Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167

Volume 5, Number 1, 2019, 13–31

ON YOUNG INEQUALITY UNDER EUCLIDEAN JORDAN ALGEBRA

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Abstract. Recently, some inequalities and trace inequalities associated with second-order cone are established. Most of them are very useful in optimization. In particular, in our recent work [14], we build up some trace versions of Young inequality in the SOC setting and indicate that the Young inequality does not hold in general. In this paper, we pay attention to Young inequality under Euclidean Jordan algebra. By using spectral decomposition, we extend one trace version of Young inequality to the general setting of symmetric cone. In addition, we provide conditions under which the Young inequality holds in the SOC setting. Accordingly, one can construct counterexamples in general case.

1. Introduction

The Young inequality states that for any $a, b \geq 0$, and p, q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, there holds

$$\frac{a^p}{p} + \frac{b^q}{q} \;\geq\; ab,$$

where the equality holds if and only if $a^p = b^q$. The Arithmetic-Geometric-Mean inequality is a special case of Young inequality:

$$\frac{a^2 + b^2}{2} \;\geq\; ab.$$

It is very useful in real analysis, as a tool to prove the Hölder's inequality. In addition, it can be used to estimate the norm of nonlinear terms in PDE theory.

As indicated in [7, 10], the second-order cone (SOC) is often involved in many optimization problems, particularly in the context of applications and solutions methods for the second-order-cone program (SOCP) and second-order-cone complementarity problem (SOCCP) [5, 6, 8, 9, 10]. For designing those solutions methods, spectral decomposition associated with SOC is required. Recently, Chang et al.[4] defined various means associated with Lorentz cones (also known as second-order cones), which are new concepts and natural extensions of traditional arithmetic mean, harmonic mean, and geometric mean, logarithmic mean. Based on these

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2010 *Mathematics Subject Classification.*

Key words and phrases. Euclidean Jordan algebra, Second-order Cone, Lorentz Cone, Young Inequality.

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means defined on Lorentz cone, some inequalities and trace inequalities are established, see [10, Chpater 4].

In our recent work [14], we build up some trace versions of Young inequality in the SOC setting and indicate that the Young inequality does not hold in general. In this paper, we pay attention to Young inequality under Euclidean Jordan algebra. By using spectral decomposition, we extend one trace version of Young inequality to the general setting of symmetric cone. In addition, we provide conditions under which the Young inequality holds in the SOC setting. More specifically, we conclude that the Young inequality associated with SOC holds under one of the following two conditions holds: (i) $p = q = 2$ or (ii) any two vectors share the same Jordan frame. Accordingly, one can construct counterexamples in general case.

2. Preliminaries

A *Euclidean Jordan algebra* [11] is a finite dimensional inner product space $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ ($\mathbb{V}$ for short) over the field of real numbers $\mathbb{R}$ equipped with a bilinear map $(x, y) \mapsto x \circ y : \mathbb{V} \times \mathbb{V} \longrightarrow \mathbb{V}$, which satisfies the following conditions:

(i) $x \circ y = y \circ x$ for all $x, y \in \mathbb{V}$;
(ii) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$ for all $x, y \in \mathbb{V}$;
(iii) $\langle x \circ y, z \rangle = \langle x, y \circ z \rangle$ for all $x, y, z \in \mathbb{V}$,

where $x^2 := x \circ x$, and $x \circ y$ is called the *Jordan product* of x and y. If a Jordan product only satisfies the conditions (i) and (ii) in the above definition, the algebra $\mathbb{V}$ is said to be a *Jordan algebra*. Moreover, if there is an (unique) element $e \in \mathbb{V}$ such that $x \circ e = x$ for all $x \in \mathbb{V}$, the element e is called the *identity element* in $\mathbb{V}$. Note that a Jordan algebra does not necessarily have an identity element. Throughout this paper, we assume that $\mathbb{V}$ is a Euclidean Jordan algebra with an identity element e.

In a given Euclidean Jordan algebra $\mathbb{V}$, the set of squares $\mathcal{K} := \{x^2 : x \in \mathbb{V}\}$ is a *symmetric cone* [11, Theorem III.2.1]. This means that $\mathcal{K}$ is a self-dual closed convex cone and, for any two elements $x, y \in \text{int}(\mathcal{K})$, there exists an invertible linear transformation $\Gamma : \mathbb{V} \longrightarrow \mathbb{V}$ such that $\Gamma(x) = y$ and $\Gamma(\mathcal{K}) = \mathcal{K}$. Accordingly, there is a natural partial order in $\mathbb{V}$. We write $x \succeq_{\mathcal{K}} y$ if $x - y \in \mathcal{K}$, and $x \succ_{\mathcal{K}} y$ if $x - y \in \text{int}\mathcal{K}$.

For any given $x \in \mathbb{V}$, we denote $m(x)$ the *degree* of the minimal polynomial of x, that is,

$$m(x) := \left\{ k > 0 \,|\, \{e, x, \dots, x^k\} \text{ is linearly dependent} \right\}.$$

Since $m(x) \leq \dim(\mathbb{V})$ where $\dim(\mathbb{V})$ is the dimension of $\mathbb{V}$, the *rank* of $\mathbb{V}$ is well-defined by $r := \max\{m(x) \,|\, x \in \mathbb{V}\}$. In Euclidean Jordan algebra $\mathbb{V}$, an element $e^{(i)} \in \mathbb{V}$ is an *idempotent* if $(e^{(i)})^2 = e^{(i)}$, and it is a *primitive idempotent* if it is nonzero and cannot be written as a sum of two nonzero idempotents. The idempotents $e^{(i)}$ and $e^{(j)}$ are said to be *orthogonal* if $e^{(i)} \circ e^{(j)} = 0$. In addition, we say that a finite set $\{e^{(1)}, e^{(2)}, \ldots, e^{(r)}\}$ of primitive idempotents in $\mathbb{V}$ is a *Jordan frame* if

$$e^{(i)} \circ e^{(j)} = 0 \quad \text{for} \quad i \neq j, \quad \text{and} \quad \sum_{i=1}^{r} e^{(i)} = e.$$

Note that $\langle e^{(i)}, e^{(j)} \rangle = \langle e^{(i)} \circ e^{(j)}, e \rangle$ whenever $i \neq j$. There also exist the so-called spectral decomposition for any element x in $\mathbb{V}$, see below theorem.

Theorem 2.1. [11, Theorem III.1.2] *Let $\mathbb{V}$ be a Euclidean Jordan algebra. Then there is a number r such that, for every $x \in \mathbb{V}$, there exists a Jordan frame $\{e^{(1)}, \ldots, e^{(r)}\}$ and real numbers $\lambda_1(x), \ldots, \lambda_r(x)$ with*

$$x = \lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}.$$

Here, the numbers $\lambda_i(x)$ $(i = 1, \ldots, r)$ are the spectral values of x, the expression $\lambda_1(x)e^{(1)} + \cdots + \lambda_r(x)e^{(r)}$ is the spectral decomposition of x. Moreover, $\mathrm{tr}\ x := \sum_{i=1}^{r} \lambda_i(x)$ *is called the trace of x, and* $\det(x) = \prod_{i=1}^{r} \lambda_i(x)$ *is call the determinant of x.*

Given a Euclidean Jordan algebra $\mathbb{V}$ with $\dim(\mathbb{V}) = n > 1$, from Proposition III 4.4-4.5 and Theorem V.3.7 in [11], we know that any Euclidean Jordan algebra $\mathbb{V}$ and its corresponding symmetric cone $\mathcal{K}$ are, in a unique way, a direct sum of simple Euclidean Jordan algebras and the constituent symmetric cones therein, respectively, i.e.,

$$\mathbb{V} = \mathbb{V}_1 \times \cdots \times \mathbb{V}_m \quad \text{and} \quad \mathcal{K} = \mathcal{K}^1 \times \cdots \times \mathcal{K}^m,$$

where every $\mathbb{V}_i$ is a simple Euclidean Jordan algebra (that cannot be a direct sum of two Euclidean Jordan algebras) with the corresponding symmetric cone $\mathcal{K}^i$ for $i = 1, \ldots, m$, and $n = \sum_{i=1}^{m} n_i$ (n_i is the dimension of $\mathbb{V}_i$). Therefore, for any $x = (x_1, \ldots, x_m)^T$ and $y = (y_1, \ldots, y_m)^T \in \mathbb{V}$ with $x_i, y_i \in \mathbb{V}_i$, we have

$$x \circ y = (x_1 \circ y_1, \ldots, x_m \circ y_m)^T \in \mathbb{V} \quad \text{and} \quad \langle x, y \rangle = \langle x_1, y_1 \rangle + \cdots + \langle x_m, y_m \rangle.$$

For simplicity, we focus on the single symmetric cone $\mathcal{K}$ because all the analysis can be carried over to the setting of Cartesian product.

3. Trace version of Young inequality under Euclidean Jordan algebra

In a recent work [14], we established three trace versions of Young inequality in the setting of second-order cone; and also made a conjecture that the eigenvalue version of Young inequality in the SOC setting holds. However, only two trace versions of Young inequality were extended to the setting of symmetric cone (under Euclidean Jordan algebra). In this section, we build up the third trace version of Young inequality based on Gowda's proof in [13]. To proceed, we first recall the below crucial inequality which was achieved in [1, Theorem 23].

