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A FIXED POINT THEOREM AND THE SOLVABILITY OF SOME NONLINEAR EQUATIONS

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Abstract. In this note, we first present three fixed point theorems related our argument. One of the fixed point theorems is a version of Schauder's fixed point theorem. Then, we give another proof for a version of Schauder's fixed point theorem. Furthermore, we prove a new type of fixed point theorem (Theorem 3.1) and then apply it to consider the solvability of some nonlinear equations.

1. Introduction

The Leray-Schauder theorem [5] was proved initially in a Banach Space. Then, Nagumo [8] extended the theorem to that in a locally convex space. These were proved by considering degree theory; for instance, see Lloyd [6]. Under advantageous conditions, some Leray-Schauder type theorems were proved without degree theory; see Browder [2], Schaefer [11], and Potter [9]. Later Morales [7], Hirano [3], Kartsatos [4], and others studied the solvability of some nonlinear equations concerning m–accretive operators. Then, some researchers used Leray-Schauder type theorems to have their results. However, in their articles, handling of Leray-Schauder type theorem is slightly complicated. Also, for students, it is not so easy to understand degree theory completely.

In this note, by studying the works as above, we prove Theorem 3.1 which is a new type of fixed point theorem. In a sense, it is a refined version of Rothe's theorem [10]. Then, to consider the problem, we choose another way of using neither Leray-Schauder type theorems nor existing fixed point theorems. Instead, we use a version of Theorem 3.1. Maybe, for students, it is easier to read this note.

2. Preliminaries

Basic concepts and notations.

We prepare some concepts and notations; sometimes we use them without notice. C always denotes a non–empty set; normally, "non–empty" is omitted. E denotes a real Banach space; normally, "real" is omitted. $\|\cdot\|$ denotes the norm of E, and E^* denotes the topological dual space of E. For $x \in E$ and $y^* \in E^*$, $\langle x, y^* \rangle$ denotes $y^*(x)$. J denotes the normalized duality mapping from E into 2^{E^*}:

$$Jx = \{x^* \in E : \langle x, x^* \rangle = \|x\|\|x^*\|,\ \|x^*\| = \|x\|\}$$

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for all $x \in E$; see [12].

Let C be a subset of E. Then, $\overline{C}$, ∂C, and $\mathrm{Int}(C)$ denote the closure, the boundary, and the interior of C, respectively. Also, $\mathrm{co}(C)$ and $\mathrm{cc}(C)$ denote the convex hull and the closed convex hull of C, respectively. By a well–known theorem of Mazur, $\mathrm{cc}(C)$ is compact if C is compact. $B_r[x]$ denotes the closed ball of E with center at $x \in E$ and radius $r > 0$. $B_r(x)$ denotes the corresponding open ball. Then, $B_r[x]$ is closed and convex, and $\mathrm{Int}(B_r[x])$ is $B_r(x)$. For simplicity, we use B_r instead of $B_r[0]$.

Let S be a mapping from C into E and let A be an operator from C into 2^E. Sometimes we denote by $D(A)$ and $R(A)$ the domain and the range of A, respectively. So, $C = D(S)$ and $S(C) = \{Sx : x \in C\} = R(S)$. $F(S)$ denotes the fixed point set of S, that is, $F(S) = \{v \in C : Sv = v\}$.

S is called bounded if it maps bounded subsets of C onto bounded sets. S is called compact if S is continuous and it maps bounded subsets of C onto relatively compact sets. A is called accretive if there is $j \in J(x-y)$ satisfying $\langle u - v, j\rangle \geq 0$ for $x, y \in C$, $u \in Ax$ and $v \in Ay$. Let A be accretive and let $a \in (0, \infty)$. The Yosida resolvent J_a from $R(I + aA)$ onto $C = D(A)$ and the Yosida approximant A_a from $R(I + aA)$ into E are defined respectively by

$$J_a = (I + aA)^{-1}, \qquad A_a = \tfrac{1}{a}(I - J_a).$$

So, $D(J_a) = R(I + aA)$ and $R(J_a) = D(A)$. For J_a and A_a, the following hold:

- ○ $\|J_a x - J_a y\| \leq \|x - y\|$ for all $x, y \in D(J_a) = R(I + aA)$.
- ○ $A_a x \in AJ_a x$ for all $x \in R(I + aA)$.

