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WEAK CONVERGENCE THEOREMS FOR NEW MAPPINGS IN A BANACH SPACE

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Dedicated to Professor Jong Soo Jung on the occasion of his 65th birthday

Abstract. In a Hilbert space, concepts of attractive point and acute point studied by many researchers. Moreover these concepts extended to Banach space. In previous paper we introduced a new class of mappings on Banach space corresponding to the class of all widely more generalized hybrid mappings on Hilbert space. In this paper we introduce some extensions of weak convergence theorems.

1. Introduction

In [23] Takahashi and Takeuchi introduced a concept of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H. $x \in H$ is called an attractive point of T if

$$\|x - Ty\| \leq \|x - y\|$$

for any $y \in C$. Let

$$A(T) = \{x \in H \mid \|x - Ty\| \leq \|x - y\| \text{ for any } y \in C\}.$$

Moreover they proved that the Baillon type ergodic theorem [2] for generalized hybrid mappings [18] without convexity of C. A mapping T from C into H is said to be generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for any $x, y \in C$. Such a mapping is said to be (α, β)-generalized hybrid. The class of all generalized hybrid mappings is a new class of nonlinear mappings including nonexpansive mappings, nonspreading mappings [20] and hybrid mappings [22]. A mapping T from C into H is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for any $x, y \in C$; nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$; hybrid if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

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for any $x, y \in C$. Any nonexpansive mapping is $(1,0)$-generalized hybrid; any nonspreading mapping is $(2,1)$-generalized hybrid; any hybrid mapping is $\left(\frac{3}{2}, \frac{1}{2}\right)$-generalized hybrid.

Motivated these mappings, in [15] Kawasaki and Takahashi introduced a new very wider class of mappings, called widely more generalized hybrid mappings, than the class of all generalized hybrid mappings. A mapping T from C into H is widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ +\varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid. This class includes the class of all generalized hybrid mappings and also the class of all k-pseudo-contractions [3] for $k \in [0,1]$. A mapping T from C into H is said to be k-pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2$$

for any $x, y \in C$. Any (α, β)-generalized hybrid mapping is $(\alpha, 1-\alpha, -\beta, \beta-1, 0, 0, 0)$-widely more generalized hybrid; any k-pseudo-contraction is $(1, 0, 0, -1, 0, 0, -k)$-widely more generalized hybrid. Moreover they proved some fixed point theorems [5–10, 14–17] and some ergodic theorems [5, 6, 14–16].

There are some studies on Banach space related to these results. In [24] Takahashi, Wong and Yao introduced the generalized nonspreading mapping and the skew-generalized nonspreading mapping in a Banach space. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping T from C into E is said to be generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ &\leq \varepsilon(\phi(Ty, Tx) - \phi(Ty, x)) + \zeta(\phi(y, Tx) - \phi(y, x))\end{aligned}$$

for any $x, y \in C$, where J is the duality mapping on E and

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2.$$

Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-generalized nonspreading. A mapping T from C into E is said to be skew-generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ &\leq \varepsilon(\phi(Ty, Tx) - \phi(y, Tx)) + \zeta(\phi(Ty, x) - \phi(y, x))\end{aligned}$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-skew-generalized nonspreading. These classes include the class of generalized hybrid mappings in a Hilbert space, however, it does not include the class of widely more generalized hybrid mappings. Moreover they introduced some extensions of attractive point and proved some attractive point theorems. $x \in E$ is an attractive point of T if

$$\phi(x, Ty) \leq \phi(x, y)$$

for any $y \in C$; $x \in E$ is a skew-attractive point of T if

$$\phi(Ty, x) \le \phi(y, x)$$

for any $y \in C$. Let

$$\begin{aligned} A(T) &= \{x \in E \mid \phi(x, Ty) \le \phi(x, y) \text{ for any } y \in C\}; \\ B(T) &= \{x \in E \mid \phi(Ty, x) \le \phi(y, x) \text{ for any } y \in C\}. \end{aligned}$$

Let C be a nonempty subset of a smooth Banach space E. A mapping T from C into E is said to be generalized nonexpansive [4] if the set of all fixed points of T is nonempty and

$$\phi(Tx, y) \le \phi(x, y)$$

for any $x \in C$ and for any fixed point y of T. Let C be a nonempty subset of E of a Banach space E. A mapping R from E onto C is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for any $x \in E$ and for any $t \in [0, \infty)$. A mapping R from E onto C is called a retraction or a projection if $Rx = x$ for any $x \in C$.

Takahashi, Wong and Yao also proved the following weak convergence theorem.

Theorem 1.1. *Let E be a uniformly convex Banach space with a uniformly Fréchet differentiable norm, let C be a nonempty convex subset of E and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-generalized nonspreading mapping from C into itself satisfying $\alpha+\beta+\gamma+\delta \ge 0$ and $\alpha + \beta > 0$. Suppose that $A(T) = B(T) \neq \emptyset$. Let R be the sunny generalized nonexpansive retraction of E onto $B(T)$ and let $\{\alpha_n\}$ be a sequence of real numbers with $\alpha_n \in (0, 1)$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n$$

for any $n \in \mathbb{N}$ is weakly convergent to an element $q \in A(T)$, where $q = \lim_{n\to\infty} Rx_n$.

On the other hand, in [1] Atsushiba, Iemoto, Kubota and Takeuchi introduced a concept of acute point as an extension of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H and $k \in [0, 1]$. $x \in H$ is called a k-acute point of T if

$$\|x - Ty\|^2 \le \|x - y\|^2 + k\|y - Ty\|^2$$

for any $y \in C$. Let

$$\mathscr{A}_k(T) = \{x \in H \mid \|x - Ty\|^2 \le \|x - y\|^2 + k\|y - Ty\|^2 \text{ for any } y \in C\}.$$

Moreover, using a concept of acute point, they proved convergence theorems without convexity of C.

Motivated these results, in previous paper [11] we introduced a new class of mappings on Banach space corresponding to the class of all widely more generalized hybrid mappings on Hilbert space. In this paper we introduce some extensions of weak convergence theorems.

2. Preliminaries

We know that the following hold; for instance, see [21].

(T1) If a Banach space E is unformly convex, then E is reflexive.

(T2) Let E be a Banach space and let J be the duality mapping on E defined by

$$J(x) = \{x^* \in E^* \mid \|x\|^2 = \langle x, x^* \rangle = \|x^*\|^2\}$$

for any $x \in E$. Then E is strictly convex if and only if J is injective, that is, $x \neq y$ implies $J(x) \cap J(y) = \emptyset$.

(T3) Let E be a Banach space, let E^* be the topological dual space of E and let J be the duality mapping on E. Then E is reflexive if and only if J is surjective, that is, $\bigcup_{x \in E} J(x) = E^*$.

(T4) Let E be a Banach space and let J be the duality mapping on E. Then E is smooth if and only if J is single-valued.

(T5) Let E be a Banach space and let J be the duality mapping on E. Then

$$\langle x - y, x^* - y^* \rangle \geq 0$$

holds for any $x, y \in E$, for any $x^* \in J(x)$ and for any $y^* \in J(y)$.

(T6) Let E be a Banach space and let J be the duality mapping on E. If J is single-valued, then J is norm-to-weak* continuous.

(T7) Let E be a Banach space and let J be the duality mapping on E. If E has the Fréche differentiable norm, then J is norm-to-norm continuous.

(T8) Let E be a Banach space and let J be the duality mapping on E. Then E is strictly convex if and only if

$$1 - \langle x, y^* \rangle > 0$$

for any $x, y \in E$ with $x \neq y$ and $\|x\| = \|y\| = 1$ and for any $y^* \in J(y)$.

(T9) Let E be a Banach space and let E^* be the topological dual space of E. Then E is reflexive if and only if E^* is reflexive.

(T10) Let E be a Banach space and let E^* be the topological dual space of E. If E^* is strictly convex, then E is smooth. Conversely, E is reflexive and smooth, then E^* is strictly convex.

(T11) Let E be a Banach space and let E^* be the topological dual space of E. If E^* is smooth, then E is strictly convex. Conversely, E is reflexive and strictly convex, then E^* is smooth.

(T12) Let E be a Banach space and let E^* be the topological dual space of E. E has uniformly Frécht differentiable norm if and only if E^* is uniformly convex.

Let E be a smooth Banach space, let J be the duality mapping on E and let ϕ be the mapping from $E \times E$ into $[0, \infty)$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2$$

for any $x, y \in E$. Since by (T4) J is single-valued, ϕ is well-defined. It is obvious that $x = y$ implies $\phi(x, y) = 0$. Conversely, by (T8)

(T13) If E is also strictly convex, then $\phi(x, y) = 0$ implies $x = y$.