Theorem 3.1. [1, Theorem 23] *Let $\mathbb{V}$ be a simple Euclidean Jordan algebra with rank r. For any $x, y \in \mathcal{K}$, there holds*

$$\mathrm{tr}(x \circ y) \leq \sum_{i=1}^{r} \lambda_i(x)\lambda_i(y),$$

where $\lambda_i(x)$ and $\lambda_i(y)$ are the spectral values of x and y with decreasing order, respectively.

Theorem 3.2. (EJA Young inequality-Type III) *Let $\mathbb{V}$ be a simple Euclidean Jordan algebra with rank r. For any $x, y \in \mathbb{V}$, there holds*

$$\mathrm{tr}(|x \circ y|) \leq \mathrm{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q}\right)$$

where $1 < p, q < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $|x| = |\lambda_1(x)|e^{(1)} + \cdots + |\lambda_r(x)|e^{(r)}$.

Proof. For $x \circ y \in \mathcal{K} \cup (-\mathcal{K})$, the desired result follows from [14, Theorem 3.11]. For the remainder case, we suppose $x \circ y$ is decomposed as

$$x \circ y = \left(\lambda_1 e^{(1)} + \cdots + \lambda_k e^{(k)}\right) - \left(\lambda_{k+1} e^{(k+1)} + \cdots + \lambda_r e^{(r)}\right),$$

where $\lambda_i \geq 0$ and k is some positive integer with $1 \leq k \leq r$. For convenience, we denote

$$\begin{aligned} c &:= e^{(1)} + \cdots + e^{(k)} \\ d &:= e^{(1)} + \cdots + e^{(k)} - \left(e^{(k+1)} + \cdots + e^{(r)}\right) = 2c - e. \end{aligned}$$

Applying the Peirce decomposition [11], we know that

$$\mathbb{V} = \mathbb{V}(c,1) \oplus \mathbb{V}(c,1/2) \oplus \mathbb{V}(c,0),$$

where $\mathbb{V}(c,1)$ is a Euclidean Jordan algebra of rank k containing the subspace spanned by $\{e^{(1)}, \ldots, e^{(k)}\}$ and $\mathbb{V}(c,0)$ is a Euclidean Jordan algebra of rank $r-k$ containing the subspace spanned by $\{e^{(k+1)}, \ldots, e^{(r)}\}$. Moreover, we write $y = u + v + w$, where $u \in \mathbb{V}(c,1)$, $v \in \mathbb{V}(c,1/2)$, and $w \in \mathbb{V}(c,0)$. We notice that $\mathbb{V}(c,1) \cap \mathbb{V}(c,0) = \{0\}$, $|x \circ y| = (x \circ y) \circ d$, and $y \circ d = u - w$. On the other hand, suppose that the spectral decomposition of u, w are in the forms of

$$\begin{aligned} u &:= \lambda_1(u)\tilde{e}^{(1)} + \cdots + \lambda_1(u)\tilde{e}^{(k)} \\ w &:= \lambda_1(w)\tilde{e}^{(k+1)} + \cdots + \lambda_{r-k}(w)\tilde{e}^{(r)}. \end{aligned}$$

Then, we observe that

$$y \circ d = u - w = \lambda_1(u)\tilde{e}^{(1)} + \cdots + \lambda_1(u)\tilde{e}^{(k)} - \lambda_1(w)\tilde{e}^{(k+1)} - \cdots - \lambda_{r-k}(w)\tilde{e}^{(r)}.$$

It follow from the proof of Theorem 1.1 in [13] that $\mathrm{tr}(|y \circ d|^q) \leq \mathrm{tr}(|y|^q)$. Therefore, we obtain

$$\mathrm{tr}(|x \circ y|) = \langle |x \circ y|, e \rangle = \langle (x \circ y) \circ d, e \rangle = \langle x, y \circ d \rangle.$$

In light of Theorem 3.1, we further have

$$\begin{aligned}
\mathrm{tr}(|x \circ y|) &\leq \sum_{i=1}^{r} \lambda_i(x)\lambda_i(y \circ d) \\
&\leq \sum_{i=1}^{r} \left(\frac{|\lambda_i(x)|^p}{p} + \frac{|\lambda_i(y \circ d)|^q}{q} \right) \\
&= \frac{\mathrm{tr}(|x|^p)}{p} + \frac{\mathrm{tr}(|y \circ d|^q)}{q} \\
&\leq \frac{\mathrm{tr}(|x|^p)}{p} + \frac{\mathrm{tr}(|y|^q)}{q} \\
&= \mathrm{tr}\left(\frac{|x|^p}{p} + \frac{|y|^q}{q} \right).
\end{aligned}$$

Hence, we conclude the desired inequality. □

Remark 3.3. In the setting of second-order cone, Huang et al. [14] obtained the desired conclusion via establishing the following inequality

$$\mathrm{tr}(|x \circ y|) \leq |\lambda_1(x)\lambda_1(y)| + |\lambda_2(x)\lambda_2(y)|.$$

However, we do not know yet whether the similar inequality in Euclidean Jordan algebra hold or not. In other words, it is a future direction to prove or disprove

$$\mathrm{tr}(|x \circ y|) \leq \sum_{i=1}^{r} |\lambda_i(x)\lambda_i(y)|,$$

for any $x, y \in \mathbb{V}$, where $\lambda_i(x)$ and $\lambda_i(y)$ are the spectral values of x and y with decreasing order, respectively.

4. Counterexample of Young inequality

In this section, we show that the general Young inequality does not hold under Euclidean Jordan algebra. We will show how to construct counterexamples in the SOC setting. According, they serve as counterexamples in the symmetric cone setting under Euclidean Jordan algebra. From the construction procedure, we also conclude under what conditions the Young inequality will hold in the SOC setting.

To proceed, we recall some materials regarding the SOC in $\mathbb{R}^n$, an important example of symmetric cones. Officially, the SOC is defined as follows:

$$\mathcal{K}^n := \left\{ x = (x_0, \bar{x}) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_0 \geq \|\bar{x}\| \right\},$$

and the corresponding Jordan product of x and y in $\mathbb{R}^n$ with $x = (x_0, \bar{x}), y = (y_0, \bar{y}) \in \mathbb{R} \times \mathbb{R}^{n-1}$ is given by

$$x \circ y := \begin{bmatrix} x^T y \\ x_0 \bar{y} + y_0 \bar{x} \end{bmatrix}.$$

We note that $e = (1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1}$ acts as the Jordan identity. Later, we need to verify Young inequality in the SOC setting as two vectors belong to $\mathcal{K}^n$ with different Jordan frame. To do this, we use vector decomposition and then compare it to the condition with same Jordan frame.

For each $x \in \mathbb{R}^n$, it follows from [10, 11, 12] that the spectral decomposition associated with $\mathcal{K}^n$ is of the form

$$x = \lambda_1(x)u_x^{(1)} + \lambda_2(x)u_x^{(2)}. \tag{4.1}$$

where $\lambda_i(x)$ and $u_i(x)$ are called the spectral values and the spectral vectors of x, respectively, which defined by

$$\lambda_i(x) = x_1 + (-1)^{i-1}\|x_2\|, \tag{4.2}$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}(1, (-1)^{i-1}\frac{x_2}{\|x_2\|}) & \text{if} \quad x_2 \neq 0, \\ \frac{1}{2}(1, (-1)^{i-1}w) & \text{if} \quad x_2 = 0, \end{cases} \tag{4.3}$$

for $i = 1, 2$ with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. When $x_2 \neq 0$, the spectral factorization is unique.

Let m be any real number and $x \in \mathcal{K}^n$. Then the m^{th} power of x is defined by

$$x^m = \big(\lambda_1(x)\big)^m u_x^{(1)} + \big(\lambda_2(x)\big)^m u_x^{(2)}. \tag{4.4}$$

With this definition, we are interested in the properties of Young inequality in the SOC setting. For $x \succeq_{\mathcal{K}^n} 0, y \succeq_{\mathcal{K}^n} 0$, and p, q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, it is clear that the Young inequality holds for $p = q = 2$ since

$$\begin{aligned} \frac{x^2 + y^2}{2} \succeq_{\mathcal{K}^n} x \circ y \quad &\Longleftrightarrow \quad x^2 + y^2 \succeq_{\mathcal{K}^n} 2x \circ y \\ &\Longleftrightarrow \quad x^2 - 2x \circ y + y^2 \succeq_{\mathcal{K}^n} 0 \\ &\Longleftrightarrow \quad (x - y)^2 \succeq_{\mathcal{K}^n} 0. \end{aligned}$$

Now, we first consider $x, y \in \mathcal{K}^n$ such that x and y shart the same Jordan frame.

Theorem 4.1. *Suppose $x \succeq_{\mathcal{K}^n} 0, y \succeq_{\mathcal{K}^n} 0$, and p, q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Let x, y share the same Jordan frame, that is, x, y have spectral decomposition*

$$x = \lambda_1 u_1 + \lambda_2 u_2, \;\; y = \lambda_3 u_3 + \lambda_4 u_4$$

with $u_1 = u_3, u_2 = u_4$ or $u_1 = u_4, \;\; u_2 = u_3$. Then, we have $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^n} x \circ y$. Moreover, the equality holds if and only if $x^p = y^q$.

