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A VISCOUS ITERATION PROCESS FOR A FINITE FAMILY OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Dedicated to Professor Jong Soo Jung on the occasion of his 65th birthday

Abstract. A viscous iteration process for a finite family of nonexpansive mappings in a uniformly smooth Banach space is studied and the strong convergence of the process is proved.

1. Introduction

Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, let C be a nonempty closed convex subset of H, and let $T : C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$). Let $\mathrm{Fix}(T)$ denote the set of fixed points of T; i.e., $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. Assume $\mathrm{Fix}(T) \neq \emptyset$.

In 1967, Halpern [2] introduced an iteration process that converges in norm to a fixed point of T. More precisely, Halpern's algorithm generates a sequence (x_n) as follows: The initial guess $x_0 \in C$ is selected arbitrarily and the iteration process is proceeded according to the rule:

$$x_{n+1} = \lambda_n u + (1 - \lambda_n) T x_n \tag{1.1}$$

where $u \in C$ is referred to as anchor and (λ_n) is a sequence in $(0, 1)$.

The convergence of Halpern's algorithm has been proved by many researchers [2, 6, 10, 11, 14, 16]. In summary, we have the following convergence result for Halpern's algorithm (the reader is referred to the survey [7] for more details updated until 2010).

Theorem 1.1. *Suppose that the sequence (α_n) in $(0, 1)$ satisfies the conditions $(C1)$, $(C2)$, and $(C3)$ or $(C4)$. Then the sequence (x_n) generated by Halpern's algorithm (1.1) converges in norm to the fixed point q of T that is closest to u from* $\mathrm{Fix}(T)$*; namely, $q = P_{\mathrm{Fix}(T)}(u)$, the nearest point projection of u onto* $\mathrm{Fix}(T)$.

Here the conditions $(C1)$-$(C4)$ are stated as follows:

(C1) $\lim_{n\to\infty} \lambda_n = 0$ [2],
(C2) $\sum_{n=0}^{\infty} \lambda_n = \infty$ [2],
(C3) $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$ [11, 14],
(C4) $\lim_{n\to\infty} |\lambda_n - \lambda_{n+1}|/\lambda_{n+1} = 0$; equivalently, $\lim_{n\to\infty} \lambda_n/\lambda_{n+1} = 1$ [16].

The extension of Halpern's algorithm to a finite family of nonexpansive mappings was initiated in [6] and the following result was proved in [1].

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Theorem 1.2 ([1]). *Let $N \geq 1$ be an integer and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of C with a nonempty common fixed point set $F := \cap_{i=1}^N \mathrm{Fix}(T_i)$. Assume in addition*

$$F = \mathrm{Fix}(T_N \dots T_2 T_1) = \mathrm{Fix}(T_1 T_N \dots T_2) = \dots = \mathrm{Fix}(T_{N-1} \dots T_1 T_N). \tag{1.2}$$

Given $x_0, u \in C$. Define a sequence (x_n) by the iteration process

$$x_{n+1} = \lambda_{n+1} u + (1 - \lambda_{n+1}) T_{n+1} x_n, \; n \geq 0, \tag{1.3}$$

where $(\lambda_n) \subset (0,1)$ and $T_n = T_{n \bmod N}$, with the mod N function taking values in $\{1, 2, \dots, N\}$. Assume the conditions $(C1)$ and $(C2)$ are satisfied. Assume in addition the condition $(C5)$ below is also satisfied:

(C5) $\sum_{n=0}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$.

Then (x_n) converges in norm to a point in F.

The result of Theorem 1.2 was extended to a Banach space setting [9] where the space is either uniformly smooth or has a weakly continuous generalized duality map.

On the other hand, Moudafi [8] introduced the viscosity methods to nonexpansive mappings. He considered the following iteration process for single nonexpansive mapping:

$$x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n) T x_n, \tag{1.4}$$

where $f : C \to C$ is a γ-contraction for some constant $\gamma \in [0, 1)$; that is, f satisfies the contraction property: $\|f(x) - f(y)\| \leq \gamma \|x - y\|$ for all $x, y \in C$.

This viscosity method for single mapping was extended to the uniformly smooth Banach space setting in [15].

The purpose of the present paper is to extend the result of [15] to the case of a finite family of nonexpansive mappings in uniformly smooth Banach spaces.