Let E be a strictly convex and smooth Banach space. By (T2) an (T4) J is a bijective mapping from E onto $J(E)$. In particular, if E is also reflective, then by (T3) J is a bijective mapping from E onto E^*. Suppose that E is strictly convex, reflective and smooth. Let ϕ_* be the mapping from $E^* \times E^*$ into $[0, \infty)$ defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for any $x^*, y^* \in E^*$. Then

$$\phi_*(x^*, y^*) = \phi(J^{-1}y^*, J^{-1}x^*) \tag{2.1}$$

holds. Therefore

(T13)* $\phi_*(x^*, y^*) = 0$ if and only if $x^* = y^*$.

We use the following lemmas in this paper.

The following showed in [25].

Lemma 2.1. *Let E be a uniformly convex Banach space and let $r \in (0, \infty)$. Then there exists a strictly increasing, continuous and convex function g from $[0, \infty)$ into $[0, \infty)$ with $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for any $x, y \in B_r \overset{def}{=} \{z \in E \mid \|z\| \leq r\}$ and for any $\lambda \in [0, 1]$.

The following showed in [4].

Lemma 2.2. *Let E be a strictly convex and smooth Banach space and let C be a nonempty closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction of E onto C. Then the sunny generalized nonexpansive retraction is uniquely determined.*

Lemma 2.3. *Let E be a strictly convex and smooth Banach space and let C be a nonempty closed subset of E. Suppose that there exists a sunny generalized nonexpansive retraction of E onto C. Then the following hold.*

(i) *$z = R_C x$ if and only if $\langle x - z, Jz - Jy\rangle \geq 0$ for any $y \in C$;*

(ii) *$\phi(R_C x, y) + \phi(x, R_C x) \leq \phi(x, y)$ for any $y \in C$.*

The following showed in [19].

Lemma 2.4. *Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(i) *There exists a sunny generalized nonexpansive retraction of E onto C;*

(ii) *There exists a generalized nonexpansive retraction of E onto C;*

(iii) *$J(C)$ is closed and convex.*

Lemma 2.5. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty closed subset of E and $(x, z) \in E \times C$. Suppose that there exists a sunny generalized nonexpansive retraction R_C of E onto C. Then the following are equivalent:*

(i) *$z = R_C x$;*

(ii) $\phi(x,z) = \min_{y\in C} \phi(x,y)$.

The following showed in [24].

Lemma 2.6. *Let E be a uniformly convex and smooth Banach space, let C be a nonempty convex subset of E and let T be a mapping from C into itself. Suppose that $B(T) \neq \emptyset$. Let R be the sunny generalized nonexpansive retraction of E onto $B(T)$, let $\{\alpha_n\}$ be a sequence of real numbers with $\alpha_n \in (0,1)$ and let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$$

for any $n \in \mathbb{N}$. Then $\{Rx_n\}$ is strongly convergent to an element in $B(T)$.

3. Acute point and skew-acute point

Most of this section are included in [11], however, the following are described for completeness.

Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k,\ell \in \mathbb{R}$. $x \in E$ is called a (k,ℓ)-acute point of T if

$$\phi(x,Ty) \le \phi(x,y) + k\phi(y,Ty) + \ell\phi(Ty,y) \tag{3.1}$$

for any $y \in C$. $x \in E$ is called a (k,ℓ)-skew-acute point of T if

$$\phi(Ty,x) \le \phi(y,x) + k\phi(y,Ty) + \ell\phi(Ty,y) \tag{3.2}$$

for any $y \in C$. Let

$$\begin{aligned}&\mathscr{A}_{k,\ell}(T)\\ &= \{x \in E \mid \phi(x,Ty) \le \phi(x,y) + k\phi(y,Ty) + \ell\phi(Ty,y) \text{ for any } y \in C\};\\ &\mathscr{B}_{k,\ell}(T)\\ &= \{x \in E \mid \phi(Ty,x) \le \phi(y,x) + k\phi(y,Ty) + \ell\phi(Ty,y) \text{ for any } y \in C\}.\end{aligned}$$

It is obvious that

$$\mathscr{A}_{k_1,\ell_1}(T) \subset \mathscr{A}_{k_2,\ell_2}(T),\ \mathscr{B}_{k_1,\ell_1}(T) \subset \mathscr{B}_{k_2,\ell_2}(T)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{R}$ with $k_1 \le k_2$ and $\ell_1 \le \ell_2$.

The following lemmas are important property characterizing them.

Lemma 3.1. *Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k,\ell \in \mathbb{R}$. Then $\mathscr{A}_{k,\ell}(T)$ is closed and convex.*

Proof. (3.1) is equivalent to

$$2\langle x, Jy - JTy\rangle \le (k-1)\phi(y,Ty) + \ell\phi(Ty,y) + 2\langle y, Jy - JTy\rangle.$$

Since

$$\phi(u,v) = \phi(u,w) + \phi(w,v) + 2\langle u-w, Jw - Jv\rangle \tag{3.3}$$

for any $u,v,w \in E$, $\mathscr{A}_{k,\ell}(T)$ is closed and convex. □

Lemma 3.2. *Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k,\ell \in \mathbb{R}$. Then $\mathscr{B}_{k,\ell}(T)$ is closed.*

Proof. (3.2) is equivalent to

$$2\langle y - Ty, Jx\rangle \leq k\phi(y, Ty) + (\ell - 1)\phi(Ty, y) + 2\langle y - Ty, Jy\rangle$$

from (3.3). Moreover by (T6) J is norm-to-weak* continuous. Therefore $\mathscr{B}_{k,\ell}(T)$ is closed. □

Let E^* be the dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell \in \mathbb{R}$. $x^* \in E^*$ is called a (k, ℓ)-*-acute point of T^* if

$$\phi_*(x^*, T^*y^*) \leq \phi_*(x^*, y^*) + k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \tag{3.4}$$

for any $y^* \in C^*$. $x^* \in E^*$ is called a (k, ℓ)-*-skew-acute point of T^* if

$$\phi_*(T^*y^*, x^*) \leq \phi_*(y^*, x^*) + k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \tag{3.5}$$

for any $y^* \in C^*$. Let

$$\begin{aligned}&\mathscr{A}^*_{k,\ell}(T^*)\\ &= \left\{ x^* \in E^* \;\middle|\; \begin{array}{l} \phi_*(x^*, T^*y^*) \leq \phi_*(x^*, y^*) + k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \\ \text{for any } y^* \in C^* \end{array} \right\};\\ &\mathscr{B}^*_{k,\ell}(T^*)\\ &= \left\{ x^* \in E^* \;\middle|\; \begin{array}{l} \phi_*(T^*y^*, x^*) \leq \phi_*(y^*, x^*) + k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) \\ \text{for any } y^* \in C^* \end{array} \right\}.\end{aligned}$$

Lemma 3.3. *Let E^* be the dual space of a strictly convex, reflective and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell \in \mathbb{R}$. Then $\mathscr{A}^*_{k,\ell}(T^*)$ is closed and convex.*

Proof. (3.4) is equivalent to

$$\begin{aligned}&2\langle J^{-1}y^* - J^{-1}T^*y^*, x^*\rangle\\ &\quad \leq (k-1)\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) + 2\langle J^{-1}y^* - J^{-1}T^*y^*, y^*\rangle\end{aligned}$$

from (3.3) and (2.1), $\mathscr{A}^*_{k,\ell}(T^*)$ is closed and convex. □

Lemma 3.4. *Let E^* be the dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell \in \mathbb{R}$. Then $\mathscr{B}^*_{k,\ell}(T^*)$ is closed.*

Proof. (3.5) is equivalent to

$$\begin{aligned}&2\langle J^{-1}x^*, y^* - T^*y^*\rangle\\ &\quad \leq k\phi_*(y^*, T^*y^*) + (\ell - 1)\phi_*(T^*y^*, y^*) + 2\langle J^{-1}y^*, y^* - T^*y^*\rangle\end{aligned}$$

from (3.3) and (2.1). Moreover by (T6) J^{-1} is norm-to-weak* continuous. Therefore $\mathscr{B}^*_{k,\ell}(T^*)$ is closed. □

Lemma 3.5. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E, let $T^* = JTJ^{-1}$ and let $k, \ell \in \mathbb{R}$. Then*

$$\mathscr{A}^*_{k,\ell}(T^*) = J(\mathscr{B}_{\ell,k}(T)),\ \mathscr{B}^*_{k,\ell}(T^*) = J(\mathscr{A}_{\ell,k}(T)).$$

In particular, $J(\mathscr{B}_{k,\ell}(T))$ is closed and convex and $J(\mathscr{A}_{k,\ell}(T))$ is closed.

