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APPROXIMATE OPTIMALITY CONDITIONS FOR ROBUST CONVEX OPTIMIZATION WITHOUT CONVEXITY OF CONSTRAINTS

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Dedicated to Professor Jong Soo Jung on the occasion of his 65th birthday

Abstract. This paper dedicates to the representation of the feasible set in robust convex optimization problems. We consider a convex optimization problem which minimizes a convex function over a convex feasible set, which is given by locally Lipschitz constraints (not necessarily convex or differentiable) in the face of data uncertainty. Using the robust optimization approach (worst-case approach), we study the Karush–Kuhn–Tucker optimality conditions for the robust convex optimization problem under a non-degeneracy condition and the Slater constraint qualification. Finally, we apply the obtained results to study the KKT optimality conditions for a quasi ϵ-solution to the robust convex optimization problem.

1. Introduction

A standard constrained convex programming problem is minimizing a convex function over a convex feasible set C, which is usually given by convex inequality constraints, that is,

$$\text{(P)} \qquad \min_{x\in\mathbb{R}^n} f(x) \ \text{ subject to } \ x \in C,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $C := \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \ldots, m\}$, here $g_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \ldots, m$ are convex functions.

However, the convex feasible set C of problem (P) may not be described by convex inequality constraints. In 2010, Lasserre [16] studied a convex optimization problem, whose objective function is differentiable and convex, and constraint functions are differentiable but not necessarily convex (surely the feasible set shall be convex), and obtained the Karush–Kuhn–Tucker (KKT) optimality conditions (both necessary and sufficient) with the help of the Slater constraint qualification and an additional non-degeneracy condition. Motivated by the aforesaid work of Lasserre [16], in 2013, Dutta and Lalitha [11] extended the study to a nonsmooth scenario involving the locally Lipschitz functions, say concretely, they considered a convex optimization problem, whose objective function is convex (not necessarily differentiable) and the constraint functions are locally Lipschitz (not necessarily convex or differentiable). They showed that if the Slater constraint qualification and a simple non-degeneracy condition were satisfied then the KKT type optimality condition was both necessary

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and sufficient. Moreover, in 2016, Yamamoto and Kuroiwa [21] investigated several constraint qualifications for KKT optimality conditions in convex optimization with locally Lipschitz inequality constraints.

Very recently, the topic of convex optimization without convexity of constraints has also been studied in [5, 13, 19, 20]. It is worth mentioning that, among them, Sisarat *et al.* [20] obtained some results on the representation of the feasible set in robust convex optimization problems; in addition, they also gave the KKT optimality conditions for such a *robust convex optimization problem.*

In this paper, we mainly apply some results of Sisarat *et al.* [20] to study the KKT optimality conditions for a quasi ϵ-solution to the robust convex optimization problem.

1.1. **Problem Statement.** Consider the following convex optimization problem:

$$\text{(CP)} \qquad \min\ f(x)\ \text{ s.t. }\ g_i(x) \le 0,\ i = 1, \dots, m,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function and $g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, are locally Lipschitz functions such that the set $S_i := \{x \in R^n : g_i(x) \le 0\}$ is convex, and then the feasible set $S = \cap_{i=1}^m S_i$ is also convex. Recently many researchers have studied the convex programs of the above form and have obtained some interesting results; see, for example, [11, 21].

The convex optimization problem (CP) in the face of data uncertainty in the constraints can be written by the following problem:

$$\text{(UCP)} \qquad \min\ f(x)\ \text{ s.t. }\ g_i(x, v_i) \le 0, i = 1, \dots, m,$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function, $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $g_i(\cdot, v_i)$ is a locally Lipschitz function and $g_i(x, \cdot)$ is a concave function, and $v_i \in \mathbb{R}^q$ is an uncertain parameter which belongs to the compact convex set $\mathcal{V}_i \subset \mathbb{R}^q$, $i = 1, \dots, m$.