2. Preliminaries

Let X be a uniformly smooth Banach space, C a nonempty closed convex subset of X, and $T : C \to C$ a nonexpansive mapping. Assume $\mathrm{Fix}(T)$ is nonempty. Let Π_C denote the collection of all contractions from C into itself. That is, $f \in \Pi_C$ if and only if f is a self-mapping of C and satisfies the contraction property: $\|f(x) - f(y)\| \leq \gamma \|x - y\|$ for all $x, y \in C$ and some $\gamma \in [0, 1)$. In this case, f is said to be a γ-contraction.

The following result, known as viscosity method of nonexpansive mappings, will be needed in the proof of the main result of the paper to be presented in the next section.

Theorem 2.1 ([15]). *Under the above setting, there exists a unique sunny nonexpansive retraction $Q : \Pi_C \to \mathrm{Fix}(T)$ which is given by*

$$Q(f) = \lim_{t \to 0+} z_t, \quad f \in \Pi_C, \tag{2.1}$$

where z_t *is the unique fixed point of the contraction* $C \ni x \mapsto tf(x) + (1-t)Tx$. *Moreover,* $Q(f)$ *solves the variational inequality (VI)*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in \mathrm{Fix}(T). \tag{2.2}$$

Here $J : X \to X^*$ *is the normalized duality map from* X *into* X^* *and* X^* *is the dual space of* X.

Recall that the normalized duality map J is defined as

$$J(x) = \{x^* \in X^* : \|x\|^2 = \|x^*\|^2 = \langle x, x^*\rangle\}, \quad x \in X.$$

It is known that for a uniformly smooth Banach space X, the duality map J is single-valued and norm-to-norm uniformly continuous over every bounded subset of X. [See [17] for more information on uniform smoothness of Banach spaces.]

We also need two technical tools stated in the lemmas below.

Lemma 2.2 ([16])**.** *Assume* (s_n) *is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \sigma_n, \quad n \geq 0$$

where (α_n) *is a sequence in* $(0,1)$ *and* (σ_n) *is a sequence of real numbers such that*

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n/\alpha_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. *Let* J *be the normalized duality map of a smooth Banach space* X. *Then, for all* $x, y \in X$, *the following inequality holds*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

3. Viscous Iteration in Banach Spaces

Consider a closed convex subset C of a Banach space X and let $N \geq 1$ be a given integer. Suppose $T_i : C \to C$ for $i = 1, 2, \ldots, N$ are nonexpansive mappings which satisfy the following consistency condition:

$$\emptyset \neq \bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_N T_{N-1} \ldots T_1). \tag{3.1}$$

Note that this consistency condition implies that

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_N \ldots T_2 T_1) = \mathrm{Fix}(T_1 T_N \ldots T_2) = \cdots = \mathrm{Fix}(T_{N-1} \ldots T_1 T_N). \tag{3.2}$$

Remark 3.1. The consistency condition (3.1) is not restrictive. As a matter of fact, if we replace T_i with the averaged mapping $\hat{T}_i := (1-\alpha_i)I + \alpha_i T_i$, where $\alpha_i \in (0,1)$ for $1 \leq i \leq N$. Then $\mathrm{Fix}(\hat{T}_i) = \mathrm{Fix}(T_i)$ for each i and the consistency condition (3.1) holds for the family of the mappings $\hat{T}_i's$ which has the same set of common fixed points with the original family of mappings $T_i's$.

Now we consider the following viscous iteration process:

$$x_{n+1} = \lambda_{n+1} f(x_n) + (1 - \lambda_{n+1}) T_{[n+1]} x_n, \quad n = 0, 1, \dots, \tag{3.3}$$

where the initial guess $x_0 \in C$ is arbitrary, and $f \in \Pi_C$ is a γ-contraction with $\gamma \in [0, 1)$. Here $[n+1]$ is defined by $[n+1] := (n \mod N) + 1$ for $n \geq 0$.

The main result of this paper is the following result.

Theorem 3.2. *Let X be a uniformly smooth Banach space, C a nonempty closed convex subset of X, and $\{T_i\}_{=1}^N$ be $N \geq 1$ nonexpansive self-mappings of C. Assume the consistency condition (3.1) and the conditions $(C1)$-$(C2)$. Assume, in addition, (λ_n) satisfies either $(C5)$ or $(C6)$ which is stated below:*

(C6) $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+N}}{\lambda_{n+N}} = 0$; *equivalently,* $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

Then the sequence (x_n) generated by the viscous iteration process (3.3) converges in norm to $Q(f)$, where $Q : \Pi_C \to F$ is the unique sunny nonexpansive retraction from Π_C onto F. Here $F = \cap_{i=1}^N \mathrm{Fix}(T_i)$.

Proof. As standard, the proof consists of six steps.