A is called m–accretive if A is accretive and $R(I + aA) = E$ for all $a \in (0, \infty)$; see [12, 13] for more details.

Some basic fixed point theorems.

We present some basic fixed point theorems related to our argument.

Theorem 2.1. *Let C be a compact convex subset of an Euclidean space. Let S be a continuous self–mapping on C. Then, there is $v \in C$ satisfying $Sv = v$.*

This theorem is referred to as Brouwer's fixed point theorem. Then, Schauder proved the following well–known extension of the theorem.

Theorem 2.2. *Let C be a compact convex subset of E. Let S be a continuous self–mapping on C. Then, there is $v \in C$ satisfying $Sv = v$.*

The following is a version of Schauder's fixed point theorem.

Theorem 2.3. *Let C be a closed convex subset of E. Let S be a continuous self–mapping on C such that $S(C)$ is relatively compact. Then, there exists $v \in C$ satisfying $Sv = v$.*

Proof. By assumptions, $\mathrm{cc}(S(C))$ is compact and convex, and $\mathrm{cc}(S(C)) \subset C$. So, we can regard S as a continuous self–mapping on $\mathrm{cc}(S(C))$. Then, by Theorem 2.2, there is $v \in \mathrm{cc}(S(C)) \subset C$ satisfying $Sv = v$. □

This theorem is important in this note. In a locally convex space, the corresponding assertion also holds. Bonsall [1] says that Singbal gave a simple proof of the assertion; the proof is in [1. Appendix]. In the proof, he used only Brouwer's fixed point theorem. Also, recently, some elementally proofs of Brouwer's theorem appeared; for example, see Takeuchi and Suzuki [14, 15]. From these, in his direction, we give another proof. In this way, Schauder's theorem, Mazur's theorem, and degree theory are unnecessary.

Another proof of Theorem 2.3. Let N be the set of positive integers. Set $K = \overline{S(C)}$. Then, K is compact. Fix any $r > 0$. Then, $\{B_r(x)\}_{x\in K}$ is an open cover of K and has a finite subcover $\{B_r(x_i)\}_{i=1}^n$. Set $I = \{1, 2, \cdots, n\}$. For simplicity, we denote $\{x_i\}_{i=1}^n$ by $\{x_i\}$. We can regard $\mathrm{co}(\{x_i\})$ as a compact convex subset of an Euclidean space. Also, $\mathrm{co}(\{x_i\}) \subset C$.

Following Nagumo [8], for $i \in I$, define a continuous mapping d_i from E into $[0, r]$ by $d_i(x) = \max\{0, r - \|x - x_i\|\}$ for $x \in E$. For $x \in K$, there is $i \in I$ satisfying $\|x - x_i\| < r$, that is, $d_i(x) > 0$. Also, the following hold:

(1) For $x \in K$ and $i \in I$, $d_i(x) > 0$ if and only if $\|x - x_i\| < r$.

(2) For $x \in K$, $\frac{d_i(x)}{\sum_{i=1}^n d_i(x)} \in [0, 1]$ for $i \in I$, and $\sum_{i=1}^n \left(\frac{d_i(x)}{\sum_{i=1}^n d_i(x)}\right) = 1$.

Then, we consider a continuous mapping T_r from K into $\mathrm{co}(\{x_i\})$ such that

(3) $T_r x = \sum_{i=1}^n \frac{d_i(x)}{\sum_{i=1}^n d_i(x)} x_i$ for $x \in K$.