Proof. Without loss of generality, we assume that $u_1 = u_3, u_2 = u_4$, and the proof can be carried over to the similar case when $u_1 = u_4, \;\; u_2 = u_3$. First, we observe that

$$\frac{x^p}{p} + \frac{y^q}{q} \quad = \quad \frac{1}{p}(\lambda_1^p u_1 + \lambda_2^p u_2) + \frac{1}{q}(\lambda_3^q u_1 + \lambda_4^q u_2)$$

$$= \left(\frac{\lambda_1^p}{p} + \frac{\lambda_3^q}{q}\right) u_1 + \left(\frac{\lambda_2^p}{p} + \frac{\lambda_4^q}{q}\right) u_2, \tag{4.5}$$

and

$$\begin{aligned} x \circ y &= (\lambda_1 u_1 + \lambda_2 u_2) \circ (\lambda_3 u_1 + \lambda_4 u_2) \\ &= \lambda_1\lambda_3 u_1^2 + \lambda_1\lambda_4 u_1 \circ u_2 + \lambda_2\lambda_3 u_2 \circ u_1 + \lambda_2\lambda_4 u_2^2 \\ &= \lambda_1\lambda_3 u_1 + \lambda_2\lambda_4 u_2. \end{aligned} \tag{4.6}$$

Subtracting (4.5) from (4.6), we see that

$$\begin{aligned} \frac{x^p}{p} + \frac{y^q}{q} - x \circ y &= \left[\left(\frac{\lambda_1^p}{p} + \frac{\lambda_3^q}{q}\right) u_1 + \left(\frac{\lambda_2^p}{p} + \frac{\lambda_4^q}{q}\right) u_2\right] - (\lambda_1\lambda_3 u_1 + \lambda_2\lambda_4 u_2) \\ &= \left(\frac{\lambda_1^p}{p} + \frac{\lambda_3^q}{q} - \lambda_1\lambda_3\right) u_1 + \left(\frac{\lambda_2^p}{p} + \frac{\lambda_4^q}{q} - \lambda_2\lambda_4\right) u_2. \end{aligned} \tag{4.7}$$

Since $x \succeq_{\mathcal{K}^n} 0, y \succeq_{\mathcal{K}^n} 0$, we know that $x_1 \geq \|x_2\|$, $y_1 \geq \|y_2\|$, and hence $\lambda_i \geq 0$ for all $i = 1, 2, 3, 4$. Thus, by the traditional Young inequality for numbers, we have

$$\frac{\lambda_1^p}{p} + \frac{\lambda_3^q}{q} - \lambda_1\lambda_3 \geq 0 \quad \text{and} \quad \frac{\lambda_2^p}{p} + \frac{\lambda_4^q}{q} - \lambda_2\lambda_4 \geq 0.$$

This means $\frac{x^p}{p} + \frac{y^q}{q} - x \circ y \in \mathcal{K}^n$, i.e., $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^n} x \circ y$. Moreover, since u_1, u_2 are linearly independent. From (4.7), we have

$$\begin{aligned} \frac{x^p}{p} + \frac{y^q}{q} = x \circ y &\iff \frac{x^p}{p} + \frac{y^q}{q} - x \circ y = 0 \\ &\iff \left(\frac{\lambda_1^p}{p} + \frac{\lambda_3^q}{q} - \lambda_1\lambda_3\right) u_1 + \left(\frac{\lambda_2^p}{p} + \frac{\lambda_4^q}{q} - \lambda_2\lambda_4\right) u_2 = 0 \\ &\iff \frac{\lambda_1^p}{p} + \frac{\lambda_3^q}{q} - \lambda_1\lambda_3 = \frac{\lambda_2^p}{p} + \frac{\lambda_4^q}{q} - \lambda_2\lambda_4 = 0 \\ &\iff \frac{\lambda_1^p}{p} + \frac{\lambda_3^q}{q} = \lambda_1\lambda_3 \text{ and } \frac{\lambda_2^p}{p} + \frac{\lambda_4^q}{q} = \lambda_2\lambda_4 \\ &\iff \lambda_1^p = \lambda_3^q \text{ and } \lambda_2^p = \lambda_4^q \\ &\iff x^p = \lambda_1^p u_1 + \lambda_2^p u_2 = \lambda_3^q u_3 + \lambda_4^q u_4 = y^q, \end{aligned}$$

where the fifth equivalence holds by Young inequality for real number. In other words, the equality holds if and only if $x^p = y^q$. □

Up to now, we show that the Young inequality holds in the SOC setting under the condition that x, y share the same Jordan frame. However, it is much harder to verify two vectors belong to $\mathcal{K}^n$ with different Jordan frame. For notations simplicity, we denote

$$\frac{x^p}{p} + \frac{y^q}{q} - x \circ y := (\alpha, \beta) \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

Then, it is clear that $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}} x \circ y$ holds if and only if $\alpha^2 - \|\beta\|^2 \geq 0$. In addition, the equality holds if and only if $\alpha^2 - \|\beta\|^2 = 0$. In order to develop some tools for the case when two vectors belong to $\mathcal{K}^n$ with different Jordan frame. As below, we start with calculating $\alpha^2 - \|\beta\|^2$.

Theorem 4.2. *Suppose $x \succeq_{\mathcal{K}^n} 0, y \succeq_{\mathcal{K}^n} 0$, and p, q are positive real numbers such that $\frac{1}{p}+\frac{1}{q}=1$. Suppose that $x=(x_1,x_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$, $y=(y_1,y_2)\in\mathbb{R}\times\mathbb{R}^{n-1}$, and*

$$\theta = \begin{cases} \arccos\left(\frac{x_2^T y_2}{\|x_2\|\|y_2\|}\right) & \text{if} \|x_2\|\|y_2\| \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

that is, θ is the angle between two vectors x_2, y_2, the spectral decompositions of x, y associated with $\mathcal{K}^n$ are

$$\begin{aligned} x &= \lambda_1 u_1 + \lambda_2 u_2, \\ y &= \lambda_3 u_3 + \lambda_4 u_4. \end{aligned}$$

Then, there holds

$$\begin{aligned} \alpha^2 - \|\beta\|^2 &= f_1 + (1-\cos\theta)\big[\gamma - (1+\cos\theta)\delta\big] \\ &= f_2 - (1+\cos\theta)\big[\gamma + (1-\cos\theta)\delta\big], \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} f_1 &= \frac{\lambda_1^p\lambda_2^p}{p^2} + \frac{\lambda_3^q\lambda_4^q}{q^2} + \lambda_1\lambda_2\lambda_3\lambda_4 + \frac{1}{pq}(\lambda_1^p\lambda_4^q + \lambda_2^p\lambda_3^q) \\ &\quad -\big[(\frac{\lambda_1^p}{p} + \frac{\lambda_3^q}{q})\lambda_2\lambda_4 + (\frac{\lambda_2^p}{p} + \frac{\lambda_4^q}{q})\lambda_1\lambda_3\big], \\ f_2 &= \frac{\lambda_1^p\lambda_2^p}{p^2} + \frac{\lambda_3^q\lambda_4^q}{q^2} + \lambda_1\lambda_2\lambda_3\lambda_4 + \frac{1}{pq}(\lambda_1^p\lambda_3^q + \lambda_2^p\lambda_4^q) \\ &\quad -\big[(\frac{\lambda_1^p}{p} + \frac{\lambda_4^q}{q})\lambda_2\lambda_3 + (\frac{\lambda_2^p}{p} + \frac{\lambda_3^q}{q})\lambda_1\lambda_4\big], \\ \gamma &= \frac{1}{2pq}(\lambda_1^p - \lambda_2^p)(\lambda_3^q - \lambda_4^q) - \frac{\lambda_1\lambda_2}{2p}(\lambda_1^{p-1} - \lambda_2^{p-1})(\lambda_3 - \lambda_4) \\ &\quad - \frac{\lambda_3\lambda_4}{2q}(\lambda_3^{q-1} - \lambda_4^{q-1})(\lambda_1 - \lambda_2), \\ \delta &= \frac{(\lambda_1-\lambda_2)^2(\lambda_3-\lambda_4)^2}{16}. \end{aligned}$$

Proof. According to the spectral decomposition associated with $\mathcal{K}^n$, we have

$$x = \lambda_1 u_1 + \lambda_2 u_2, \quad \text{and} \quad y = \lambda_3 u_3 + \lambda_4 u_4,$$

where

$$\begin{aligned} &\lambda_1 = x_1 + \|x_2\|, \lambda_2 = x_1 - \|x_2\|,\ u_1 = \left(\frac{1}{2}, \frac{x_2}{2\|x_2\|}\right),\ u_2 = \left(\frac{1}{2}, \frac{-x_2}{2\|x_2\|}\right), \\ &\lambda_3 = y_1 + \|y_2\|, \lambda_4 = y_1 - \|y_2\|,\ u_3 = \left(\frac{1}{2}, \frac{y_2}{2\|y_2\|}\right),\ u_4 = \left(\frac{1}{2}, \frac{-y_2}{2\|y_2\|}\right). \end{aligned}$$

Then, we write out

$$\begin{aligned}
\frac{x^p}{p} &= \frac{\lambda_1^p}{p}u_1 + \frac{\lambda_2^p}{p}u_2 = \left(\frac{1}{2p}(\lambda_1^p + \lambda_2^p), \frac{1}{2p}(\lambda_1^p - \lambda_2^p)\frac{x_2}{\|x_2\|}\right), \\
\frac{y^q}{q} &= \frac{\lambda_3^q}{q}u_3 + \frac{\lambda_4^q}{q}u_4 = \left(\frac{1}{2q}(\lambda_3^q + \lambda_4^q), \frac{1}{2q}(\lambda_3^q - \lambda_4^q)\frac{y_2}{\|y_2\|}\right), \\
x \circ y &= \big(x_1y_1 + x_2^Ty_2, x_1y_2 + y_1x_2\big).
\end{aligned}$$