Step 1: (x_n) is bounded.

To see this, we take $p \in F$ to get by using (3.3)

$$\begin{aligned} \|x_{n+1} - p\| &\leq \lambda_n \|f(x_n) - p\| + (1 - \lambda_n)\|T_{[n+1]} x_n - p\| \\ &\leq \lambda_n(\|f(x_n) - f(p)\| + \|f(p) - p\|) + (1 - \lambda_n)\|x_n - p\| \\ &\leq (1 - (1-\gamma)\lambda_n)\|x_n - p\| + \lambda_n \|f(p) - p\| \\ &\leq \max\left\{\|x_n - p\|, \frac{1}{1-\gamma}\|f(p) - p\|\right\}. \end{aligned}$$

By induction, we obtain

$$\|x_n - p\| \leq \max\left\{\|x_0 - p\|, \frac{1}{1-\gamma}\|f(p) - p\|\right\}$$

for all $n \geq 0$. It turns out that (x_n) is bounded.

Step 2: $\|x_{n+1} - T_{[n+1]} x_n\| \to 0$ as $n \to \infty$.

This is quite straightforward. Indeed we have from (3.3) that

$$\|x_{n+1} - T_{[n+1]} x_n\| = \lambda_{n+1} \|f(x_n) - T_{[n+1]} x_n\|.$$

The boundedness of (x_n) and the fact that $\lambda_n \to 0$ (as $n \to \infty$) immediately imply the conclusion of Step 2.

Step 3: $\|x_{n+N} - x_n\| \to 0$ as $n \to \infty$.

In fact, noticing the fact that $T_{[n+1+N]} = T_{[n+1]}$, we derive that

$$\|x_{n+1+N} - x_{n+1}\| = \|\lambda_{n+1+N} f(x_{n+N}) + (1 - \lambda_{n+1+N}) T_{[n+1+N]} x_{n+N}$$

$$
\begin{aligned}
&- [\lambda_{n+1} f(x_n) + (1-\lambda_{n+1}) T_{[n+1]} x_n] \\
&= \|\lambda_{n+1+N}(f(x_{n+N}) - f(x_n)) \\
&\quad + (1-\lambda_{n+1+N})(T_{[n+1]} x_{n+N} - T_{[n+1]} x_n) \\
&\quad + (\lambda_{n+1+N} - \lambda_{n+1})(f(x_n) - T_{[n+1]} x_n)\|.
\end{aligned}
$$

Since (x_n) is bounded, we have a constant $\alpha \geq 0$ such that $\|f(x_n) - T_{[n+1]} x_n\| \leq \alpha$ for all $n \geq 0$. Also since f is a γ-contraction, it turns out that

$$
\begin{aligned}
\|x_{n+1+N} - x_{n+1}\| &\leq (1-(1-\gamma)\lambda_{n+1+N})\|x_{n+N} - x_n\| + \alpha|\lambda_{n+1+N} - \lambda_{n+1}| \\
&= (1-(1-\gamma)\lambda_{n+1+N})\|x_{n+N} - x_n\| + (1-\gamma)\lambda_{n+1+N}\beta_n
\end{aligned}
$$

where $\beta_n = \alpha|\lambda_{n+1+N} - \lambda_{n+1}|/((1-\gamma)\lambda_{n+1+N})$. Now under the conditions $(C1)$-$(C2)$ and $(C5)$ or $(C6)$, we can apply Lemma 2.2 to immediately obtain Step 3.

Step 4: $\|x_n - T_{[n+N]} \dots T_{[n+1]} x_n\| \to 0$ as $n \to \infty$.

To prove Step 4, we set $U_0 = I$ and $U_i = T_{[n+N]} \dots T_{[n+N-i+1]}$ for $i = 1, 2, \dots, N$. Then U_i are all nonexpansive. Moreover, it is easily seen that $U_1 = T_{[n+N]}, \dots, U_N = T_{[n+N]} \dots T_{[n+1]}$, and $U_{i+1} = U_i T_{[n+N-i]}$ for $i = 0, 1, \dots, N-1$. We then get

$$
\begin{aligned}
&\|x_n - T_{[n+N]} \dots T_{[n+1]} x_n\| = \|x_n - U_N x_n\| \\
&\leq \|x_n - x_{n+N}\| + \|x_{n+N} - U_N x_n\| \\
&= \|x_n - x_{n+N}\| + \left\| \sum_{i=0}^{N-1} (U_i x_{n+N-i} - U_{i+1} x_{n+N-i-1}) \right\| \\
&= \|x_n - x_{n+N}\| + \left\| \sum_{i=0}^{N-1} (U_i x_{n+N-i} - U_i T_{[n+N-i]} x_{n+N-i-1}) \right\| \\
&\leq \|x_n - x_{n+N}\| + \sum_{i=0}^{N-1} \|x_{n+N-i} - T_{[n+N-i]} x_{n+N-i-1}\|.
\end{aligned}
$$

Now by Step 2, we see that each term under the above summation tends to zero when $n \to \infty$. This combining with Step 3 yields the result of Step 4.