In this work, we treat the robust approach for (UCP), which is the worst case approach for (UCP); see, for example, [1, 2, 3, 4, 18]. Now, we associate with (UCP) its robust counterpart:

$$\text{(RCP)} \qquad \min\ f(x)\ \text{ s.t. }\ g_i(x, v_i) \le 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \dots, m.$$

Denote by $F := \{x \in \mathbb{R}^n : g_i(x, v_i) \le 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \dots, m\}$ as the feasible set of (RCP), and assume here the feasible set F is convex. Set $F = \bigcap_{i=1}^m \bigcap_{v_i \in \mathcal{V}_i} F_i(v_i)$, where $F_i(v_i) := \{x \in \mathbb{R}^n : g_i(x, v_i) \le 0\}$, $v_i \in \mathcal{V}_i$, $i = 1, \dots, m$.

Let $x \in F$, $I := \{1, \dots, m\}$ and define functions $\psi_i \colon \mathbb{R}^n \to \mathbb{R}$ by $\psi_i(x) := \max\{g_i(x, v_i) : v_i \in \mathcal{V}_i\}$, $i \in I$. Let $I(x) := \{i \in I : \psi_i(x) = 0\}$. We put for each $i \in I(x)$,

$$\mathcal{V}_i(x) := \{v_i \in \mathcal{V}_i : g_i(x, v_i) = \psi_i(x)\}.$$

1.2. **Non-degeneracy condition.**

Definition 1.1. Consider the problem (RCP). We say that the *non-degeneracy condition* holds at $x \in F$ if for all $i \in I(x)$ and all $v_i \in \mathcal{V}_i(x)$

$$0 \notin \partial^\circ g_i(x, v_i).$$

The feasible set F is said to satisfy the *non-degeneracy condition* if it holds for every $x \in F$.

Remark 1.2. This condition was introduced firstly by Lasserre [16] in the case that g_i is differentiable. Motivated by this idea, Dutta and Lalitha [11] extended the non-degeneracy condition to the nonsmooth case.

2. Preliminaries

In this section, we recall some notations and give preliminary results for next sections. Throughout this paper, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ and the associated Euclidean norm $\| \cdot \|$. We say that a set Γ in $\mathbb{R}^n$ is *convex* whenever $\mu a_1 + (1-\mu)a_2 \in \Gamma$ for all $\mu \in [0,1]$, $a_1, a_2 \in \Gamma$. We denote the domain of f by $\operatorname{dom} f$, that is, $\operatorname{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$. f is said to be *convex* if for all $\lambda \in [0,1]$,

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y)$$

for all $x, y \in \mathbb{R}^n$. The function f is said to be *concave* whenever $-f$ is convex. The (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^n \mid \langle x^*, y - x \rangle \leq f(y) - f(x), \ \forall y \in \mathbb{R}^n\}, \text{ if } x \in \operatorname{dom} f, \\ \emptyset, \text{ otherwise.} \end{cases}$$

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function, that is, for each $x \in \mathbb{R}^n$, there exist an open neighborhood U and a constant $L > 0$ such that for all y and z in U,

$$|g(y) - g(z)| \leq L\|y - z\|.$$

Definition 2.1. For each $d \in \mathbb{R}^n$, the *Clarke directional derivative* of g at $x \in \mathbb{R}^n$ in the direction d, denoted by $g^\circ(x; d)$, is given by

$$g^\circ(x; d) \ = \limsup_{h \to 0, \ t \to 0+} \frac{g(x + h + td) - g(x + h)}{t}.$$

We also denote the usual *one-sided directional derivative* of g at x by $g'(x; d)$. Thus

$$g'(x; d) = \lim_{t \to 0+} \frac{g(x + td) - g(x)}{t},$$

whenever this limit exists.