Proof. Let $x^* \in \mathscr{A}^*_{k,\ell}(T^*)$. Then

$$\phi_*(x^*, T^*y^*) \le \phi_*(x^*, y^*) + k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*)$$

for any $y^* \in J(C)$. From (2.1)

$$\begin{aligned}&\phi(J^{-1}T^*y^*, J^{-1}x^*)\\ &\quad \le \phi(J^{-1}y^*, J^{-1}x^*) + k\phi(J^{-1}T^*y^*, J^{-1}y^*) + \ell\phi(J^{-1}y^*, J^{-1}T^*y^*)\end{aligned}$$

for any $y^* \in J(C)$. Since $J^{-1}T^* = TJ^{-1}$, putting $y = J^{-1}y^*$, we obtain

$$\phi(Ty, J^{-1}x^*) \le \phi(y, J^{-1}x^*) + \ell\phi(y, Ty) + k\phi(Ty, y).$$

Therefore $J^{-1}x^* \in \mathscr{B}_{\ell,k}(T)$ and hence $\mathscr{A}^*_{k,\ell}(T^*) = J(\mathscr{B}_{\ell,k}(T))$.

$\mathscr{B}^*_{k,\ell}(T^*) = J(\mathscr{A}_{\ell,k}(T))$ can be shown similarly.

Moreover, by Lemma 3.3 $J(\mathscr{B}_{k,\ell}(T))$ is closed and convex and by Lemma 3.4 $J(\mathscr{A}_{k,\ell}(T))$ is closed. □

Lemma 3.6. *Let E be a strictly convex and smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell \in \mathbb{R}$. Then the following hold.*

(1) *If $(k, \ell) \in (-\infty, 1] \times (-\infty, 0] \setminus \{(1, 0)\}$, then $C \cap \mathscr{A}_{k,\ell}(T)$ is included in the set of all fixed points of T;*

(2) *If $(k, \ell) \in (-\infty, 0] \times (-\infty, 1] \setminus \{(0, 1)\}$, then $C \cap \mathscr{B}_{k,\ell}(T)$ is included in the set of all fixed points of T.*

Proof. Let $x \in C \cap \mathscr{A}_{k,\ell}(T)$. Then (3.1) holds for any $y \in C$. Putting $y = x$, we obtain $(1-k)\phi(x, Tx) - \ell\phi(Tx, x) \le 0$. If $(k, \ell) \in (-\infty, 1] \times (-\infty, 0] \setminus \{(1, 0)\}$, then by (T13) we obtain $x = Tx$.

Let $x \in C \cap \mathscr{B}_{k,\ell}(T)$. Then (3.2) holds for any $y \in C$. Putting $y = x$, we obtain $-k\phi(x, Tx) + (1-\ell)\phi(Tx, x) \le 0$. If $(k, \ell) \in (-\infty, 0] \times (-\infty, 1] \setminus \{(0, 1)\}$, then by (T13) we obtain $x = Tx$. □

Lemma 3.7. *Let E^* be a strictly convex and smooth topological dual space of a Banach space, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell \in \mathbb{R}$. Then the following hold.*

(1) *If $(k, \ell) \in (-\infty, 1] \times (-\infty, 0] \setminus \{(1, 0)\}$, then $C \cap \mathscr{A}^*_{k,\ell}(T^*)$ is included in the set of all fixed points of T^*;*

(2) *If $(k, \ell) \in (-\infty, 0] \times (-\infty, 1] \setminus \{(0, 1)\}$, then $C \cap \mathscr{B}^*_{k,\ell}(T^*)$ is included in the set of all fixed points of T^*.*

Proof. Let $x^* \in C^* \cap \mathscr{A}^*_{k,\ell}(T^*)$. Then (3.4) holds for any $y^* \in C^*$. Putting $y^* = x^*$, by we obtain $(1-k)\phi_*(x^*, T^*x^*) - \ell\phi_*(T^*x^*, x^*) \le 0$. If $(k, \ell) \in (-\infty, 1] \times (-\infty, 0] \setminus \{(1, 0)\}$, then by (T13)* we obtain $x^* = T^*x^*$.

Let $x^* \in C^* \cap \mathscr{B}^*_{k,\ell}(T^*)$. Then (3.5) holds for any $y^* \in C^*$. Putting $y^* = x^*$, by we obtain $-k\phi_*(x^*, T^*x^*) + (1-\ell)\phi_*(T^*x^*, x^*) \le 0$. If $(k, \ell) \in (-\infty, 0] \times (-\infty, 1] \setminus \{(0, 1)\}$, then by (T13)* we obtain $x^* = T^*x^*$. □

4. Weak convergence theorems

Lemma 4.1. *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm and let $\{x_n\}$ and $\{y_n\}$ be sequences in E. If $\{x_n\}$ is bounded and $\{x_n - y_n\}$ is strongly convergent to 0, then $\{Jx_n - Jy_n\}$ is weakly convergent to 0.*

Proof. Since $\{x_n\}$ is bounded and $\{x_n - y_n\}$ is strongly convergent to 0, $\{y_n\}$ is also bounded. Firstly we show in the case of $\{x_n\}, \{y_n\} \subset S(E) \overset{\text{def}}{=} \{z \in E \mid \|z\| = 1\}$. Note that

$$\lim_{t \to 0} \frac{\|x + ty\| - 1}{t} = \langle y, Jx \rangle$$

for any $x, y \in S(E)$ if the norm $\|\cdot\|$ of E is Gâteaux differentiable. Since the norm $\|\cdot\|$ of E is uniformly Gâteaux differentiable, we have that, for any $w \in E$ with $w \neq 0$ and for any $\varepsilon > 0$, there exists $\delta > 0$ such that, if $0 < |t| < \delta$, then

$$\left| \frac{\left\| x_n + \frac{t}{\|w\|} w \right\| - 1}{t} - \frac{1}{\|w\|} \langle w, Jx_n \rangle \right| < \varepsilon,$$

$$\left| \frac{\left\| y_n + \frac{t}{\|w\|} w \right\| - 1}{t} - \frac{1}{\|w\|} \langle w, Jy_n \rangle \right| < \varepsilon$$

for any $n \in \mathbb{N}$. Therefore we obtain

$$\left\| x_n + \frac{t}{\|w\|} w \right\| < 1 + t\varepsilon + \frac{t}{\|w\|} \langle w, Jx_n, \rangle,$$

$$\left\| x_n - \frac{t}{\|w\|} w \right\| < 1 + t\varepsilon - \frac{t}{\|w\|} \langle w, Jx_n, \rangle,$$

$$\left\| y_n + \frac{t}{\|w\|} w \right\| < 1 + t\varepsilon + \frac{t}{\|w\|} \langle w, Jy_n, \rangle,$$

$$\left\| y_n - \frac{t}{\|w\|} w \right\| < 1 + t\varepsilon - \frac{t}{\|w\|} \langle w, Jy_n, \rangle$$

for any $t \in (0, \delta)$. Since

$$\begin{aligned} |\langle x_n, Jy_n \rangle - 1| &= |\langle x_n - y_n, Jy_n \rangle| \\ &\leq \|x_n - y_n\|, \\ |\langle y_n, Jx_n \rangle - 1| &= |\langle y_n - x_n, Jx_n \rangle| \\ &\leq \|y_n - x_n\|, \end{aligned}$$

we have that there exists $N \in \mathbb{N}$ such that

$$|\langle x_n, Jy_n \rangle - 1| < t\varepsilon,$$
$$|\langle y_n, Jx_n \rangle - 1| < t\varepsilon$$

for any $n > N$. Therefore

$$-t\varepsilon \quad < \quad \langle x_n, Jy_n \rangle - 1$$

$$\begin{aligned} &= \left\langle x_n + \frac{t}{\|w\|}w, Jy_n \right\rangle + \left\langle x_n - \frac{t}{\|w\|}w, Jx_n \right\rangle - \frac{t}{\|w\|}\langle w, Jy_n - Jx_n\rangle - 2 \\ &\leq \left\| x_n + \frac{t}{\|w\|}w \right\| + \left\| x_n - \frac{t}{\|w\|}w \right\| - \frac{t}{\|w\|}\langle w, Jy_n - Jx_n\rangle - 2 \\ &< 2t\varepsilon - \frac{t}{\|w\|}\langle w, Jy_n - Jx_n\rangle \end{aligned}$$