Step 5: $\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle \leq 0$. Here $q = Q(f)$ and Q is the unique sunny nonexpansive retraction from Π_C onto $F = \cap_{i=1}^N \mathrm{Fix}(T_i)$.

To verify Step 5, we take a subsequence (x_{n_k}) of (x_n) with the property:

$$
\limsup_{n\to\infty} \langle f(q) - q, J(x_n - q)\rangle = \lim_{k\to\infty} \langle f(q) - q, J(x_{n_k} - q)\rangle. \tag{3.4}
$$

With no loss of generality we may assume that (x_{n_k}) converges weakly to a point $\hat{x}$ in C. Since the pool of the mappings of the family is finite, we may further assume that $[n_k] \equiv i$ for all k and some $i \in \{1, 2, \dots, N\}$ so that $T_{[n_k]} = T_i$ for all k. Let $U = T_{i+N} \dots T_{[i+1]}$. Then $F = \mathrm{Fix}(U)$ by (3.2). Moreover, it turns out from Step 4 that $\|x_{n_k} - Ux_{n_k}\| \to 0$ as $k \to \infty$.

By the uniqueness of the sunny nonexpansive retraction from Π_C onto F, we have that $q = Q(f) = \lim_{t\to 0} z_t$, where $z_t = tf(z_t) + (1-t)Uz_t$ for $0 < t < 1$. By Lemma 2.3, we get

$$\begin{aligned}\|z_t - x_{n_k}\|^2 &= \|(1-t)(Uz_t - x_{n_k}) + t(f(z_t) - x_{n_k})\|^2 \\ &\le (1-t)^2\|Uz_t - x_{n_k}\|^2 + 2t\langle f(z_t) - x_{n_k}, J(z_t - x_{n_k})\rangle \\ &\le (1-t)^2\left(\|Uz_t - Ux_{n_k}\| + \|Ux_{n_k} - x_{n_k}\|\right)^2 \\ &\quad + 2t\langle f(z_t) - z_t + z_t - x_{n_k}, J(z_t - x_{n_k})\rangle \\ &\le (1-t)^2\left(\|z_t - x_{n_k}\| + \|Ux_{n_k} - x_{n_k}\|\right)^2 \\ &\quad + 2t\left(\|z_t - x_{n_k}\|^2 + \langle f(z_t) - z_t, J(z_t - x_{n_k})\rangle\right) \\ &\le (1+t^2)\|z_t - x_{n_k}\|^2 + 2t\langle f(z_t) - z_t, J(z_t - x_{n_k})\rangle \\ &\quad + (1-t)^2\|Ux_{n_k} - x_{n_k}\|(\|Ux_{n_k} - x_{n_k}\| + 2\|z_t - x_{n_k}\|).\end{aligned}$$

It turns out that, for all k and $t \in (0,1)$,

$$\begin{aligned}&\langle f(z_t) - z_t, J(x_{n_k} - z_t)\rangle \\ &\le \frac{t}{2}\|z_t - x_{n_k}\|^2 + \frac{1}{2t}\|Ux_{n_k} - x_{n_k}\|(\|Ux_{n_k} - x_{n_k}\| + 2\|z_t - x_{n_k}\|).\end{aligned} \tag{3.5}$$

Let $\alpha > 0$ be a constant bigger than $\|z_t - x_{n_k}\|^2$ and $\|Ux_{n_k} - x_{n_k}\| + 2\|z_t - x_{n_k}\|$. Then (3.5) is reduced to, for all k and $t \in (0,1)$,

$$\langle f(z_t) - z_t, J(x_{n_k} - z_t)\rangle \le \alpha t + \frac{\alpha}{t}\|Ux_{n_k} - x_{n_k}\|. \tag{3.6}$$

Since $\|Ux_{n_k} - x_{n_k}\| \to 0$ as $k \to \infty$, we can take the limit by letting $k \to \infty$ in (3.6) to get, for all $t \in (0,1)$,

$$\limsup_{k\to\infty}\langle f(z_t) - z_t, J(x_{n_k} - z_t)\rangle \le \alpha t. \tag{3.7}$$

However, since $z_t \to q$ in norm and J is uniformly continuous in the norm topology over a bounded set that contains $(z_t) \cup (x_n)$, we can take the limit as $t \to 0$ in (3.7) to get

$$\limsup_{k\to\infty}\langle f(q) - q, J(x_{n_k} - q)\rangle \le 0.$$

This together with (3.4) verifies Step 5.