Definition 2.2. The *Clarke subdifferential* of g at x, denoted by $\partial^\circ g(x)$, is the (nonempty) set of all ξ in $\mathbb{R}^n$ satisfying the following condition:

$$g^\circ(x; d) \geq \langle \xi, d \rangle, \quad \text{for all} \ \ d \in \mathbb{R}^n.$$

We summarize some fundamental results in the calculus of the Clarke subdifferential (for more details, see [6, 7, 8, 9, 17]):

- $\partial^\circ g(x)$ is a nonempty, convex, compact subset of $\mathbb{R}^n$;
- The function $d \mapsto g^\circ(x; d)$ is convex;
- For every d in $\mathbb{R}^n$, one has

$$g^\circ(x; d) = \max\{\langle \xi, d \rangle : \xi \in \partial^\circ g(x)\}. \tag{2.1}$$

Let $\mathcal{V} \subset \mathbb{R}^q$ be a compact set and let $g\colon \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$ be a given function. Here after all, we assume that the following assumptions hold:

- (A1) $g(x, v)$ is upper semicontinuous in (x, v).
- (A2) g is locally Lipschitz in x, uniformly for v in $\mathcal{V}$, that is, for each $x \in \mathbb{R}^n$, there exist an open neighborhood U of x and a constant $L > 0$ such that for all y and z in U, and $v \in \mathcal{V}$,

$$|g(y, v) - g(z, v)| \leq L\|y - z\|.$$

- (A3) $g_x^\circ(x, v; \cdot) = g_x'(x, v; \cdot)$, the derivatives being with respect to x.

We define a function $\psi\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\psi(x) := \max\{g(x, v) : v \in \mathcal{V}\},$$

and observe that our assumptions (A1)-(A2) imply that ψ is defined and finite (with the maximum defining ψ attained) on $\mathbb{R}^n$. Let

$$\mathcal{V}(x) := \{v \in \mathcal{V} : g(x, v) = \psi(x)\},$$

then for each $x \in \mathbb{R}^n$, $\mathcal{V}(x)$ is a nonempty closed set.

The following lemma, which is a nonsmooth version of Danskin's theorem [10] for max-functions, makes connection between the functions $\psi'(x; d)$ and $g_x^\circ(x, v; d)$.

Lemma 2.3. *Under the assumptions* (A1)–(A3), *the usual one-sided directional derivative* $\psi'(x; d)$ *exists, and satisfies*

$$\begin{aligned} \psi'(x; d) = \psi^\circ(x; d) &= \max\{g_x^\circ(x, v; d) : v \in \mathcal{V}(x)\} \\ &= \max\{\langle \xi, d\rangle : \xi \in \partial_x^\circ g(x, v),\ v \in \mathcal{V}(x)\}. \end{aligned}$$

Proof. See [8, Theorem 2] (see also [6, Theorem 2.1], [10]). □

The following result will be useful in the sequel.

Lemma 2.4. [18] *In addition to the basic assumptions* (A1)–(A3), *suppose that* $\mathcal{V}$ *is convex, and that* $g(x, \cdot)$ *is concave on* $\mathcal{V}$, *for each* $x \in U$. *Then the following statements hold*:

(i) *The set* $\mathcal{V}(x)$ *is convex and compact.*
(ii) *The set*

$$\partial_x^\circ g(x, \mathcal{V}(x)) := \{\xi : \exists v \in \mathcal{V}(x) \ \ \textit{such that} \ \ \xi \in \partial_x^\circ g(x, v)\}$$

is convex and compact.
(iii) $\partial^\circ \psi(x) = \{\xi : \exists v \in \mathcal{V}(x) \ \ \textit{such that} \ \ \xi \in \partial_x^\circ g(x, v)\}$.

It is worth noting that the concavity of $g_i(x, \cdot)$ plays an important role, since our main results (Theorem 3.1, 3.2 and 3.6) shall be obtained with the aid of the above Lemma 2.4.