By (1)–(3), we see $\|x - T_r x\| \le \frac{1}{\sum_{i=1}^n d_i(x)} \sum_{i=1}^n d_i(x)\|x - x_i\| < r$ for $x \in K$.

Furthermore, we consider the continuous self–mapping $T_r S$ on $\mathrm{co}(\{x_i\})$. By Brouwer's fixed point theorem, there is $u \in \mathrm{co}(\{x_i\})$ satisfying $T_r S u = u$. By $Su \in K$, we already know $\|Su - T_r Su\| < r$, that is, $\|Su - u\| < r$.

By the argument so far, we see that there is a sequence $\{u_m\} \subset C$ satisfying $\|Su_m - u_m\| < 1/m$ for $m \in N$. Since K is compact and $\{Su_m\} \subset K$, there is a subsequence $\{Su_{m_j}\}$ of $\{Su_m\}$ which converges to some $v \in K \subset C$. Then, by $\lim_m \|Su_m - u_m\| = 0$, $\{u_{m_j}\}$ also converges to v. We know

$$\|v - Sv\| \le \|v - Su_{m_j}\| + \|Su_{m_j} - Sv\| \quad \text{for} \quad j \in N.$$

From these, since S is continuous, we see $\|v - Sv\| = 0$, that is, $v = Sv$.

Remark. Consider the finite dimensional linear space L spanned by $\{x_i\}_{i=1}^n$. Then, only one topology of L makes it a linear topological space. We may consider the topology as familiar Euclidean topology. Then, Euclidean topology of L and the relative topology of L as a subspace of E have to be coincide. □

3. A theorem

Theorem 3.1. *Let C be a subset of E with $0 \in Int(C)$. Let $r > 0$ satisfy $B_r \subset C$. Let S be a continuous mapping from C into E such that $S(B_r)$ is relatively compact. Define mappings f_r from E into $(0, 1]$ and M_r from E into B_r respectively by $f_r(y) = \frac{r}{\max\{r, \|y\|\}}$ and $M_r y = f_r(y)y$ for $y \in E$. Define a self–mapping V_r on B_r by*

$$V_r y = M_r S y = f_r(Sy)Sy \in B_r \qquad \text{for} \quad y \in B_r.$$

Then, there is $y_r \in B_r \subset C$ satisfying $V_r y_r = y_r$. Also, the following hold:

(1) $Sy_r = y_r$ *if* $Sy_r \in B_r$.
In particular, $Sy_r = y_r$ *if* $y_r \in \partial B_r$ *and* $Sy_r \in B_r$.

(2) $Sy_r = y_r$ ($f_r(Sy_r) = 1$) *if* $y_r \in \mathrm{Int}(B_r) = B_r(0)$.

Remark. *By considering a suitable translation, we can have the corresponding results if $\emptyset \neq Int(C)$. In the definition of V_r, we consider S as a mapping from B_r into E. By (2), there is $y_r \in B_r(0)$ satisfying $Sy_r = y_r$ if V_r is fixed point free on ∂B_r. We do not know the relation between y_r and $F(S)$ if $Sy_r \notin B_r$.*

Proof. By $0 \in \mathrm{Int}(C)$, we can choose such $r > 0$. We can easily see the following:

(i) $f_r(y) = 1$ and $M_r y = y \in B_r$ for $y \in B_r$.

(ii) $f_r(y) \in (0, 1)$ and $M_r y = \frac{r}{\|y\|} y \in B_r$ for $y \notin B_r$.

(iii) f_r and M_r are continuous.