Step 6: $x_n \to Q(f)$ in norm (as $n \to \infty$). Here again Q is the unique sunny nonexpansive retraction from Π_C onto $F = \cap_{i=1}^N \mathrm{Fix}(T_i)$.

We will use Lemma 2.2 to prove Step 6. Let $q = Q(f)$. It follows from Lemma 2.3 that

$$\begin{aligned}\|x_{n+1} - q\|^2 &= \|\lambda_{n+1}(f(x_n) - q) + (1-\lambda_{n+1})(T_{[n+1]}x_n - q)\|^2 \\ &\le (1-\lambda_{n+1})^2\|T_{[n+1]}x_n - q\|^2 + 2\lambda_{n+1}\langle f(x_n) - q, J(x_{n+1} - q)\rangle \\ &\le (1-\lambda_{n+1})^2\|x_n - q\|^2 + 2\lambda_{n+1}\langle f(x_n) - f(q), J(x_{n+1} - q)\rangle\end{aligned}$$

$$
\begin{aligned}
&+ 2\lambda_{n+1}\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\le (1-\lambda_{n+1})^2\|x_n - q\|^2 + 2\lambda_{n+1}\gamma\|x_n - q\| \cdot \|J(x_{n+1} - q)\| \\
&\quad + 2\lambda_{n+1}\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&= (1-\lambda_{n+1})^2\|x_n - q\|^2 + 2\lambda_{n+1}\gamma\|x_n - q\| \cdot \|x_{n+1} - q\| \\
&\quad + 2\lambda_{n+1}\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\le (1-\lambda_{n+1})^2\|x_n - q\|^2 + \lambda_{n+1}\gamma(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) \\
&\quad + 2\lambda_{n+1}\langle f(q) - q, J(x_{n+1} - q)\rangle.
\end{aligned}
$$

It turns out that

$$\|x_{n+1} - q\|^2 \le \frac{(1-\lambda_{n+1})^2 + \gamma\lambda_{n+1}}{1-\gamma\lambda_{n+1}}\|x_n - q\|^2 + \frac{2\lambda_{n+1}}{1-\gamma\lambda_{n+1}}\langle f(q) - q, J(x_{n+1} - q)\rangle.$$

Setting $s_n = \|x_n - q\|^2$,

$$\alpha_n = 1 - \frac{(1-\lambda_{n+1})^2 + \gamma\lambda_{n+1}}{1-\gamma\lambda_{n+1}} = \frac{(2-\gamma-\lambda_{n+1})\lambda_{n+1}}{1-\gamma\lambda_{n+1}} = O(\lambda_{n+1}),$$

and

$$\sigma_n = \frac{2\lambda_{n+1}}{1-\gamma\lambda_{n+1}}\langle f(q) - q, J(x_{n+1} - q)\rangle,$$

we may rewrite the last inequality as

$$s_{n+1} \le (1-\alpha_n)s_n + \sigma_n. \tag{3.8}$$

It is evidently clear that $\alpha_n \to 0$ since $\lambda_n \to 0$. In addition, since

$$\frac{\sigma_n}{\alpha_n} = \frac{2}{2-\gamma-\lambda_{n+1}}\langle f(q) - q, J(x_{n+1} - q)\rangle,$$

it turns out from Step 5 that

$$\limsup_{n\to\infty}\frac{\sigma_n}{\alpha_n} = \frac{2}{2-\gamma}\limsup_{n\to\infty}\langle f(q) - q, J(x_{n+1} - q)\rangle \le 0.$$

Consequently, Lemma 2.2 is applicable and we obtain from (3.8) that $s_n \to 0$, namely, $x_n \to q$ in norm. The proof is complete. □

Remark 3.3. (i) The result of Theorem 2.1 remains valid if the uniform smoothness is replaced with the assumption that the space X either is uniformly Gateaux differentiable with the fixed point property for nonexpansive mappings or has a weakly continuous duality mapping with some gauge φ.

(ii) Theorem 2.1 has been proved in some papers with an additional condition that the space X be strictly convex as well (see [3–5, 12, 13]).

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