Since we have denoted $\frac{x^p}{p} + \frac{y^q}{q} - x \circ y$ by (α, β), we have

$$\begin{aligned}
\alpha &= \frac{1}{2p}(\lambda_1^p + \lambda_2^p) + \frac{1}{2q}(\lambda_3^q + \lambda_4^q) - (x_1y_1 + x_2^Ty_2), \\
\beta &= \frac{1}{2p}(\lambda_1^p - \lambda_2^p)\frac{x_2}{\|x_2\|} + \frac{1}{2q}(\lambda_3^q - \lambda_4^q)\frac{y_2}{\|y_2\|} - (x_1y_2 + y_1x_2).
\end{aligned}$$

Thus, we see that

$$\begin{aligned}
\alpha^2 &= \frac{1}{4p^2}(\lambda_1^{2p} + \lambda_2^{2p} + 2\lambda_1^p\lambda_2^p) + \frac{1}{4q^2}(\lambda_3^{2q} + \lambda_4^{2q} + 2\lambda_3^q\lambda_4^q) \\
&\quad + (x_1^2y_1^2 + (x_2^Ty_2)^2 + 2x_1y_1x_2^Ty_2) + \frac{1}{2pq}(\lambda_1^p\lambda_3^q + \lambda_1^p\lambda_4^q + \lambda_2^p\lambda_3^q + \lambda_2^p\lambda_4^q) \\
&\quad - \frac{1}{p}(\lambda_1^px_1y_1 + \lambda_1^px_2^Ty_2 + \lambda_2^px_1y_1 + \lambda_2^px_2^Ty_2) \\
&\quad - \frac{1}{q}(\lambda_3^qx_1y_1 + \lambda_3^qx_2^Ty_2 + \lambda_4^qx_1y_1 + \lambda_4^qx_2^Ty_2), \qquad (4.9)
\end{aligned}$$

and

$$\begin{aligned}
\|\beta\|^2 &= \frac{1}{4p^2}(\lambda_1^{2p} + \lambda_2^{2p} - 2\lambda_1^p\lambda_2^p) + \frac{1}{4q^2}(\lambda_3^{2q} + \lambda_4^{2q} - 2\lambda_3^q\lambda_4^q) \\
&\quad + (x_1^2\|y_2\|^2 + y_1^2\|x_2\|^2 + 2x_1y_1x_2^Ty_2) + \frac{1}{2pq}(\lambda_1^p\lambda_3^q - \lambda_1^p\lambda_4^q - \lambda_2^p\lambda_3^q + \lambda_2^p\lambda_4^q) \\
&\quad - \frac{1}{p}(\lambda_1^px_1\frac{x_2^Ty_2}{\|x_2\|} + \lambda_1^py_1\|x_2\| - \lambda_2^px_1\frac{x_2^Ty_2}{\|x_2\|} - \lambda_2^py_1\|x_2\|) \\
&\quad - \frac{1}{q}(\lambda_3^qx_1\|y_2\| + \lambda_3^qy_1\frac{x_2^Ty_2}{\|y_2\|} - \lambda_4^qx_1\|y_2\| + \lambda_4^qy_1\frac{x_2^Ty_2}{\|y_2\|}). \qquad (4.10)
\end{aligned}$$

Subtracting (4.10) from (4.9) yields

$$\begin{aligned}
&\alpha^2 - \|\beta\|^2 \\
= \; & \frac{\lambda_1^p\lambda_2^p}{p^2} + \frac{\lambda_3^q\lambda_4^q}{q^2} + \left[x_1^2y_1^2 + (x_2^Ty_2)^2 - x_1^2\|y_2\|^2 - y_1^2\|x_2\|^2\right] \\
& + \frac{1}{2pq}[(\lambda_1^p\lambda_3^q + \lambda_2^p\lambda_4^q)(1 - \frac{x_2^Ty_2}{\|x_2\|\|y_2\|}) + (\lambda_1^p\lambda_4^q + \lambda_2^p\lambda_3^q)(1 + \frac{x_2^Ty_2}{\|x_2\|\|y_2\|})] \\
& + \frac{1}{p}[\lambda_1^px_1(\frac{x_2^Ty_2}{\|x_2\|} - y_1) + \lambda_1^p\|x_2\|(y_1 - \frac{x_2^Ty_2}{\|x_2\|})] \\
& - \frac{1}{p}[\lambda_2^px_1(\frac{x_2^Ty_2}{\|x_2\|} + y_1) + \lambda_2^p\|x_2\|(y_1 + \frac{x_2^Ty_2}{\|x_2\|})]
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{q}[\lambda_3^q x_1(\|y_2\| - y_1) + \lambda_3^q x_2^T y_2(\frac{y_1}{\|y_2\|} - 1)] \\
& -\frac{1}{q}[\lambda_4^q x_1(\|y_2\| + y_1) + \lambda_4^q x_2^T y_2(\frac{y_1}{\|y_2\|} + 1)] \\
= \; & \frac{\lambda_1^p\lambda_2^p}{p^2} + \frac{\lambda_3^q\lambda_4^q}{q^2} + [x_1^2 y_1^2 + \|x_2\|^2\|y_2\|^2\cos^2\theta - x_1^2\|y_2\|^2 - y_1^2\|x_2\|^2] \\
& +\frac{1}{2pq}(\lambda_1^p\lambda_3^q + \lambda_2^p\lambda_4^q)(1-\cos\theta) + \frac{1}{2pq}(\lambda_1^p\lambda_4^q + \lambda_2^p\lambda_3^q)(1+\cos\theta) \\
& +\frac{1}{p}\,[(\lambda_1^p x_1(\|y_2\|\cos\theta - y_1) + \lambda_1^p\|x_2\|(y_1 - \|y_2\|\cos\theta)] \\
& -\frac{1}{p}\,[(\lambda_2^p x_1(\|y_2\|\cos\theta + y_1) + \lambda_2^p\|x_2\|(y_1 + \|y_2\|\cos\theta)] \\
& +\frac{1}{q}\,[\lambda_3^q x_1(\|y_2\| - y_1) + \lambda_3^q\|x_2\|\cos\theta(y_1 - \|y_2\|)] \\
& -\frac{1}{q}\,[\lambda_4^q x_1(\|y_2\| + y_1) + \lambda_4^q\|x_2\|\cos\theta(y_1 + \|y_2\|)] \\
= \; & \frac{\lambda_1^p\lambda_2^p}{p^2} + \frac{\lambda_3^q\lambda_4^q}{q^2} + \left[x_1^2(y_1^2 - \|y_2\|^2) - \|x_2\|^2(y_1^2 - \|y_2\|^2\cos^2\theta)\right] \\
& +\frac{1}{2pq}\left[(\lambda_1^p\lambda_3^q + \lambda_2^p\lambda_4^q)(1-\cos\theta) + (\lambda_1^p\lambda_4^q + \lambda_2^p\lambda_3^q)(1+\cos\theta)\right] \\
& -\frac{\lambda_1^p}{p}\lambda_2(y_1 - \|y_2\|\cos\theta) - \frac{\lambda_2^p}{p}\lambda_1(y_1 + \|y_2\|\cos\theta) \\
& -\frac{\lambda_3^q}{q}\lambda_4(x_1 - \|x_2\|\cos\theta) - \frac{\lambda_4^q}{q}\lambda_3(x_1 + \|x_2\|\cos\theta). \qquad (4.11)
\end{aligned}
$$

It is difficult to estimate that the value is positive or negative as the equality contains the term $\cos\theta$ since $-1 \le \cos\theta \le 1$. For the case that two vectors have the same Jordan frame, we can replace the term $\cos\theta$ by 1 or -1 since two vectors have the same Jordan frame if and only if $|\cos\theta| = 1$. To proceed, we need to discuss two subcases.