For $y \notin B_r$, $M_r y = f_r(y)y = \frac{r}{\|y\|} y$ and $\|\frac{r}{\|y\|} y\| = \frac{r}{\|y\|}\|y\| = r$. So, $M_r y \in B_r$. We confirmed only the latter of (ii). By (i) and (ii), we see

$$M_r(E) \subset B_r, \quad V_r(B_r) = M_r S(B_r) \subset M_r(\overline{S(B_r)}) \subset B_r.$$

By assumptions, $\overline{S(B_r)}$ is compact. Then, by (iii), $V_r(B_r) \subset B_r$, V_r is continuous, and $V_r(B_r)$ is relatively compact. Thus, by Theorem 2.3, there is $y_r \in B_r$ satisfying $V_r y_r = M_r S y_r = y_r$. We show (1). Suppose $Sy_r \in B_r$. Then, by (i), we see $Sy_r = M_r S y_r = y_r$. To prove (2), by (1), it suffices to show $Sy_r \in B_r$.

Arguing by contradiction, assume $Sy_r \notin B_r$. By $y_r \in \mathrm{Int}(B_r) = B_r(0)$, $Sy_r \notin B_r$, and (ii), we immediately have a contradiction:

$$r > \|y_r\| = \|M_r S y_r\| = \|\frac{r}{\|Sy_r\|} S y_r\| = \frac{r}{\|Sy_r\|}\|Sy_r\| = r.$$

Of course, $Sy_r \in B_r$ implies $f_r(Sy_r) = 1$. □

In a sense, Theorem 3.1 is a refined version of Rohte's fixed point theorem.

Theorem 3.2. *Let $r > 0$ and let S be a compact mapping from B_r into E satisfying $S(\partial B_r) \subset B_r$. Then there is $v \in B_r$ satisfying $Sv = v$.*

Proof. In Theorem 3.1, set $C = B_r$. Then, the result is immediate. □

4. Applications

Let E be a Banach space. Let A be an m-accretive operator from a subset $D(A)$ of E into 2^E, and let S be a mapping from a subset $D(S)$ of E into E with $D(A) \subset D(S)$. Let $p \in E$. Consider the nonlinear equations of the form

(P) $$p \in Ax + Sx.$$

Obviously, (P) has a solution if and only if $p \in R(A + S)$.

Recall properties of J_1 and A_1. Then, for $y \in E$, the following are equivalent:

(a) $A_1 y + SJ_1 y - p = (I - J_1)y + SJ_1 y - p = 0$.

(b) $y = p + J_1 y - SJ_1 y$.

Define a mapping U from E into E by

$$Uy = p + J_1 y - SJ_1 y \quad \text{for} \quad y \in E.$$

Suppose there is $u \in F(U)$. Then, since (a) and (b) are equivalent, we see $p = A_1 u + SJ_1 u$. By $A_1 u \in AJ_1 u$, set $x = J_1 u \in D(A)$. So, we see $p \in Ax + Sx$. Thus, we confirmed that (P) has a solution if U has a fixed point.

This problem has been studied by Morales [7], Hirano [3], Kartsatos [4], and others. Maybe, for students, their arguments and handling of Leray–Shauder type theorems are slightly complicated. By the reason and theoretical interest, we try to clarify a structure of the problem; for simplicity, we consider the case that A has a compact resolvent. Then, to do this, we show Theorem 4.2. Also, we present Theorems 4.3 whose expression is similar to their results.

In advance, we prepare the following version of Theorem 3.1, in which M_r is omitted because maybe readers are accustomed to the situation.

Theorem 4.1. *Let A be an m-accretive operator from $D(A) \subset E$ into 2^E. Let S be a continuous mapping from $\overline{D(A)}$ into E such that $J_1 = (I + A)^{-1}$ is compact. Let $p \in E$. Define a mapping U from E into E by*

$$Uy = p + J_1 y - SJ_1 y \quad \text{for} \quad y \in E.$$

Let $r > 0$. Define a mapping f_r from E into $(0, 1]$ by $f_r(y) = \frac{r}{\max\{r, \|y\|\}}$ for $y \in E$, and define a self–mapping V_r on B_r by

$$V_r y = f_r(Uy)Uy \in B_r \qquad \text{for} \quad y \in B_r.$$

Then, there is $y_r \in B_r$ satisfying $V_r y_r = y_r$. Also, the following hold:

(1) *$Uy_r = y_r$ if $y_r \in \partial B_r$ and $Uy_r \in B_r$.*
(2) *$Uy_r = y_r$ ($f_r(Uy_r) = 1$) if $y_r \in \mathrm{Int}(B_r) = B_r(0)$.*

Proof. Since J_1 is compact and S is continuous, $\overline{J_1(B_r)}$ is relatively compact and U is continuous. Since $\overline{J_1(B_r)}$ is compact and $\overline{J_1(B_r)} \subset \overline{D(A)}$, $S(\overline{J_1(B_r)})$ is also compact. By $S(J_1(B_r)) \subset S(\overline{J_1(B_r)})$, $SJ_1(B_r)$ and $U(B_r)$ are relatively compact. From these, U is a continuous mapping from E into E and $U(B_r)$ is relatively compact. Thus, by Theorem 3.1, we have the results. ☐

Theorem 4.2. *Let A be an m-accretive operator from $D(A) \subset E$ into 2^E. Let S be a continuous mapping from $\overline{D(A)}$ into E such that $J_1 = (I + A)^{-1}$ is compact. Let $p \in E$ and $r > 0$. Then, there exists $u \in B_r$ such that $cu = p + J_1 u - SJ_1 u$, where $c = \frac{\max\{r, \|p + J_1 u - SJ_1 u\|\}}{r} \in [1, \infty)$. Furthermore, the following hold:*

(1) *$c = 1$ and $u = p + J_1 u - SJ_1 u$ are equivalent.*
(2) *$c = 1$ is a sufficient condition for establishing $p \in R(A + S)$.*
(3) *Suppose $u \in \mathrm{Int}(B_r) = B_r(0)$. Then, $c = 1$.*

Set $v = J_1 u$. From traditional view points, the following hold:

(4) *Assume that $j \in E^*$ satisfies $\langle u, j \rangle \neq 0$. Then,*
(i) *$c = 1$ and* (ii) *$\langle A_1 u - p + Sv, j \rangle = 0$ are equivalent.*

Proof. Let U be as in Theorem 4.1, that is, $Uy = p + J_1y - SJ_1y$ for $y \in E$. Then, we already know that $p \in R(A + S)$ holds if $u \in F(U)$ exists.

Also, consider f_r and V_r such that $f_r(y) = \frac{r}{\max\{r, \|y\|\}}$ for $y \in E$, and

$$V_ry = f_r(Uy)Uy \in B_r \qquad \text{for} \quad y \in B_r.$$

By Theorem 4.1, there is $u \in B_r$ satisfying $u = V_ru = f_r(Uu)Uu$. We know $f_r(Uu) \in (0, 1]$. Then, we can easily see $c = \frac{1}{f_r(Uu)} \in [1, \infty)$, and

$$cu = Uu = p + J_1u - SJ_1u. \tag{4.1}$$

We show the latter half. By (4.1), $c = 1$ if and only if $u = p+J_1u-SJ_1u$. Then, (1) holds. Also, $c = 1$ if and only if $u \in F(U)$. Then, $c = 1$ implies $p \in R(A+S)$. So, (2) holds. Suppose $u \in \mathrm{Int}(B_r) = B_r(0)$. By Theorem 4.1 (2), we know $f_r(Uu) = 1$, that is, $c = 1$. Then, (3) holds.