and hence

$$\langle w, Jy_n - Jx_n\rangle < 3\|w\|\varepsilon;$$

$$\begin{aligned} -t\varepsilon &< \langle y_n, Jx_n\rangle - 1 \\ &= \left\langle y_n + \frac{t}{\|w\|}w, Jx_n \right\rangle + \left\langle y_n - \frac{t}{\|w\|}w, Jy_n \right\rangle - \frac{t}{\|w\|}\langle w, Jx_n - Jy_n\rangle - 2 \\ &\leq \left\| y_n + \frac{t}{\|w\|}w \right\| + \left\| y_n - \frac{t}{\|w\|}w \right\| - \frac{t}{\|w\|}\langle w, Jx_n - Jy_n\rangle - 2 \\ &< 2t\varepsilon - \frac{t}{\|w\|}\langle w, Jx_n - Jy_n\rangle \end{aligned}$$

and hence

$$\langle w, Jx_n - Jy_n\rangle < 3\|w\|\varepsilon.$$

Therefore we obtain

$$|\langle w, Jx_n - Jy_n\rangle| < 3\|w\|\varepsilon$$

and hence $\{Jx_n - Jy_n\}$ is weakly convergent to 0. In the general case, if $x_n = 0$ or $y_n = 0$, then

$$\begin{aligned} |\langle w, Jx_n - Jy_n\rangle| &\leq \|w\|\|Jx_n - Jy_n\| \\ &= \|w\|\|x_n - y_n\|; \end{aligned}$$

otherwise

$$\begin{aligned} &|\langle w, Jx_n - Jy_n\rangle| \\ &\quad = \left| \left\langle w, \|x_n\| \left(J\left(\frac{1}{\|x_n\|}x_n\right) - J\left(\frac{1}{\|y_n\|}y_n\right)\right) + (\|x_n\| - \|y_n\|)J\left(\frac{1}{\|y_n\|}y_n\right) \right\rangle \right| \\ &\quad \leq \|x_n\| \left| \left\langle w, J\left(\frac{1}{\|x_n\|}x_n\right) - J\left(\frac{1}{\|y_n\|}y_n\right) \right\rangle \right| + \|w\| \big| \|x_n\| - \|y_n\| \big| \\ &\quad \leq \|x_n\| \left| \left\langle w, J\left(\frac{1}{\|x_n\|}x_n\right) - J\left(\frac{1}{\|y_n\|}y_n\right) \right\rangle \right| + \|w\|\|x_n - y_n\|. \end{aligned}$$

Since $\{x_n\}$ is bounded, $\left\{ J\left(\frac{1}{\|x_n\|}x_n\right) - J\left(\frac{1}{\|y_n\|}y_n\right) \right\}$ is weakly convergent to 0 and $\{x_n - y_n\}$ is strongly convergent to 0, $\{Jx_n - Jy_n\}$ is weakly convergent to 0. □

Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping T from C into E is called a generalized pseudocontraction [11] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1\phi(Tx,Ty) + \alpha_2\phi(Ty,Tx) + \beta_1\phi(x,Ty) + \beta_2\phi(Ty,x) \\ &\quad +\gamma_1\phi(Tx,y) + \gamma_2\phi(y,Tx) + \delta_1\phi(x,y) + \delta_2\phi(y,x) \\ &\quad +\varepsilon_1\phi(Tx,x) + \varepsilon_2\phi(x,Tx) + \zeta_1\phi(y,Ty) + \zeta_2\phi(Ty,y) \\ &\le 0 \end{aligned} \tag{4.1}$$

for any $x, y \in C$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction.

Lemma 4.2. *Let E be a smooth Banach space, let C be a nonempty subset of E, let D be a nonempty convex subset of E, let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into D and let $\lambda \in [0,1]$. Then T is a $((1-\lambda)\alpha_1+\lambda\alpha_2), \lambda\alpha_1+(1-\lambda)\alpha_2, (1-\lambda)\beta_1+\lambda\gamma_2, \lambda\gamma_1+(1-\lambda)\beta_2, (1-\lambda)\gamma_1+\lambda\beta_2, \lambda\beta_1+(1-\lambda)\gamma_2, (1-\lambda)\delta_1+\lambda\delta_2, \lambda\delta_1+(1-\lambda)\delta_2, (1-\lambda)\varepsilon_1+\lambda\zeta_2, \lambda\zeta_1+(1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1+\lambda\varepsilon_2, \lambda\varepsilon_1+(1-\lambda)\zeta_2)$-generalized pseudocontraction from C into D.*

Proof. Changing the variables x and y in (4.1), we obtain

$$\begin{aligned} &\alpha_2\phi(Tx,Ty) + \alpha_1\phi(Ty,Tx) + \gamma_2\phi(x,Ty) + \gamma_1\phi(Ty,x) \\ &\quad +\beta_2\phi(Tx,y) + \beta_1\phi(y,Tx) + \delta_2\phi(x,y) + \delta_1\phi(y,x) \\ &\quad +\zeta_2\phi(Tx,x) + \zeta_1\phi(x,Tx) + \varepsilon_2\phi(y,Ty) + \varepsilon_1\phi(Ty,y) \\ &\le 0. \end{aligned} \tag{4.2}$$

Adding (4.1) multiplied by $1-\lambda$ and (4.2) multiplied by λ, we obtain

$$\begin{aligned} &((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty) + (\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx) \\ &\quad +((1-\lambda)\beta_1+\lambda\gamma_2)\phi(x,Ty) + (\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x) \\ &\quad +((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y) + (\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx) \\ &\quad +((1-\lambda)\delta_1+\lambda\delta_2)\phi(x,y) + (\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x) \\ &\quad +((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x) + (\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx) \\ &\quad +((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi(y,Ty) + (\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y) \\ &\le 0. \end{aligned}$$

Therefore T is a $((1-\lambda)\alpha_1+\lambda\alpha_2), \lambda\alpha_1+(1-\lambda)\alpha_2, (1-\lambda)\beta_1+\lambda\gamma_2, \lambda\gamma_1+(1-\lambda)\beta_2, (1-\lambda)\gamma_1+\lambda\beta_2, \lambda\beta_1+(1-\lambda)\gamma_2, (1-\lambda)\delta_1+\lambda\delta_2, \lambda\delta_1+(1-\lambda)\delta_2, (1-\lambda)\varepsilon_1+\lambda\zeta_2, \lambda\zeta_1+(1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1+\lambda\varepsilon_2, \lambda\varepsilon_1+(1-\lambda)\zeta_2)$-generalized pseudocontraction. □

Theorem 4.3. *Let E be a strictly convex Banach space with a uniformly Gâteaux differentiable norm, let C be a nonempty subset of E, let $\{x_n\}$ be a sequence in C and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into E. Suppose that there exists $\lambda \in [0,1]$ such that*

$$(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1) + \lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \ge 0;$$
$$\lambda(\alpha_1+\gamma_1) + (1-\lambda)(\alpha_2+\beta_2) \ge 0;$$
$$\lambda(\beta_1+\delta_1) + (1-\lambda)(\gamma_2+\delta_2) \ge 0;$$
$$(1-\lambda)\varepsilon_1 + \lambda\zeta_2 \ge 0;$$

$$\lambda\zeta_1 + (1-\lambda)\varepsilon_2 \geq 0;$$
$$(1-\lambda)(\alpha_1+\beta_1) + \lambda(\alpha_2+\gamma_2) > 0.$$

If $\{x_n\}$ is weakly convergent to q and $\{x_n - Tx_n\}$ is strongly convergent to 0*, then*

$$q \in \mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T).$$

In particular, any fixed point of T belongs to

$$\mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T).$$

Proof. By Lemma 4.2 T is a $((1-\lambda)\alpha_1+\lambda\alpha_2), \lambda\alpha_1+(1-\lambda)\alpha_2, (1-\lambda)\beta_1+\lambda\gamma_2, \lambda\gamma_1+(1-\lambda)\beta_2, (1-\lambda)\gamma_1+\lambda\beta_2, \lambda\beta_1+(1-\lambda)\gamma_2, (1-\lambda)\delta_1+\lambda\delta_2, \lambda\delta_1+(1-\lambda)\delta_2, (1-\lambda)\varepsilon_1+\lambda\zeta_2, \lambda\zeta_1+(1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1+\lambda\varepsilon_2, \lambda\varepsilon_1+(1-\lambda)\zeta_2)$-generalized pseudocontraction. From (3.3) we obtain

$$\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty) + (\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&\quad +((1-\lambda)\beta_1+\lambda\gamma_2)\phi(x,Ty) + (\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&\quad +((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y) + (\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&\quad +((1-\lambda)\delta_1+\lambda\delta_2)\phi(x,y) + (\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&\quad +((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x) + (\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&\quad +((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi(y,Ty) + (\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&= ((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty) + (\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&\quad -((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(x,Ty)\\
&\quad +((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))(\phi(x,y)+\phi(y,Ty)+2\langle x-y, Jy-JTy\rangle)\\
&\quad +(\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&\quad +((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y) + (\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&\quad +((1-\lambda)\delta_1+\lambda\delta_2)\phi(x,y) + (\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&\quad +((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x) + (\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&\quad +((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi(y,Ty) + (\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&= ((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty) + (\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&\quad -((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(x,Ty) + (\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&\quad +((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y) + (\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&\quad +((1-\lambda)(\alpha_1+\beta_1+\delta_1)+\lambda(\alpha_2+\gamma_2+\delta_2))\phi(x,y)\\
&\quad +(\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&\quad +((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x) + (\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&\quad +((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad +(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad +2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y, Jy-JTy\rangle.
\end{aligned}$$