Case 1. Change $\cos\theta$ to 1, by applying (4.11), we have

$$
\begin{aligned}
& \alpha^2 - \|\beta\|^2 \\
= \; & \frac{\lambda_1^p\lambda_2^p}{p^2} + \frac{\lambda_3^q\lambda_4^q}{q^2} + \left[x_1^2(y_1^2 - \|y_2\|^2) - \|x_2\|^2(y_1^2 - \|y_2\|^2(1 + \cos^2\theta - 1))\right] \\
& +\frac{1}{2pq}(\lambda_1^p\lambda_3^q + \lambda_2^p\lambda_4^q)(1 - 1 + (1-\cos\theta)) \\
& +\frac{1}{2pq}(\lambda_1^p\lambda_4^q + \lambda_2^p\lambda_3^q)(1 + 1 + (\cos\theta - 1)) \\
& -\frac{\lambda_1^p}{p}\lambda_2(y_1 - \|y_2\|(1 + (\cos\theta - 1))) - \frac{\lambda_2^p}{p}\lambda_1(y_1 + \|y_2\|(1 + (\cos\theta - 1))) \\
& -\frac{\lambda_3^q}{q}\lambda_4(x_1 - \|x_2\|(1 + (\cos\theta - 1))) - \frac{\lambda_4^q}{q}\lambda_3(x_1 + \|x_2\|(1 + (\cos\theta - 1)))
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \Big\{ \frac{\lambda_1^p \lambda_2^p}{p^2} + \frac{\lambda_3^q \lambda_4^q}{q^2} + \big[x_1^2 (y_1^2 - \|y_2\|^2) - \|x_2\|^2 (y_1^2 - \|y_2\|^2) \big] \\
& + \frac{1}{2pq} \big[(\lambda_1^p \lambda_3^q + \lambda_2^p \lambda_4^q)(1-1) + (\lambda_1^p \lambda_4^q + \lambda_2^p \lambda_3^q)(1+1) \big] \\
& - \big[\frac{\lambda_1^p}{p} \lambda_2 (y_1 - \|y_2\|) + \frac{\lambda_2^p}{p} \lambda_1 (y_1 + \|y_2\|) \\
& + \frac{\lambda_3^q}{q} \lambda_4 (x_1 - \|x_2\|) + \frac{\lambda_4^q}{q} \lambda_3 (x_1 + \|x_2\|) \big] \Big\} \\
& + \Big\{ \|x_2\|^2 \|y_2\|^2 (\cos^2\theta - 1) \\
& + \frac{1}{2pq} \big[(\lambda_1^p \lambda_3^q + \lambda_2^p \lambda_4^q)(1 - \cos\theta) + (\lambda_1^p \lambda_4^q + \lambda_2^p \lambda_3^q)(\cos\theta - 1) \big] \\
& - \big[- \frac{\lambda_1^p}{p} \lambda_2 \|y_2\| (\cos\theta - 1) + \frac{\lambda_2^p}{p} \lambda_1 \|y_2\| (\cos\theta - 1) \\
& - \frac{\lambda_3^q}{q} \lambda_4 \|x_2\| (\cos\theta - 1) + \frac{\lambda_4^q}{q} \lambda_3 \|x_2\| (\cos\theta - 1) \big] \Big\} \\
= \quad & \Big\{ \frac{\lambda_1^p \lambda_2^p}{p^2} + \frac{\lambda_3^q \lambda_4^q}{q^2} + \lambda_1 \lambda_2 \lambda_3 \lambda_4 + \frac{1}{pq} (\lambda_1^p \lambda_4^q + \lambda_2^p \lambda_3^q) \\
& - \big(\frac{\lambda_1^p}{p} \lambda_2 \lambda_4 + \frac{\lambda_2^p}{p} \lambda_1 \lambda_3 + \frac{\lambda_3^q}{q} \lambda_4 \lambda_2 + \frac{\lambda_4^q}{q} \lambda_3 \lambda_1 \big) \Big\} \\
& + (1 - \cos\theta) \Big\{ - \|x_2\|^2 \|y_2\|^2 (1 + \cos\theta) \\
& + \frac{1}{2pq} \big[(\lambda_1^p \lambda_3^q + \lambda_2^p \lambda_4^q) - (\lambda_1^p \lambda_4^q + \lambda_2^p \lambda_3^q) \big] \\
& - \big[\frac{\lambda_1^p}{p} \lambda_2 \|y_2\| - \frac{\lambda_2^p}{p} \lambda_1 \|y_2\| + \frac{\lambda_3^q}{q} \lambda_4 \|x_2\| - \frac{\lambda_4^q}{q} \lambda_3 \|x_2\| \big] \Big\}.
\end{aligned}
$$

After direct calculation, we further have

$$
\begin{aligned}
& \alpha^2 - \|\beta\|^2 \\
= \quad & \frac{\lambda_1^p \lambda_2^p}{p^2} + \frac{\lambda_3^q \lambda_4^q}{q^2} + \lambda_1 \lambda_2 \lambda_3 \lambda_4 + \frac{1}{pq} (\lambda_1^p \lambda_4^q + \lambda_2^p \lambda_3^q) \\
& - \big[\big(\frac{\lambda_1^p}{p} + \frac{\lambda_3^q}{q} \big) \lambda_2 \lambda_4 + \big(\frac{\lambda_2^p}{p} + \frac{\lambda_4^q}{q} \big) \lambda_1 \lambda_3 \big] \\
& + (1 - \cos\theta) \Big\{ - \|x_2\|^2 \|y_2\|^2 (1 + \cos\theta) + \frac{1}{2pq} \big[\lambda_1^p (\lambda_3^q - \lambda_4^q) + \lambda_2^p (\lambda_4^q - \lambda_3^q) \big] \\
& - \big[\big(\frac{\lambda_1^p \lambda_2}{p} - \frac{\lambda_1 \lambda_2^p}{p} \big) \|y_2\| + \big(\frac{\lambda_3^q \lambda_4}{q} - \frac{\lambda_3 \lambda_4^q}{q} \big) \|x_2\| \big] \Big\} \\
= \quad & \frac{\lambda_1^p \lambda_2^p}{p^2} + \frac{\lambda_3^q \lambda_4^q}{q^2} + \lambda_1 \lambda_2 \lambda_3 \lambda_4 + \frac{1}{pq} (\lambda_1^p \lambda_4^q + \lambda_2^p \lambda_3^q) \\
& - \big[\big(\frac{\lambda_1^p}{p} + \frac{\lambda_3^q}{q} \big) \lambda_2 \lambda_4 + \big(\frac{\lambda_2^p}{p} + \frac{\lambda_4^q}{q} \big) \lambda_1 \lambda_3 \big]
\end{aligned}
$$

$$
\begin{aligned}
& +(1-\cos\theta)\big\{-\|x_2\|^2\|y_2\|^2(1+\cos\theta)+\frac{1}{2pq}\big[\lambda_1^p(\lambda_3^q-\lambda_4^q)+\lambda_2^p(\lambda_4^q-\lambda_3^q)\big] \\
& -\big[\frac{\lambda_1\lambda_2}{p}(\lambda_1^{p-1}-\lambda_2^{p-1})\|y_2\|+\frac{\lambda_3\lambda_4}{q}(\lambda_3^{q-1}-\lambda_4^{q-1})\|x_2\|\big]\big\} \\
= \; & \frac{\lambda_1^p\lambda_2^p}{p^2}+\frac{\lambda_3^q\lambda_4^q}{q^2}+\lambda_1\lambda_2\lambda_3\lambda_4+\frac{1}{pq}(\lambda_1^p\lambda_4^q+\lambda_2^p\lambda_3^q) \\
& -\big[(\frac{\lambda_1^p}{p}+\frac{\lambda_3^q}{q})\lambda_2\lambda_4+(\frac{\lambda_2^p}{p}+\frac{\lambda_4^q}{q})\lambda_1\lambda_3\big] \\
& +(1-\cos\theta)\big\{-(\frac{\lambda_1-\lambda_2}{2})^2(\frac{\lambda_3-\lambda_4}{2})^2(1+\cos\theta)+\frac{1}{2pq}(\lambda_1^p-\lambda_2^p)(\lambda_3^q-\lambda_4^q) \\
& -\frac{\lambda_1\lambda_2}{p}(\lambda_1^{p-1}-\lambda_2^{p-1})\frac{\lambda_3-\lambda_4}{2}-\frac{\lambda_3\lambda_4}{q}(\lambda_3^{q-1}-\lambda_4^{q-1})\frac{\lambda_1-\lambda_2}{2}\big\} \\
= \; & \frac{\lambda_1^p\lambda_2^p}{p^2}+\frac{\lambda_3^q\lambda_4^q}{q^2}+\lambda_1\lambda_2\lambda_3\lambda_4+\frac{1}{pq}(\lambda_1^p\lambda_4^q+\lambda_2^p\lambda_3^q) \\
& -\big[(\frac{\lambda_1^p}{p}+\frac{\lambda_3^q}{q})\lambda_2\lambda_4+(\frac{\lambda_2^p}{p}+\frac{\lambda_4^q}{q})\lambda_1\lambda_3\big] \\
& +(1-\cos\theta)\big\{\big[\frac{1}{2pq}(\lambda_1^p-\lambda_2^p)(\lambda_3^q-\lambda_4^q)-\frac{\lambda_1\lambda_2}{2p}(\lambda_1^{p-1}-\lambda_2^{p-1})(\lambda_3-\lambda_4) \\
& -\frac{\lambda_3\lambda_4}{2q}(\lambda_3^{q-1}-\lambda_4^{q-1})(\lambda_1-\lambda_2)\big]-(1+\cos\theta)\frac{(\lambda_1-\lambda_2)^2(\lambda_3-\lambda_4)^2}{16}\big\} \\
:= \; & f_1+(1-\cos\theta)\big[\gamma-(1+\cos\theta)\delta\big],
\end{aligned}
$$

where the third equality holds since $\|x_2\|=\frac{\lambda_1-\lambda_2}{2}$, $\|y_2\|=\frac{\lambda_3-\lambda_4}{2}$.

Case 2. Change $\cos\theta$ to -1, by applying the same argument in Case 1, we have

$$
\begin{aligned}
& \alpha^2-\|\beta\|^2 \\
= \; & \big\{\frac{\lambda_1^p\lambda_2^p}{p^2}+\frac{\lambda_3^q\lambda_4^q}{q^2}+\lambda_1\lambda_2\lambda_3\lambda_4+\frac{1}{pq}(\lambda_1^p\lambda_3^q+\lambda_2^p\lambda_4^q) \\
& -\big[(\frac{\lambda_1^p}{p}+\frac{\lambda_4^q}{q})\lambda_2\lambda_3+(\frac{\lambda_2^p}{p}+\frac{\lambda_3^q}{q})\lambda_1\lambda_4\big]\big\} \\
& -(1+\cos\theta)\big\{\big[\frac{1}{2pq}(\lambda_1^p-\lambda_2^p)(\lambda_3^q-\lambda_4^q)-\frac{\lambda_1\lambda_2}{2p}(\lambda_1^{p-1}-\lambda_2^{p-1})(\lambda_3-\lambda_4) \\
& -\frac{\lambda_3\lambda_4}{2q}(\lambda_3^{q-1}-\lambda_4^{q-1})(\lambda_1-\lambda_2)\big]+(1-\cos\theta)\frac{(\lambda_1-\lambda_2)^2(\lambda_3-\lambda_4)^2}{16}\big\} \\
:= \; & f_2-(1+\cos\theta)\big[\gamma+(1-\cos\theta)\delta\big].
\end{aligned}
$$

Then, the desired result follows.. □

Notice that f_1 and f_2 equal to $\alpha^2-\|\beta\|^2$ while x,y have the same Jordan Frame. In fact, $\alpha^2-\|\beta\|^2=f_1$ as $\cos\theta=1$ and $\alpha^2-\|\beta\|^2=f_2$ as $\cos\theta=-1$. We shall establish some properties of them in following lemma.