We show (4). Assume that $j \in E^*$ satisfies $\langle u, j\rangle \neq 0$. Then, by $v = J_1u$ and (4.1), we know

$$A_1u = u - v, \quad cu = p + J_1u - SJ_1u = p + v - Sv. \tag{4.2}$$

We show (i)→(ii). Suppose $c = 1$. Then, by (4.2), we see

$$0 = u - v - p + Sv = A_1u - p + Sv, \quad \langle A_1u - p + Sv, j\rangle = 0.$$

We show (ii)→(i). Suppose $\langle u - v - p + Sv, j\rangle = 0$. Then, by $\langle u, j\rangle \neq 0$, we may show $(c - 1)\langle u, j\rangle = 0$. By (4.2), we see

$$0 = \langle cu - v - p + Sv, j\rangle = (c - 1)\langle u, j\rangle + 0 = (c - 1)\langle u, j\rangle.$$

We confirmed that (4) holds. □

Theorem 4.3. *Let A be an m-accretive operator from $D(A) \subset E$ into 2^E such that $J_1 = (I + A)^{-1}$ is compact. Let S be a continuous mapping from $\overline{D(A)}$ into E. Let $p \in E$ and $r > 0$. Set $v_y = J_1y$ for $y \in E$. Assume that, for each $y \in \partial B_r$, there is $j \in E^*$ satisfying*

$$\langle y, j\rangle \neq 0, \quad \langle y, j\rangle\langle A_1y - p + Sv_y, j\rangle \geq 0. \tag{$*$}$$

Then $p \in R(A + S)$.

Proof. Theorem 4.2 asserts that there is $u \in B_r$ satisfying $cu = p + J_1u - SJ_1u$ and $p \in R(A + S)$ holds if $c = 1$, where $c = \frac{\max\{r, \|p+J_1u-SJ_1u\|\}}{r} \in [1, \infty)$.

Suppose $u \in \partial B_r$. We show $c = 1$. Arguing by contradiction, assume $c > 1$. By $v_u = J_1u$ and $(*)$, there is $j \in E^*$ satisfying

$$\langle u, j\rangle \neq 0, \quad \langle u, j\rangle\langle u - v_u - p + Sv_u, j\rangle \geq 0. \tag{4.3}$$

In the case of $\langle u, j\rangle > 0$, by $cu = p + J_1u - SJ_1u$ and $(c - 1)\langle u, j\rangle > 0$, we see

$$\begin{aligned} 0 &= \langle cu - v_u - p + Sv_u, j\rangle \\ &= (c - 1)\langle u, j\rangle + \langle u - v_u - p + Sv_u, j\rangle > \langle u - v_u - p + Sv_u, j\rangle. \end{aligned}$$

By $\langle u, j\rangle > 0$, this contradicts to (4.3). In the case of $\langle u, j\rangle < 0$, we see

$$0 = \langle cu - v_u - p + Sv_u, j\rangle$$

$$= (c-1)\langle u, j\rangle + \langle u - v_u - p + Sv_u, j\rangle < \langle u - v_u - p + Sv_u, j\rangle.$$

So, in both cases, we have a contradiction. Thus, $c = 1$.

There remains the case of $u \notin \partial B_r$. In this case, $u \in \mathrm{Int}(B_r) = B_r(0)$. Then, by Theorem 4.2 (3), we know $c = 1$. □

Finally, for reference, we present Theorem 5 in Kartsatos [4].

Theorem 4.4. *Let A be an m-accretive operator from $D(A) \subset E$ into 2^E such that $J_1 = (I + A)^{-1}$ is compact. Let S be a continuous and bounded mapping from $\overline{D(A)}$ into E. Let $p \in E$. Assume that $b > 0$ satisfies the following: For every $x \in D(A)$ with $\|x\| \geq b$, there is $j \in Jx$ satisfying*

$$(**) \qquad \langle u - p + Sx, j\rangle \geq 0 \qquad \text{for} \quad u \in Ax.$$

Then $p \in R(A + S)$.

5. Appendix

X denotes a real locally convex linear Hausdorff topological space; in short, locally convex space. For a subset C of X, $\overline{C}$, ∂C, and $\mathrm{Int}(C)$ denote respectively the closure of C, the boundary of C, and the interior of C.