Since

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\delta_1)+\lambda(\alpha_2+\beta_2+\delta_2) \geq -((1-\lambda)\gamma_1+\lambda\gamma_2);\\
&\lambda\gamma_1+(1-\lambda)\beta_2 \geq -(\lambda\alpha_1+(1-\lambda)\alpha_2);\\
&\lambda\delta_1+(1-\lambda)\delta_2 \geq -(\lambda\beta_1+(1-\lambda)\gamma_2);\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2 \geq 0,
\end{aligned}$$

we obtain

$$\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty)+(\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&\quad -((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(x,Ty)+(\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&\quad +((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y)+(\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&\quad +((1-\lambda)(\alpha_1+\beta_1+\delta_1)+\lambda(\alpha_2+\gamma_2+\delta_2))\phi(x,y)\\
&\quad +(\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&\quad +((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x)+(\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&\quad +((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad +(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad +2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y,Jy-JTy\rangle\\
&\geq ((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi(Tx,Ty)-\phi(x,Ty))\\
&\quad +(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi(Ty,Tx)-\phi(Ty,x))\\
&\quad +((1-\lambda)\gamma_1+\lambda\beta_2)(\phi(Tx,y)-\phi(x,y))\\
&\quad +(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi(y,Tx)-\phi(y,x))\\
&\quad +((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad +(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad +2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y,Jy-JTy\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi(Tx,Ty)-\phi(x,Ty))\\
&\quad +(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi(Ty,Tx)-\phi(Ty,x))\\
&\quad +((1-\lambda)\gamma_1+\lambda\beta_2)(\phi(Tx,y)-\phi(x,y))\\
&\quad +(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi(y,Tx)-\phi(y,x))\\
&\quad +((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad +(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad +2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y,Jy-JTy\rangle\\
&\leq 0.
\end{aligned}$$

Let $\{x_n\}$ be a sequence in C. Suppose that $\{x_n\}$ is weakly convergent to q and $\{x_n - Tx_n\}$ is strongly convergent to 0. Replacing x by x_n, we obtain

$$((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi(Tx_n,Ty)-\phi(x_n,Ty))$$

$$
\begin{aligned}
&+(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi(Ty,Tx_n)-\phi(Ty,x_n))\\
&+((1-\lambda)\gamma_1+\lambda\beta_2)(\phi(Tx_n,y)-\phi(x_n,y))\\
&+(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi(y,Tx_n)-\phi(y,x_n))\\
&+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x_n-y,Jy-JTy\rangle\\
\leq{}& 0.
\end{aligned}
$$

From (3.3) we obtain

$$
\begin{aligned}
\phi(Tx_n,w)-\phi(x_n,w)&=2\langle x_n-Tx_n,Jw\rangle+\|Tx_n\|^2-\|x_n\|^2,\\
\phi(w,Tx_n)-\phi(w,x_n)&=2\langle w,Jx_n-JTx_n\rangle+\|Tx_n\|^2-\|x_n\|^2.
\end{aligned}
$$

Since $\{x_n-Tx_n\}$ is strongly convergent to 0, we obtain

$$\lim_{n\to\infty}\langle x_n-Tx_n,Jw\rangle=0.$$

Since $\{x_n\}$ is weakly convergent, $\{x_n\}$ is bounded. Moreover, since $\{x_n-Tx_n\}$ is strongly convergent to 0, by Lemma 4.1 we obtain

$$\lim_{n\to\infty}\langle w,Jx_n-JTx_n\rangle=0.$$

Since

$$
\begin{aligned}
\big|\|Tx_n\|^2-\|x_n\|^2\big| &= (\|Tx_n\|+\|x_n\|)\big|\|Tx_n\|-\|x_n\|\big|\\
&\leq (\|Tx_n\|+\|x_n\|)\|Tx_n-x_n\|
\end{aligned}
$$

and $\{x_n-Tx_n\}$ is strongly convergent to 0, we obtain

$$\lim_{n\to\infty}(\|Tx_n\|^2-\|x_n\|^2)=0.$$

Therefore we obtain

$$
\begin{aligned}
&((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle q-y,Jy-JTy\rangle\\
&\leq 0.
\end{aligned}
$$

From (3.3) we obtain

$$
\begin{aligned}
&((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi(y,Ty)+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))(\phi(q,Ty)-\phi(q,y))\\
&\leq 0.
\end{aligned}
$$

Since $(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)>0$, we obtain

$$
\begin{aligned}
\phi(q,Ty)\leq\phi(q,y)&-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}\phi(y,Ty)\\
&-\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}\phi(Ty,y)
\end{aligned}
$$

and hence

$$q \in \mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)},-\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T).$$

□

Theorem 4.4. *Let E be a uniformly convex Banach space with a uniformly Fréchet differentiable norm, let C be a nonempty convex subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \geq 0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2) \geq 0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2 \geq 0;\\
&(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2) > 0,
\end{aligned}$$

and suppose that

$$\mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)},-\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T) \subset B(T) \neq \emptyset.$$

Let R be the sunny generalized nonexpansive retraction of E onto $B(T)$ and let $\{\alpha_n\}$ be a sequence of real numbers with $\alpha_n \in (0, 1)$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n$$

for any $n \in \mathbb{N}$ is weakly convergent to an element

$$q \in \mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)},-\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T),$$

where $q = \lim_{n\to\infty} Rx_n$.

Additionally, if C is closed and one of the following holds:

(1) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) > 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 \geq 0$;

(2) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) \geq 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 > 0$,

then q is a fixed point of T.

Proof. By the assumption E is strictly convex and smooth, and by (T1) E is reflexive. By Lemma 3.2 $B(T)$ is closed and by Lemma 3.5 $J(B(T))$ is closed and convex. Therefore by Lemmas 2.4 and 2.2 there exists a unique sunny nonexpansive retraction R of E onto $B(T)$.

Let $z \in B(T)$. Then we obtain

$$\begin{aligned}
\phi(x_{n+1}, z) &= \phi(\alpha_n x_n + (1-\alpha_n)Tx_n, z)\\
&\leq \alpha_n\phi(x_n, z) + (1-\alpha_n)\phi(Tx_n, z)\\
&\leq \alpha_n\phi(x_n, z) + (1-\alpha_n)\phi(x_n, z)\\
&= \phi(x_n, z).
\end{aligned}$$

Therefore $\{\phi(x_n, z)\}$ is non-increasing and hence $\lim_{n\to\infty} \phi(x_n, z)$ exists. Moreover $\{x_n\}$ is bounded and $\{Tx_n\}$ is also bounded. Put $r = \sup_{n\in\mathbb{N}}\{\|x_n\|, \|Tx_n\|\}$. By Lemma 2.1 we obtain

$$\begin{aligned}
\phi(x_{n+1}, z) &= \phi(\alpha_n x_n + (1-\alpha_n)Tx_n, z) \\
&= \|\alpha_n x_n + (1-\alpha_n)Tx_n\|^2 - 2\langle \alpha_n x_n + (1-\alpha_n)Tx_n, Jz\rangle + \|z\|^2 \\
&\leq \alpha_n\|x_n\|^2 + (1-\alpha_n)\|Tx_n\|^2 - \alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|) \\
&\quad -2\langle \alpha_n x_n + (1-\alpha_n)Tx_n, Jz\rangle + \|z\|^2 \\
&= \alpha_n(\|x_n\|^2 - 2\langle x_n, Jz\rangle + \|z\|^2) \\
&\quad +(1-\alpha_n)(\|Tx_n\|^2 - 2\langle Tx_n, Jz\rangle + \|z\|^2) \\
&\quad -\alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|) \\
&= \alpha_n\phi(x_n, z) + (1-\alpha_n)\phi(Tx_n, z) - \alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|) \\
&\leq \phi(x_n, z) - \alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|).
\end{aligned}$$