Lemma 4.3. *For any $x \succeq_{\mathcal{K}^n} 0, y \succeq_{\mathcal{K}^n} 0$, suppose the spectral decompositions of x, y are*

$$\begin{aligned} x &= \lambda_1 u_1 + \lambda_2 u_2, \\ y &= \lambda_3 u_3 + \lambda_4 u_4, \end{aligned}$$

and p, q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then the following hold:

(a): *$f_1 \geq 0$. Moreover, $f_1 = 0$ if and only if $\lambda_1^p = \lambda_3^q$ and $\lambda_2^p = \lambda_4^q$.*

(b): *$f_2 \geq 0$. Moreover, $f_2 = 0$ if and only if $\lambda_i = 0$ for $i = 1, 2, 3, 4$.*

Proof. For part(a), we let $\kappa = (\kappa_1, \kappa_2) = (\frac{\lambda_1+\lambda_2}{2}, \frac{\lambda_1-\lambda_2}{2}\bar{e}) \in \mathbb{R}\times\mathbb{R}^{n-1}$, $\omega = (\omega_1, \omega_2) = (\frac{\lambda_3+\lambda_4}{2}, \frac{\lambda_3-\lambda_4}{2}\bar{e}) \in \mathbb{R}\times\mathbb{R}^{n-1}$, where $\bar{e} = (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$, then $\kappa \succeq_{\mathcal{K}^n} 0, \omega \succeq_{\mathcal{K}^n} 0$. Note that $\kappa_1 = \frac{\lambda_1+\lambda_2}{2}$, $\|\kappa_2\| = \frac{\lambda_1-\lambda_2}{2}$. By the spectral decomposition (4.1)-(4.3), we have

$$\kappa = \lambda_1(\kappa) u_\kappa^{(1)} + \lambda_2(\kappa) u_\kappa^{(2)} = \lambda_1 u_\kappa^{(1)} + \lambda_2 u_\kappa^{(2)}.$$

where

$$\begin{aligned} \lambda_1(\kappa) &= \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2} = \lambda_1, \\ \lambda_2(\kappa) &= \frac{\lambda_1+\lambda_2}{2} - \frac{\lambda_1-\lambda_2}{2} = \lambda_2, \\ u_\kappa^{(1)} &= \frac{1}{2}\left(1, \frac{\frac{\lambda_1+\lambda_2}{2}e}{\frac{\lambda_1+\lambda_2}{2}}\right) = \left(\frac{1}{2}, \frac{e}{2}\right), \\ u_\kappa^{(2)} &= \frac{1}{2}\left(1, -\frac{\frac{\lambda_1+\lambda_2}{2}e}{\frac{\lambda_1+\lambda_2}{2}}\right) = \left(\frac{1}{2}, -\frac{e}{2}\right). \end{aligned}$$

Similarly, the spectral decomposition of ω is

$$\omega = \lambda_1(\omega) u_\omega^{(1)} + \lambda_2(\omega) u_\omega^{(2)} = \lambda_3 u_\kappa^{(1)} + \lambda_4 u_\kappa^{(2)}.$$

where

$$\begin{aligned} \lambda_1(\omega) &= \frac{\lambda_3+\lambda_4}{2} + \frac{\lambda_3-\lambda_4}{2} = \lambda_3, \\ \lambda_2(\omega) &= \frac{\lambda_3+\lambda_4}{2} - \frac{\lambda_3-\lambda_4}{2} = \lambda_4, \\ u_\omega^{(1)} &= \frac{1}{2}\left(1, \frac{\frac{\lambda_3+\lambda_4}{2}e}{\frac{\lambda_3+\lambda_4}{2}}\right) = \left(\frac{1}{2}, \frac{e}{2}\right) = u_\kappa^{(1)}, \\ u_\omega^{(2)} &= \frac{1}{2}\left(1, -\frac{\frac{\lambda_3+\lambda_4}{2}e}{\frac{\lambda_3+\lambda_4}{2}}\right) = \left(\frac{1}{2}, -\frac{e}{2}\right) = u_\kappa^{(2)}. \end{aligned}$$

Since x, κ have the same spectral values λ_1, λ_2 and y, ω have the same spectral values λ_3, λ_4, we get via (4.8) that $\left(\alpha^2 - \|\beta\|^2\right)_{\kappa\omega} = f_1$. Moreover κ, ω share the same Jordan frame $u_\kappa^{(1)}, u_\kappa^{(2)}$. By Theorem 4.1, it yields

$$\frac{\kappa^p}{p} + \frac{\omega^q}{q} \succeq_{\mathcal{K}^n} \kappa \circ \omega$$

and hence $f_1 = \big(\alpha^2 - \|\beta\|^2\big)_{\kappa\omega} \geq 0$. Moreover,

$$
\begin{aligned}
f_1 = \big(\alpha^2 - \|\beta\|^2\big)_{\kappa\omega} = 0 \iff & \frac{\kappa^p}{p} + \frac{\omega^q}{q} = \kappa \circ \omega \\
\iff & \kappa^p = \omega^q \\
\iff & (\lambda_1^p - \lambda_3^q)u_\kappa^{(1)} + (\lambda_2^p - \lambda_4^q)u_\kappa^{(2)} = 0 \\
\iff & \lambda_1^p - \lambda_3^q = \lambda_2^p - \lambda_4^q = 0 \qquad (4.12)\\
\iff & \lambda_1^p = \lambda_3^q \text{ and } \lambda_2^p = \lambda_4^q.
\end{aligned}
$$

where the equivalence (4.12) holds since $u_\kappa^{(1)},\ u_\kappa^{(2)}$ are linearly independent.

For part(b), we let $\zeta = (\zeta_1, \zeta_2) = (\frac{\lambda_1+\lambda_2}{2}, \frac{\lambda_1-\lambda_2}{2}e) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $\eta = (\eta_1, \eta_2) = (\frac{\lambda_4+\lambda_3}{2}, -\frac{\lambda_4-\lambda_3}{2}e) \in \mathbb{R} \times \mathbb{R}^{n-1}$, where $e = (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$, then $\zeta \succeq_{\mathcal{K}^n} 0, \eta \succeq_{\mathcal{K}^n} 0$. Similarly,

$$
\begin{aligned}
\zeta &= \lambda_1(\zeta)u_\zeta^{(1)} + \lambda_2(\zeta)u_\zeta^{(2)} = \lambda_1 u_\zeta^{(1)} + \lambda_2 u_\zeta^{(2)}. \\
\eta &= \lambda_1(\eta)u_\eta^{(1)} + \lambda_2(\eta)u_\eta^{(2)} = \lambda_3 u_\zeta^{(2)} + \lambda_4 u_\zeta^{(1)}.
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_1(\zeta) &= \frac{\lambda_1+\lambda_2}{2} + \frac{\lambda_1-\lambda_2}{2} = \lambda_1, \\
\lambda_2(\zeta) &= \frac{\lambda_1+\lambda_2}{2} - \frac{\lambda_1-\lambda_2}{2} = \lambda_2, \\
u_\zeta^{(1)} &= \frac{1}{2}\Big(1, \frac{\frac{\lambda_1+\lambda_2}{2}e}{\frac{\lambda_1+\lambda_2}{2}}\Big) = (\frac{1}{2}, \frac{e}{2}), \\
u_\zeta^{(2)} &= \frac{1}{2}\Big(1, -\frac{\frac{\lambda_1+\lambda_2}{2}e}{\frac{\lambda_1+\lambda_2}{2}}\Big) = (\frac{1}{2}, -\frac{e}{2}), \\
\lambda_1(\eta) &= \frac{\lambda_3+\lambda_4}{2} + \frac{\lambda_3-\lambda_4}{2} = \lambda_3, \\
\lambda_2(\eta) &= \frac{\lambda_3+\lambda_4}{2} - \frac{\lambda_3-\lambda_4}{2} = \lambda_4, \\
u_\eta^{(1)} &= \frac{1}{2}\Big(1, \frac{\frac{\lambda_3+\lambda_4}{2}(-e)}{\frac{\lambda_3+\lambda_4}{2}}\Big) = (\frac{1}{2}, -\frac{e}{2}) = u_\zeta^{(2)}, \\
u_\eta^{(2)} &= \frac{1}{2}\Big(1, -\frac{\frac{\lambda_3+\lambda_4}{2}(-e)}{\frac{\lambda_3+\lambda_4}{2}}\Big) = (\frac{1}{2}, \frac{e}{2}) = u_\zeta^{(1)}.
\end{aligned}
$$