In the locally convex space setting, we show an extension of Theorem 3.1. Let K be a closed convex subset of a locally convex space X with $0 \in \mathrm{Int}(K)$. Define mappings g from X into $[0, \infty)$ and f from X into $(0, 1]$ by

$$g(x) = \inf\{r > 0 : x \in rK\}, \quad f(x) = \frac{1}{\max\{1, g(x)\}} \qquad \text{for} \quad x \in X.$$

g is called the Mankowski functional associated to K. Then, the following hold:

(m_1) $g(x) \in [0, 1)$ if and only if $x \in \mathrm{Int}(K)$.
(m_2) $g(x) = 1$ if and only if $x \in \partial K$.
(m_3) $g(x) \in (1, \infty)$ if and only if $x \notin K$.
(m_4) $g(ax) = ag(x)$ for $x \in X$ and $a \in [0, \infty)$.
(m_5) $f(x) = 1$ for $x \in K$, and $f(x) = \frac{1}{g(x)} \in (0, 1)$ for $x \notin K$.
(m_6) g and f are continuous.

We already know that Singbal gave a proof of the following theorem.

Theorem 5.1. *Let C be a closed convex subset of X. Let S be a continuous self–mapping on C such that $S(C)$ is relatively compact. Then, there exists $v \in C$ satisfying $Sv = v$.*

In Theorem 3.1, $\frac{1}{r}\|\cdot\|$ is the Mankowski functional associated to B_r. Furthermore, in the proof, we only used properties (m_1)–(m_6) of $\frac{1}{r}\|\cdot\|$. Then, in a similar way, we can have the following extension of Theorem 3.1.

Theorem 5.2. *Let K and C be a subset of X such that K is closed and convex, and $0 \in \mathrm{Int}(K) \subset K \subset C$. Let S be a continuous mapping from C into X such that $S(K)$ is relatively compact. Define mappings f from X into $(0, 1]$ and M from X*

into K *by* $f(y) = \frac{1}{\max\{1, g(y)\}}$ *and* $Mx = f(y)y$ *for* $y \in X$*, where* g *is the Mankowski functional associated to* K*. Define a self–mapping* V *on* K *by*

$$Vy = MSy = f(Sy)Sy \in K \qquad \text{for} \quad y \in K.$$

Then, there is $v \in K \subset C$ *satisfying* $Vv = v$*. Also, the following hold:*

(1) $Sv = v$ *if* $Sv \in K$ ($Sv = v$ *if* $v \in \partial K$ *and* $Sv \in K$).
(2) $Sv = v$ ($f_r(Sv) = 1$) *if* $v \in \mathrm{Int}(K)$.

Proof. By (m_1)–(m_6) and the definition of M, we easily see the following:

(i) $My = y \in K$ for $y \in K$, (ii) $My = \frac{1}{g(y)}y \in K$ for $y \notin K$,
(iii) M is continuous.

For $y \notin K$, $My = f(y)y = \frac{1}{g(y)}y$ and $g(\frac{1}{g(y)}y) = \frac{1}{g(y)}g(y) = 1$. So, $My \in K$. We confirmed only (ii). By (i) and (ii), we see

$$M(X) \subset K, \quad V(K) = MS(K) \subset M(\overline{S(K)}) \subset K.$$

By assumptions, $\overline{S(K)}$ is compact. Then, by (iii), $V(K) \subset K$, V is continuous, and $V(K)$ is relatively compact. Thus, by Theorem 5.1, there is $v \in K \subset C$ satisfying $Vv = MSv = v$. We show (1). Suppose $Sv \in K$. Then, by (i), we see $Sv = MSv = v$. To prove (2), by (1), we may show $Sv \in K$.

Arguing by contradiction, assume $Sv \notin K$. By $v \in \mathrm{Int}(K)$, $Sv \notin K$, and (ii), we immediately have a contradiction:

$$1 > g(v) = g(MSv) = g\left(\frac{1}{g(Sv)}Sv\right) = \frac{1}{g(Sv)}g(Sv) = 1.$$

Of course, $Sv \in K$ implies $f(Sv) = 1$. □

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