Therefore we obtain

$$\alpha_n(1-\alpha_n)g(\|x_n - Tx_n\|) \leq \phi(x_n, z) - \phi(x_{n+1}, z).$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we obtain

$$\lim_{n\to\infty} g(\|x_n - Tx_n\|) = 0.$$

From the properties of g we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is weakly convergent to an element $p \in E$. Let $\{x_{n_j}\}$ be an another subsequence of $\{x_n\}$ and suppose that $\{x_{n_j}\}$ is weakly convergent to $p_1 \in E$. By Theorem 4.3

$$p, p_1 \in \mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T).$$

Since

$$\mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T) \subset B(T),$$

$\lim_{n\to\infty} \phi(x_n, p)$ and $\lim_{n\to\infty} \phi(x_n, p_1)$ exist. Put $a = \lim_{n\to\infty}(\phi(x_n, p) - \phi(x_n, p_1))$. Since

$$\phi(x_n, p) - \phi(x_n, p_1) = 2\langle x_n, Jp_1 - Jp\rangle + \|p\|^2 - \|p_1\|^2,$$

we obtain

$$\begin{aligned}
a &= 2\langle p, Jp_1 - Jp\rangle + \|p\|^2 - \|p_1\|^2, \\
a &= 2\langle p_1, Jp_1 - Jp\rangle + \|p\|^2 - \|p_1\|^2.
\end{aligned}$$

Therefore we obtain

$$\langle p - p_1, Jp_1 - Jp\rangle = 0.$$

From (3.3) we obtain

$$-\phi(p, p_1) - \phi(p_1, p) = 0.$$

By (T13) we obtain $p_1 = p$ and hence $\{x_n\}$ is weakly convergent to

$$p \in \mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T).$$

On the other hand, by Lemma 2.6 $\{Rx_n\}$ is strongly convergent to an element $q \in B(T)$. By Lemma 2.3 we obtain

$$\langle x_n - Rx_n, JRx_n - Ju\rangle \geq 0$$

for any $u \in B(T)$. Since by (T7) J is norm-to-norm continuous and

$$p \in \mathscr{A}_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T) \subset B(T),$$

we obtain

$$\langle p - q, Jq - Jp\rangle \geq 0.$$

From (3.3) we obtain

$$-\phi(p, q) - \phi(q, p) \geq 0.$$

By (T13) we obtain $p = q$.

Additionally, if C is closed and (1) or (2) holds, then $p \in C$. By Lemma 3.6 $q = p$ is a fixed point of T. □

Let E^* be the dual space of a strictly convex, reflexive and smooth Banach space E and let C^* be a nonempty subset of E^*. A mapping T^* from C^* into E^* is called a *-generalized pseudocontraction [11] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_1\phi_*(T^*x^*, T^*y^*) + \alpha_2\phi_*(T^*y^*, T^*x^*) + \beta_1\phi_*(x^*, T^*y^*) + \beta_2\phi_*(T^*y^*, x^*) \\
&\quad +\gamma_1\phi_*(T^*x^*, y^*) + \gamma_2\phi_*(y^*, T^*x^*) + \delta_1\phi_*(x^*, y^*) + \delta_2\phi_*(y^*, x^*) \\
&\quad +\varepsilon_1\phi_*(T^*x^*, x^*) + \varepsilon_2\phi_*(x^*, T^*x^*) + \zeta_1\phi_*(y^*, T^*y^*) + \zeta_2\phi_*(T^*y^*, y^*) \\
&\quad \leq 0
\end{aligned}$$

for any $x^*, y^* \in C^*$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-*-generalized pseudocontraction.

Lemma 4.5. *Let E^* be the dual space of a strictly convex, reflexive and smooth Banach space E, let C^* and D^* be nonempty subsets of E^*, let T^* be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-*-generalized pseudocontraction from C^* into D^* and let $\lambda \in [0, 1]$. Then T^* is a $((1-\lambda)\alpha_1 + \lambda\alpha_2), \lambda\alpha_1 + (1-\lambda)\alpha_2, (1-\lambda)\beta_1 + \lambda\gamma_2, \lambda\gamma_1 + (1-\lambda)\beta_2, (1-\lambda)\gamma_1 + \lambda\beta_2, \lambda\beta_1 + (1-\lambda)\gamma_2, (1-\lambda)\delta_1 + \lambda\delta_2, \lambda\delta_1 + (1-\lambda)\delta_2, (1-\lambda)\varepsilon_1 + \lambda\zeta_2, \lambda\zeta_1 + (1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1 + \lambda\varepsilon_2, \lambda\varepsilon_1 + (1-\lambda)\zeta_2)$-*-generalized pseudocontraction from C^* into D^*.*

Proof. The proof is similar to the proof of Lemma 4.2. □

Theorem 4.6. *Let E^* be the dual space of a strictly convex, reflexive and smooth Banach space E, where E^* has a uniformly Gâteaux differentiable norm, let C^* be a nonempty subset of E^*, let $\{x_n^*\}$ be a sequence in C^* and let T^* be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-*-generalized pseudocontraction from C^* into E^*. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \geq 0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2) \geq 0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2 \geq 0;\\
&(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2) > 0.
\end{aligned}$$

If $\{x_n^*\}$ is weakly convergent to q^* and $\{x_n^* - T^*x_n^*\}$ is strongly convergent to 0, then

$$q^* \in \mathscr{A}^*_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)},-\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T^*).$$

In particular, any fixed point of T^* belongs to

$$\mathscr{A}^*_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)},-\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T^*).$$

Proof. By Lemma 4.5 T^* is a $((1-\lambda)\alpha_1+\lambda\alpha_2), \lambda\alpha_1+(1-\lambda)\alpha_2, (1-\lambda)\beta_1+\lambda\gamma_2, \lambda\gamma_1+(1-\lambda)\beta_2, (1-\lambda)\gamma_1+\lambda\beta_2, \lambda\beta_1+(1-\lambda)\gamma_2, (1-\lambda)\delta_1+\lambda\delta_2, \lambda\delta_1+(1-\lambda)\delta_2, (1-\lambda)\varepsilon_1+\lambda\zeta_2, \lambda\zeta_1+(1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1+\lambda\varepsilon_2, \lambda\varepsilon_1+(1-\lambda)\zeta_2)$-*-generalized pseudocontraction. From (2.1) and (3.3) we obtain

$$\phi_*(u^*, v^*) = \phi_*(u^*, w^*) + \phi_*(w^*, v^*) + 2\langle J^{-1}w^* - J^{-1}v^*, u^* - w^*\rangle. \tag{4.3}$$

Note that by (T10) E^* is strictly convex. Therefore we obtain similarly to the proof of Theorem 4.3

$$\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi_*(T^*x^*, T^*y^*) - \phi_*(x^*, T^*y^*))\\
&\quad+(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi_*(T^*y^*, T^*x^*) - \phi_*(T^*y^*, x^*))\\
&\quad+((1-\lambda)\gamma_1+\lambda\beta_2)(\phi_*(T^*x^*, y^*) - \phi_*(x^*, y^*))\\
&\quad+(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi_*(y^*, T^*x^*) - \phi_*(y^*, x^*))\\
&\quad+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi_*(y^*, T^*y^*)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi_*(T^*y^*, y^*)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle J^{-1}y^* - J^{-1}T^*y^*, x^* - y^*\rangle\\
&\leq 0.
\end{aligned}$$

Let $\{x_n^*\}$ be a sequence in C. Suppose that $\{x_n^*\}$ is weakly convergent to q^* and $\{x_n^* - T^*x_n^*\}$ is strongly convergent to 0. Replacing x^* by x_n^*, we obtain

$$\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi_*(T^*x_n^*, T^*y^*) - \phi_*(x_n^*, T^*y^*))\\
&\quad+(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi_*(T^*y^*, T^*x_n^*) - \phi_*(T^*y^*, x_n^*))\\
&\quad+((1-\lambda)\gamma_1+\lambda\beta_2)(\phi_*(T^*x_n^*, y^*) - \phi_*(x_n^*, y^*))
\end{aligned}$$

$$\begin{aligned}
&+(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi_*(y^*,T^*x_n^*)-\phi_*(y^*,x_n^*))\\
&+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi_*(y^*,T^*y^*)\\
&+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi_*(T^*y^*,y^*)\\
&+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle J^{-1}y^*-J^{-1}T^*y^*,x_n^*-y^*\rangle\\
\le{}& 0.
\end{aligned}$$