Since x, ζ have the same λ_1, λ_2 and $y,\ \eta$ have the same λ_3, λ_4, we get via (4.8) that $\big(\alpha^2 - \|\beta\|^2\big)_{\zeta\eta} = f_2$. Since ζ, η share the same Jordan frame $u_\zeta^{(1)}, u_\zeta^{(2)}$. By Theorem 4.1, it yields

$$
\frac{\zeta^p}{p} + \frac{\eta^q}{q} \succeq_{\mathcal{K}^n} \zeta \circ \eta
$$

which implies $f_2 = \big(\alpha^2 - \|\beta\|^2\big)_{\zeta\eta} \geq 0$. Moreover,

$$\begin{aligned} f_2 = \big(\alpha^2 - \|\beta\|^2\big)_{\zeta\eta} = 0 \quad &\Longleftrightarrow \quad \frac{\zeta^p}{p} + \frac{\eta^q}{q} = \zeta \circ \eta \\ &\Longleftrightarrow \quad \zeta^p = \eta^q \\ &\Longleftrightarrow \quad (\lambda_1^p - \lambda_4^q)u_\zeta^{(1)} + (\lambda_2^p - \lambda_3^q)u_\zeta^{(2)} = 0 \\ &\Longleftrightarrow \quad \lambda_1^p - \lambda_4^q = \lambda_2^p - \lambda_3^q = 0 \\ &\Longleftrightarrow \quad \lambda_4^q = \lambda_1^p \geq \lambda_2^p = \lambda_3^q \\ &\Longleftrightarrow \quad \lambda_1^p = \lambda_2^p = \lambda_3^q = \lambda_4^q = 0. \end{aligned}$$

where last equivalence follows by $\lambda_3^q \geq \lambda_4^q$. □

Theorem 4.4. *Suppose $x \succeq_{\mathcal{K}^n} 0, y \succeq_{\mathcal{K}^n} 0$, and p, q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $\gamma - (1+\cos\theta)\delta \geq 0$ or $\gamma + (1-\cos\theta)\delta \leq 0$ for all $0 \leq \theta \leq 2\pi$, then $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^n} x \circ y$.*

Proof. By Lemma 4.3, we know that $f_1 \geq 0$ *and* $f_2 \geq 0$ because $1+\cos\theta$, $1-\cos\theta$ are nonnegative.

(i) If $\gamma - (1+\cos\theta)\delta \geq 0$, then (4.8) yields

$$\alpha^2 - \|\beta\|^2 = f_1 + (1-\cos\theta)\big[\gamma - (1+\cos\theta)\delta\big] \geq 0.$$

Therefore, we have $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^n} x \circ y$.

(ii) If $\gamma + (1-\cos\theta)\delta \leq 0$, then (4.8) yields

$$\alpha^2 - \|\beta\|^2 = f_2 - (1+\cos\theta)\big[\gamma + (1-\cos\theta)\delta\big] \geq 0.$$

Therefore, we have $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^n} x \circ y$. □

Theorem 4.4 provides a way to verify $\alpha^2 - \|\beta\|^2$. Unfortunately, the condition in Theorem 4.4 does not always hold. Consequently, $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^n} x \circ y$ might fail. To see this, taking $p = \frac{3}{2}$, $q = 3$ and $x = (61, 36, 48) \in \mathcal{K}^3$, $y = (6, 4, 3) \in \mathcal{K}^3$, we obtain $\frac{x^p}{p} = (444, 266, \frac{1064}{3})$, $\frac{y^q}{q} = (222, \frac{532}{3}, 133)$, and $x \circ y = (654, 460, 471)$. Thus,

$$\frac{x^p}{p} + \frac{y^q}{q} - x \circ y = \left(12, -16\frac{2}{3}, 16\frac{2}{3}\right) \notin \mathcal{K}^3.$$

In fact, for $p \neq q$ and x, y have different Jordan frame, there always exist a counterexample for $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^n} x \circ y$. To prove this, we need a technical lemma.

Lemma 4.5. *For any $x \succ_{\mathcal{K}^n} 0$, $y \succ_{\mathcal{K}^n} 0$, suppose the spectral decompositions of x, y are below:*

$$\begin{aligned} x &= \lambda_1 u_1 + \lambda_2 u_2, \\ y &= \lambda_3 u_3 + \lambda_4 u_4, \end{aligned}$$

and p, q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $\cos\theta = \frac{4}{pq}$, then there exists a positive real number $\xi \geq 1$ such that $\frac{\lambda_1}{\lambda_2} \geq \xi$, which implies $\gamma - (1+\cos\theta)\delta < 0$.

Proof. First, we observe that

$$\begin{aligned}\frac{1}{p}+\frac{1}{q}=1 \quad &\Longleftrightarrow \quad p+q=pq \Longleftrightarrow p=q(p-1) \Longleftrightarrow p-1=\frac{p}{q}, \\ &\Longleftrightarrow \quad q=p(q-1) \Longleftrightarrow q-1=\frac{q}{p},\end{aligned}$$

and

$$\begin{aligned}\lambda_1^p=\lambda_3^q \quad &\Longleftrightarrow \quad \lambda_3=\lambda_1^{\frac{p}{q}}=\lambda_1^{p-1} \Longleftrightarrow \lambda_1=\lambda_3^{\frac{q}{p}}=\lambda_3^{q-1}, \\ \lambda_2^p=\lambda_4^q \quad &\Longleftrightarrow \quad \lambda_4=\lambda_2^{\frac{p}{q}}=\lambda_2^{p-1} \Longleftrightarrow \lambda_2=\lambda_4^{\frac{q}{p}}=\lambda_4^{q-1}.\end{aligned}$$

Then, we have

$$\begin{aligned}& \gamma-(1+\cos\theta)\delta \\ = \ & \frac{1}{2pq}(\lambda_1^p-\lambda_2^p)(\lambda_3^q-\lambda_4^q)-(1+\cos\theta)\frac{(\lambda_1-\lambda_2)^2(\lambda_3-\lambda_4)^2}{16} \\ & -\frac{\lambda_1\lambda_2}{2p}(\lambda_1^{p-1}-\lambda_2^{p-1})(\lambda_3-\lambda_4)-\frac{\lambda_3\lambda_4}{2q}(\lambda_3^{q-1}-\lambda_4^{q-1})(\lambda_1-\lambda_2) \\ = \ & \frac{1}{2pq}(\lambda_1^p-\lambda_2^p)^2-(1+\cos\theta)\frac{(\lambda_1-\lambda_2)^2(\lambda_1^{p-1}-\lambda_2^{p-1})^2}{16} \\ & -\frac{1}{2p}\lambda_1\lambda_2(\lambda_1^{p-1}-\lambda_2^{p-1})^2-\frac{1}{2q}(\lambda_1\lambda_2)^{p-1}(\lambda_1-\lambda_2)^2 \\ = \ & \frac{1}{2pq}\left[(\lambda_1^p-\lambda_2^p)^2-\frac{pq(1+\cos\theta)}{8}(\lambda_1-\lambda_2)^2(\lambda_1^{p-1}-\lambda_2^{p-1})^2\right] \\ & -\left[\frac{1}{2p}\lambda_1\lambda_2(\lambda_1^{p-1}-\lambda_2^{p-1})^2+\frac{1}{2q}(\lambda_1\lambda_2)^{p-1}(\lambda_1-\lambda_2)^2\right]. \qquad (4.13)\end{aligned}$$

Consider the homogeneous part with degree $2p$ in (4.13):

$$\begin{aligned}& (\lambda_1^p-\lambda_2^p)^2-\frac{pq(1+\cos\theta)}{8}(\lambda_1-\lambda_2)^2(\lambda_1^{p-1}-\lambda_2^{p-1})^2 \\ = \ & \lambda_2^{2p}\left[((\frac{\lambda_1}{\lambda_2})^p-1)^2-\frac{pq(1+\cos\theta)}{8}(\frac{\lambda_1}{\lambda_2}-1)^2((\frac{\lambda_1}{\lambda_2})^{p-1}-1)^2\right] \\ := \ & \lambda_2^{2p}\cdot g(t),\end{aligned}$$

where $t=\frac{\lambda_1}{\lambda_2}>1$ and $g(t)=(t^p-1)^2-\frac{pq(1+\cos\theta)}{8}(t-1)^2(t^{p-1}-1)^2$. Since $\frac{1}{p}+\frac{1}{q}=1$ with $p\neq q$ and p, q are positive, it yields by Arithmetic-Geometric-Mean inequality that

$$\sqrt{\frac{1}{pq}}<\frac{\frac{1}{p}+\frac{1}{q}}{2}=\frac{1}{2} \implies 4<pq \implies 0<\frac{4}{pq}<1.$$

Thus, we may take $\cos\theta=\frac{4}{pq}$, and hence

$$\frac{pq(1+\cos\theta)}{8}=\frac{pq(1+\frac{4}{pq})}{8}=\frac{pq+4}{8}>\frac{4+4}{8}=1.$$

In other words, $g(t)$ has negative leading coefficient. Due to the degree of $g(t)$ is $2p > 1$, there exists a positive real number $\xi \geq 1$ such that $\frac{\lambda_1}{\lambda_2} = t \geq \xi$, which implies

$$g(t) < 0 \Longrightarrow \lambda_2^{2p} \cdot g(t) < 0 \Longrightarrow \gamma - (1 + \cos\theta)\delta < 0.$$

Here the last implications holds by (4.13), which is negative. Then, the proof is complete. □

Theorem 4.6. *Suppose $x \succeq_{\mathcal{K}^n} 0, y \succeq_{\mathcal{K}^n} 0$, and p, q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. If $p \neq q$ and $n \geq 3$, then there always exists a counterexample for $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^n} x \circ y$.*

