strongly nonexpansive sequence in Hilbert spaces*, J. Nonlinear Convex Anal. **8** (2007), 471–489.

[2] R. E. Bruck, *On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces*, Israel J. Math. **38** (1981), 304–314.

[3] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), 459–470.

[4] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Prob. **18** (2002), 441–453.

[5] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.

[6] J. Deepho, J. Martínez-Moreno, K. Sitthithakerngkiet and P. Kumam, *Convergence analysis of hybrid projection with Cesàro mean method for the split equilibrium and general system of finite variational inequalities*, J. Comput. Appl. Math. **318** (2017), 658–673.

[7] K. Geobel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, 1990.

[8] R. Kraikaew and S. Saejung, *On split common fixed point problems*, J. Math. Anal. Appl. **415** (2014), 513–524.

[9] A. Moudafi, *Split monotone variational inclusions*, J. Optim. Theory Appl. **150** (2011), 275-283.

[10] K. Nakajo, K. Shimoji and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings in Banach spaces*, J. Nonlinear Convex Anal. **8** (2007), 11–34.

[11] S. Takahashi and W. Takahashi, *Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space*, Nonlinear Anal. **69** (2008), 10251033.

[12] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.

[13] R. U. Verma, *On a new system of nonlinear variational inequalities and associated iterative algorithms*, Math. Sci. Res. Hot-Line **3** (1999), 65–68.

[14] E. H. Zarantonello, *Projections onto convex sets in Hilbert space, and spectral theory* Rev. Un. Mat. Argentina **26** (1972/73), 187–201 (Spanish).

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