From (4.3) we obtain

$$\begin{aligned}
\phi_*(T^*x_n^*,w^*)-\phi_*(x_n^*,w^*)&=2\langle J^{-1}w^*,x_n^*-T^*x_n^*\rangle+\|T^*x_n^*\|^2-\|x_n^*\|^2,\\
\phi_*(w^*,T^*x_n^*)-\phi_*(w^*,x_n^*)&=2\langle J^{-1}x_n^*-J^{-1}T^*x_n^*,w^*\rangle+\|T^*x_n^*\|^2-\|x_n^*\|^2.
\end{aligned}$$

Since $\{x_n^*-T^*x_n^*\}$ is strongly convergent to 0, we obtain

$$\lim_{n\to\infty}\langle J^{-1}w^*,x_n^*-T^*x_n^*\rangle=0.$$

Moreover by Lemma 4.1 we obtain

$$\lim_{n\to\infty}\langle J^{-1}x_n^*-J^{-1}T^*x_n^*,w^*\rangle=0.$$

Since $\{x_n^*\}$ is weakly convergent, $\{x_n^*\}$ is bounded. Moreover, since $\{x_n^*-T^*x_n^*\}$ is strongly convergent to 0, $\{T^*x_n^*\}$ is also bounded. Since

$$\begin{aligned}
\big|\|T^*x_n^*\|^2-\|x_n^*\|^2\big| &= (\|T^*x_n^*\|+\|x_n^*\|)\big|\|T^*x_n^*\|-\|x_n^*\|\big|\\
&\le (\|T^*x_n^*\|+\|x_n^*\|)\|T^*x_n^*-x_n^*\|
\end{aligned}$$

and $\{x_n^*-T^*x_n^*\}$ is strongly convergent to 0, we obtain

$$\lim_{n\to\infty}(\|T^*x_n^*\|^2-\|x_n^*\|^2)=0.$$

Therefore we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi_*(y^*,T^*y^*)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi_*(T^*y^*,y^*)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle q^*-y^*,J^{-1}y^*-J^{-1}T^*y^*\rangle\\
&\le 0.
\end{aligned}$$

From (4.3) we obtain

$$\begin{aligned}
&((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi_*(y^*,T^*y^*)+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi_*(T^*y^*,y^*)\\
&\quad+((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))(\phi_*(q^*,T^*y^*)-\phi_*(q^*,y^*))\\
&\le 0.
\end{aligned}$$

Since $(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)>0$, we obtain

$$\begin{aligned}
\phi_*(q^*,T^*y^*)\le\phi_*(q^*,y^*)&-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}\phi_*(y^*,T^*y^*)\\
&-\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}\phi_*(T^*y^*,y^*)
\end{aligned}$$

and hence

$$q^*\in\mathscr{A}^*_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)},-\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T).$$

□

Theorem 4.7. *Let E^* be a uniformly convex topological dual space with a uniformly Fréchet differentiable norm, let C^* be a nonempty convex subset of E^* and let T^* be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-*-generalized pseudocontraction from C^* into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \ge 0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2) \ge 0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2) \ge 0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2 \ge 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2 \ge 0;\\
&(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2) > 0,
\end{aligned}$$

and suppose that

$$\mathscr{A}^*_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)},-\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T^*) \subset \mathscr{B}^*_{0,0}(T^*) \ne \emptyset.$$

Let R^* be the sunny generalized nonexpansive retraction of E^* onto $\mathscr{B}^*_{0,0}(T^*)$ and let $\{\alpha_n\}$ be a sequence of real numbers with $\alpha_n \in (0, 1)$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then a sequence $\{x^*_n\}$ generated by $x^*_1 = x^* \in C^*$ and

$$x^*_{n+1} = \alpha_n x^*_n + (1-\alpha_n)T^*x^*_n$$

for any $n \in \mathbb{N}$ is weakly convergent to an element

$$q^* \in \mathscr{A}^*_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)},-\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T^*),$$

where $q^* = \lim_{n\to\infty} R^*x^*_n$.

Additionally, if C^* is closed and one of the following holds:

(1) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) > 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 \ge 0$*;*

(2) $(1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2) \ge 0$ *and* $\lambda\varepsilon_1+(1-\lambda)\zeta_2 > 0$*,*

then q^* is a fixed point of T^*.

Proof. By (T1) and (T12) E is a uniformly convex Banach space with a uniformly Fréchet differentiable norm. Therefore ϕ_* is well-defined. By the assumption E^* is strictly convex and smooth, and by (T1) E^* is reflexive. By Lemma 3.2 $\mathscr{B}^*_{0,0}(T^*)$ is closed and by Lemma 3.5 $J(\mathscr{B}^*_{0,0}(T^*))$ is closed and convex. Therefore by Lemmas 2.4 and 2.2 there exists a unique sunny nonexpansive retraction R^* of E^* onto $\mathscr{B}^*_{0,0}(T^*)$.

Let $z^* \in \mathscr{B}^*_{0,0}(T^*)$. Then we obtain

$$\begin{aligned}
\phi_*(x^*_{n+1}, z^*) &= \phi_*(\alpha_n x^*_n + (1-\alpha_n)T^*x^*_n, z^*)\\
&\le \alpha_n\phi_*(x^*_n, z^*) + (1-\alpha_n)\phi_*(T^*x^*_n, z^*)\\
&\le \alpha_n\phi_*(x^*_n, z^*) + (1-\alpha_n)\phi_*(x^*_n, z^*)\\
&= \phi_*(x^*_n, z^*).
\end{aligned}$$

Therefore $\{\phi_*(x_n^*, z^*)\}$ is non-increasing and hence $\lim_{n\to\infty} \phi_*(x_n^*, z^*)$ exists. Moreover $\{x_n^*\}$ is bounded and $\{T^*x_n^*\}$ is also bounded. Put $r = \sup_{n\in\mathbb{N}}\{\|x_n^*\|, \|T^*x_n^*\|\}$. By Lemma 2.1 we obtain

$$\begin{aligned}
&\phi_*(x_{n+1}^*, z^*)\\
&\quad = \phi_*(\alpha_n x_n^* + (1-\alpha_n)T^*x_n^*, z^*)\\
&\quad = \|\alpha_n x_n^* + (1-\alpha_n)T^*x_n^*\|^2 - 2\langle J^{-1}z^*, \alpha_n x_n^* + (1-\alpha_n)T^*x_n^*\rangle + \|z^*\|^2\\
&\quad \le \alpha_n\|x_n^*\|^2 + (1-\alpha_n)\|T^*x_n^*\|^2 - \alpha_n(1-\alpha_n)g(\|x_n^* - T^*x_n^*\|)\\
&\qquad -2\langle J^{-1}z^*, \alpha_n x_n + (1-\alpha_n)T^*x_n^*\rangle + \|z^*\|^2\\
&\quad = \alpha_n(\|x_n^*\|^2 - 2\langle J^{-1}z^*, x_n^*\rangle + \|z^*\|^2)\\
&\qquad +(1-\alpha_n)(\|T^*x_n^*\|^2 - 2\langle J^{-1}z^*, T^*x_n^*\rangle + \|z^*\|^2)\\
&\qquad -\alpha_n(1-\alpha_n)g(\|x_n^* - T^*x_n^*\|)\\
&\quad = \alpha_n\phi_*(x_n^*, z^*) + (1-\alpha_n)\phi_*(T^*x_n^*, z^*) - \alpha_n(1-\alpha_n)g(\|x_n^* - T^*x_n^*\|)\\
&\quad \le \phi_*(x_n^*, z^*) - \alpha_n(1-\alpha_n)g(\|x_n^* - T^*x_n^*\|).
\end{aligned}$$

Therefore we obtain

$$\alpha_n(1-\alpha_n)g(\|x_n^* - T^*x_n^*\|) \le \phi_*(x_n^*, z^*) - \phi_*(x_{n+1}^*, z^*).$$

Since $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we obtain

$$\lim_{n\to\infty} g(\|x_n^* - T^*x_n^*\|) = 0.$$

By (T1) E^* is reflexive. Since $\{x_n^*\}$ is bounded, there exists a subsequence $\{x_{n_i}^*\}$ of $\{x_n^*\}$ such that $\{x_{n_i}^*\}$ is weakly convergent to an element $p^* \in E^*$. Let $\{x_{n_j}^*\}$ be an another subsequence of $\{x_n^*\}$ and suppose that $\{x_{n_j}^*\}$ is weakly convergent to $p_1^* \in E^*$. By Theorem 4.6

$$p^*, p_1^* \in \mathscr{A}^*_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T).$$