Proof. Suppose $x \succeq_{\mathcal{K}^n} 0$, $y \succeq_{\mathcal{K}^n} 0$ with the spectral decomposition as below:

$$\begin{aligned} x &= \lambda_1 u_1 + \lambda_2 u_2, \\ y &= \lambda_3 u_3 + \lambda_4 u_4, \end{aligned}$$

Since $p \neq q$, it implies $0 < \frac{4}{pq} < 1$. Then, we take $\cos\theta = \frac{4}{pq}$. Consider those $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ belong to $\mathcal{K}^n$ with $\lambda_1^p = \lambda_3^q$, $\lambda_2^p = \lambda_4^q$. By Lemma 4.5, there exists $\xi \geq 1$ such that $\frac{\lambda_1}{\lambda_2} \geq \xi$, which implies $\gamma - (1 + \cos\theta)\delta < 0$. We choose two vectors x, y satisfying $\frac{\lambda_1}{\lambda_2} \geq \xi$, $\lambda_1^p = \lambda_3^q$, $\lambda_2^p = \lambda_4^q$, and $\cos\theta = \frac{4}{pq}$. Such kinds of x, y exist because $n \geq 3$, so there exists x, y with different Jordan frame, and the angle θ between x_2 *and* y_2 satisfies $-1 < \cos\theta < 1$. Because $\lambda_1^p = \lambda_3^q$, $\lambda_2^p = \lambda_4^q$, by applying Lemma 4.3(a) we have $f_1 = 0$. Note that $\frac{\lambda_1}{\lambda_2} \geq \xi$, which implies $\gamma - (1 + \cos\theta)\delta < 0$. Thus, it yields that

$$\alpha^2 - \|\beta\|^2 = 0 + (1 - \cos\theta)\big[\gamma - (1 + \cos\theta)\delta\big] < 0.$$

In summary, the Young inequality does not hold. □

Example 4.7. For $p = \frac{3}{2}$, $q = 3$, $n = 3$, and taking $x = (281, -96, -128) \in \mathcal{K}^3$, $y = (16, 4, 3) \in \mathcal{K}^3$, we have $\lambda_1 = 441$, $\lambda_2 = 121$, $\lambda_3 = 21$, $\lambda_4 = 11$, and $\frac{\lambda_1}{\lambda_2} \approx 3.6446$, $\cos\theta \approx -0.96$. Since $\frac{x^p}{p} = (3530\frac{2}{3}, -1586, -2114\frac{2}{3})$, $\frac{y^q}{q} = (1765\frac{1}{3}, 1057\frac{1}{3}, 793)$ and $x \circ y = (3728, -412, -1205)$, we obtain

$$\frac{x^p}{p} + \frac{y^q}{q} - x \circ y = \left(1568, -116\frac{2}{3}, -116\frac{2}{3}\right) \in \mathcal{K}^3.$$

In other words, $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^3} x \circ y$. Although $\lambda_1^p = \lambda_3^q$, $\lambda_2^p = \lambda_4^q$ implies $f_1 = 0$, and $\frac{(1+\cos\theta)pq}{8} = 0.0225 < 1$, which says that $g(t)$ has positive leading coefficient. Hence, we cannot apply Lemma 4.5 to derive $\gamma - (1 + \cos\theta)\delta < 0$. Nonetheless, in this example, there still holds $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^3} x \circ y$.

Example 4.8. For $p = \frac{3}{2}$, $q = 3$, $n = 3$, and taking $x = (281, 96, 128) \in \mathcal{K}^3$, $y = (16, 4, 3) \in \mathcal{K}^3$, we have $\lambda_1 = 441$, $\lambda_2 = 121$, $\lambda_3 = 21$, $\lambda_4 = 11$, and $\frac{\lambda_1}{\lambda_2} \approx 3.6446$, $\cos\theta \approx 0.96$. It is easy to compute that $\frac{x^p}{p} = (3530\frac{2}{3}, 1586, 2114\frac{2}{3})$, $\frac{y^q}{q} = (1765\frac{1}{3}, 1057\frac{1}{3}, 793)$, and $x \circ y = (5264, 2660, 2891)$. Therefore, we have

$$\frac{x^p}{p} + \frac{y^q}{q} - x \circ y = \left(32, -16\frac{2}{3}, 16\frac{2}{3}\right) \in \mathcal{K}^3.$$

In other words, $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^3} x \circ y$. Although $\lambda_1^p = \lambda_3^q$, $\lambda_2^p = \lambda_4^q$ implies $f_1 = 0$, and $\frac{(1+\cos\theta)pq}{8} = 1.1025 > 1$ says that $g(t)$ has negative leading coefficient, $\frac{\lambda_1}{\lambda_2} \approx 3.6446$, which is not large enough to make $g(t) < 0$. This is why $\frac{x^p}{p} + \frac{y^q}{q} \succeq_{\mathcal{K}^3} x \circ y$.

Example 4.9. For $p = \frac{3}{2}$, $q = 3$, $n = 3$, and taking $x = (61, 36, 48) \in \mathcal{K}^3$, $y = (6, 4, 3) \in \mathcal{K}^3$, we have $\lambda_1 = 121$, $\lambda_2 = 1$, $\lambda_3 = 11$, $\lambda_4 = 1$, and $\frac{\lambda_1}{\lambda_2} = 121$, $\cos\theta \approx 0.96$. Since $\frac{x^p}{p} = (444, 266, \frac{1064}{3})$, $\frac{y^q}{q} = (222, \frac{532}{3}, 133)$ and $x \circ y = (654, 460, 471)$, we see that

$$\frac{x^p}{p} + \frac{y^q}{q} - x \circ y = \left(12, -16\frac{2}{3}, 16\frac{2}{3}\right) \notin \mathcal{K}^3.$$

In this example, $\lambda_1^p = \lambda_3^q$, $\lambda_2^p = \lambda_4^q$ implies $f_1 = 0$, and $\frac{(1+\cos\theta)pq}{8} = 1.1025 > 1$ says that $g(t)$ has negative leading coefficient. Because $\frac{\lambda_1}{\lambda_2} = 121$ is large enough to enable $g(t) < 0$, we conclude that $\frac{x^p}{p} + \frac{y^q}{q} \not\succeq_{\mathcal{K}^3} x \circ y$.

As a consequence of all the above results and discussions, we have the following corollary which describes under what conditions the Young inequality holds.

Corollary 4.10. *Suppose $x \succeq_{\mathcal{K}^n} 0, y \succeq_{\mathcal{K}^n} 0$, and p, q are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then, the Young inequality holds either when $p = q$ or when $n = 2$.*

Proof. By Theorem 4.1 and Theorem 4.6, the conclusion is drawn. □

References

[1] M. Baes, *Convexity and differentiability properties of spectral functions and spectral mappings on Euclidean Jordan algebras*, Linear Algebra Appl. **422(2-3)** (2007), 664–700.

[2] H. H. Bauschke and M. Koecher, *Jordan-Algebren*, Springer, New York, 1966.

[3] Y.-L. Chang and J-S. Chen, *Convexity of symmetric cone trace functions in Euclidean Jordan algebras*, J. Nonlinear Convex Anal. **14** (2013), 53–61.

[4] Y.-L. Chang, C.-H. Huang, J.-S. Chen, and C.-C. Hu *Some inequalities for means defined on the Lorentz cone*, Math. Inequal. Appl. **21(4)** (2018), 1015–1028.

[5] J.-S. Chen, X. Chen, and P. Tseng, *Analysis of nonsmooth vector-valued functions associated with second-order cones*, Math. Program. **101** (2004), 95–117.

[6] J.-S. Chen and P. Tseng, *An unconstrained smooth minimization reformulation of second-order cone complementarity problem*, Math. Program. **104** (2005), 293–327.

[7] J.-S. Chen, *The convex and monotone functions associated with second-order cone*, Optimization **55(4)** (2006), 363–385.

[8] J.-S. Chen, *A new merit function and its related properties for the second-order cone complementarity problem*, Pac. J. Optim. **2(1)** (2006), 167–179.
[9] J.-S. Chen, *Two classes of merit functions for the second-order cone complementarity problem*, Math. Method Oper. Res. **64(3)** (2006), 495–519.
[10] J.-S. Chen, *SOC Functions and Their Applications*, Springer Optimization and Its Applications 143, Springer, Singapore, 2019.
[11] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford Mathematical Monographs Oxford University Press, New York, 1994.
[12] M. Fukushima, Z.-Q. Luo, and P. Tseng, *Smoothing functions for second- order-cone complimentarity problems*, SIAM J. Optim. **12** (2002), 436–460.
[13] M. S. Gowda *A Hölder type inequality and an interpolation theorem in Euclidean Jordan algebras* J. Math. Anal. Appl. **474** (2019), 248–263.
[14] C.-H. Huang, J.-S. Chen, and C.-C. Hu, *Trace versions of Young inequality and its applications*, J. Nonlinear Convex Anal. **20(2)** (2019), 215–238.
[15] M. Koecher, *The Minnesota Notes on Jordan Algebras and Their Applications*, edited and annotated by A. Brieg and S. Walcher, Springer, Berlin, 1999.
[16] A. Korányi, *Monotone functions on formually real Jordan algebras*, Math. Ann. **269** (1984), 73–76.
[17] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ 1970.
[18] D. Sun and J. Sun, *Löwner's operator and spectral functions in Euclidean Jordan algebras*, Math. Oper. Res. **33(2)** (2008), 421–445.

Manuscript received 10 December 2018
revised 5 March 2019

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