Since

$$\mathscr{A}^*_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T^*) \subset \mathscr{B}^*_{0,0}(T^*),$$

$\lim_{n\to\infty} \phi_*(x_n^*, p^*)$ and $\lim_{n\to\infty} \phi_*(x_n^*, p_1^*)$ exist. Put $a = \lim_{n\to\infty}(\phi_*(x_n^*, p^*) - \phi_*(x_n^*, p_1^*))$. Since

$$\phi_*(x_n^*, p^*) - \phi_*(x_n^*, p_1^*) = 2\langle J^{-1}p_1^* - J^{-1}p^*, x_n^*\rangle + \|p^*\|^2 - \|p_1^*\|^2,$$

we obtain

$$\begin{aligned}
a &= 2\langle J^{-1}p_1^* - J^{-1}p^*, p^*\rangle + \|p^*\|^2 - \|p_1^*\|^2,\\
a &= 2\langle J^{-1}p_1^* - J^{-1}p^*, p_1^*\rangle + \|p^*\|^2 - \|p_1^*\|^2.
\end{aligned}$$

Therefore we obtain

$$\langle J^{-1}p_1^* - J^{-1}p^*, p^* - p_1^*\rangle = 0.$$

From (4.3) we obtain

$$-\phi_*(p^*, p_1^*) - \phi_*(p_1^*, p^*) = 0.$$

By (T13)* we obtain $p_1^* = p^*$ and hence $\{x_n^*\}$ is weakly convergent to

$$p^* \in \mathscr{A}^*_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T^*).$$

On the other hand, by Lemma 2.6 $\{R^* x_n^*\}$ is strongly convergent to an element $q^* \in \mathscr{B}^*_{0,0}(T^*)$. By Lemma 2.3 we obtain

$$\langle J^{-1}R^*x_n^* - J^{-1}u^*, x_n^* - R^*x_n^*\rangle \geq 0$$

for any $u^* \in \mathscr{B}^*_{0,0}(T^*)$. Since by (T7) J^{-1} is norm-to-norm continuous and

$$p^* \in \mathscr{A}^*_{-\frac{(1-\lambda)\zeta_1+\lambda\varepsilon_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}, -\frac{\lambda\varepsilon_1+(1-\lambda)\zeta_2}{(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2)}}(T^*) \subset \mathscr{B}^*_{0,0}(T^*),$$

we obtain

$$\langle J^{-1}q^* - J^{-1}p^*, p^* - q^*\rangle \geq 0.$$

From (4.3) we obtain

$$-\phi_*(p^*, q^*) - \phi_*(q^*, p^*) \geq 0.$$

By (T13) we obtain $p^* = q^*$.

Additionally, if C^* is closed and (1) or (2) holds, then $p^* \in C^*$. By Lemma 3.7 $q^* = p^*$ is a fixed point of T^*. □

By Theorem 4.7 we obtain the following.

Theorem 4.8. *Let E be a uniformly convex Banach space with a uniformly Fréchet differentiable norm, let C be a nonempty subset of E satisfying $J(C)$ is convex and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2)+\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) \geq 0;\\
&\lambda(\alpha_2+\gamma_2)+(1-\lambda)(\alpha_1+\beta_1) \geq 0;\\
&\lambda(\beta_2+\delta_2)+(1-\lambda)(\gamma_1+\delta_1) \geq 0;\\
&(1-\lambda)\varepsilon_2+\lambda\zeta_1 \geq 0;\\
&\lambda\zeta_2+(1-\lambda)\varepsilon_1 \geq 0;\\
&(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1) > 0,
\end{aligned}$$

suppose that

$$\mathscr{B}_{-\frac{\lambda\varepsilon_2+(1-\lambda)\zeta_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}, -\frac{(1-\lambda)\zeta_2+\lambda\varepsilon_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}}(T) \subset A(T) \neq \emptyset$$

and suppose that J^{-1} is weakly sequentially continuous. Let R^* be the sunny generalized nonexpansive retraction of E^* onto $J(A(T))$ and let $\{\alpha_n\}$ be a sequence of real numbers with $\alpha_n \in (0, 1)$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then a sequence $\{x_n\}$ generated by $x_1 = x \in C$ and

$$x_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n)$$

for any $n \in \mathbb{N}$ is weakly convergent to an element

$$q \in \mathscr{B}_{-\frac{\lambda\varepsilon_2+(1-\lambda)\zeta_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}, -\frac{(1-\lambda)\zeta_2+\lambda\varepsilon_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}}(T),$$

where $q = \lim_{n\to\infty} J^{-1}R^*Jx_n$.

Additionally, if $J(C)$ *is closed and one of the following holds:*

(1) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2)+\lambda(\alpha_1+\gamma_1+\varepsilon_1)>0$ *and* $\lambda\varepsilon_2+(1-\lambda)\zeta_1\geq 0$*;*

(2) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2)+\lambda(\alpha_1+\gamma_1+\varepsilon_1)\geq 0$ *and* $\lambda\varepsilon_2+(1-\lambda)\zeta_1>0$*,*

then q *is a fixed point of* T*.*

Proof. By (T1) and (T12) E^* is uniformly convex with a uniformly Fréchet differentiable norm. Let $T^* = JTJ^{-1}$. Then T^* is a mapping from $J(C)$ into itself. Putting $x^* = Jx$ and $y^* = Jy$, (4.1) is equivalent to

$$\begin{aligned}
&\alpha_2\phi_*(T^*x^*,T^*y^*)+\alpha_1\phi_*(T^*y^*,T^*x^*)+\beta_2\phi_*(x^*,T^*y^*)+\beta_1\phi_*(T^*y^*,x^*)\\
&\quad+\gamma_2\phi_*(T^*x^*,y^*)+\gamma_1\phi_*(y^*,T^*x^*))+\delta_2\phi_*(x^*,y^*)+\delta_1\phi_*(y^*,x^*\\
&\quad+\varepsilon_2\phi_*(T^*x^*,x^*)+\varepsilon_1\phi_*(x^*,T^*x^*)+\zeta_2\phi_*(y^*,T^*y^*)+\zeta_1\phi_*(T^*y^*,y^*)\\
&\quad\leq 0
\end{aligned}$$

from (2.1). Therefore T^* is an $(\alpha_2,\alpha_1,\beta_2,\beta_1,\gamma_2,\gamma_1,\delta_2,\delta_1,\varepsilon_2,\varepsilon_1,\zeta_2,\zeta_1)$-*-generalized pseudocontraction from $J(C)$ into itself. Since $(T^*)^nx^* = JT^nx$, $\|(T^*)^nx^*\| = \|JT^nx\| = \|T^nx\|$ and hence $\{(T^*)^nx^* \mid n\in\mathbb{N}\cup\{0\}\}$ is bounded. By Lemma 3.5

$$\begin{aligned}
&\mathscr{A}^*_{-\frac{(1-\lambda)\zeta_2+\lambda\varepsilon_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)},-\frac{\lambda\varepsilon_2+(1-\lambda)\zeta_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_2+\gamma_2)}}(T^*)\\
&\qquad= J\left(\mathscr{B}_{-\frac{\lambda\varepsilon_2+(1-\lambda)\zeta_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_2+\gamma_2)},-\frac{(1-\lambda)\zeta_2+\lambda\varepsilon_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}}(T)\right),\\
&B^*_{0,0}(T^*) = J(A(T)).
\end{aligned}$$

By Theorem 4.7 for any $x\in C$, $\{Jx_n\}$ is weakly convergent to an element

$$q^*\in J\left(\mathscr{B}_{-\frac{\lambda\varepsilon_2+(1-\lambda)\zeta_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_2+\gamma_2)},-\frac{(1-\lambda)\zeta_2+\lambda\varepsilon_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}}(T)\right),$$

where $q^* = \lim_{n\to\infty} R^*Jx_n$. Since J^{-1} is weakly sequentially continuous and by (T7) J^{-1} is norm-to-norm continuous, $\{x_n\}$ is weakly convergent to the element

$$q = J^{-1}q^*\in\mathscr{B}_{-\frac{\lambda\varepsilon_2+(1-\lambda)\zeta_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_2+\gamma_2)},-\frac{(1-\lambda)\zeta_2+\lambda\varepsilon_1}{(1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1)}}(T),$$

where $q = \lim_{n\to\infty} J^{-1}R^*Jx_n$.

Additionally, if $J(C)$ is closed and (1) or (2) holds, then q^* is a fixed point of T^* and hence $q = J^{-1}q^*$ is a fixed point of T. □

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