3. Main results

This section presents our main results. An equivalent characterization of the convex set F under the robust counterpart scenario is given as well as a robust KKT optimality theorem for (RCP) with the help of the Slater constraint qualification and the non-degeneracy condition (see also Sisarat *et al.* [20]). Then, we apply the obtained result to the study of KKT optimality condition for a quasi ϵ-solution of (RCP). Some simple examples are also provided to illustrate the results.

3.1. **KKT optimality theorem.** First, the Slater constraint qualification along with the non-degeneracy condition gives the following equivalent characterization of the convex set F under the robust counterpart scenario.

Theorem 3.1. *Let F be given in the problem* (RCP). *Assume that each g_i satisfies the assumptions* (A1)–(A3). *Moreover, assume that the non-degeneracy condition holds at $x \in F$, and the Slater constraint qualification also holds, that is, there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0, v_i) < 0$, for all $v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$. Then F is convex if and only if for all $i \in I(x)$, there exists $\bar{v}_i \in \mathcal{V}_i(x)$ such that*

$$g_{ix}^{\circ}\big(x, \bar{v}_i; y - x\big) \leq 0, \text{ for all } x, y \in F. \tag{3.1}$$

Proof. First, let us define $\psi_i(x) := \max_{v_i \in \mathcal{V}_i} g_i(x, v_i)$, $i = 1, 2, \ldots, m$. Since g_i, $i = 1, 2, \ldots, m$ satisfy the assumptions (A1) and (A2), $\psi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are locally Lipschitz functions. By Lemma 2.3, we know that $\psi'(x; y-x) = \psi^{\circ}(x; y-x)$. Moreover, by Lemma 2.4 (iii), we have for all $i \in I(x)$, $0 \notin \partial^{\circ}\psi_i(x)$. Clearly, ψ_i satisfies the Slater constraint qualification. Therefore, according to [11, Proposition 1], F is convex if and only if for all $i \in I(x)$,

$$\psi_i^{\circ}(x; y - x) \leq 0 \text{ for all } x, y \in F.$$

In addition, by Lemma 2.3, $\psi_i^{\circ}(x; y - x) = \max\{g_{ix}^{\circ}(x, v_i; y - x) : v_i \in \mathcal{V}_i(x)\}$. It means that there exists $\bar{v}_i \in \mathcal{V}_i(x)$ such that $\psi_i^{\circ}(x; y - x) = g_{ix}^{\circ}(x, v_i; y - x)$. Thus, we conclude that F is convex if and only if for all $i \in I(x)$, there exists $\bar{v}_i \in \mathcal{V}_i(x)$ such that (3.1) holds. $\square$

The following result is a robust KKT optimality theorem for (RCP), which is a robust version of [11, Theorem 2.4].

Theorem 3.2. *Let us consider the problem* (RCP). *Assume that each g_i satisfies the assumptions* (A1)–(A3). *Moreover assume that the non-degeneracy condition holds at $\bar{x} \in F$, and the Slater constraint qualification also holds. Then $\bar{x} \in F$ is an optimal solution of f over F if and only if there exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i(\bar{x})$, $i = 1, \ldots, m$, such that*

$$0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial_x^{\circ} g_i(\bar{x}, \bar{v}_i), \tag{3.2}$$

$$0 = \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i), \ i = 1, \ldots, m. \tag{3.3}$$

Proof. Define $\psi_i(x) := \max_{v_i \in \mathcal{V}_i} g_i(x, v_i)$, $i = 1, 2, \ldots, m$. Then, based on [11, Theorem 2.4], $\bar{x} \in F$ is an optimal solution of (RCP) if and only if there exists $\bar{\lambda}_i$, $i = 1, \ldots, m$, such that

$$\begin{aligned} &0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial^\circ \psi_i(\bar{x}), \\ &0 = \bar{\lambda}_i \psi_i(\bar{x}),\ i = 1, \ldots, m. \end{aligned}$$

Moreover, by Lemma 2.4 (iii), since $\partial^\circ \psi_i(\bar{x}) = \{\xi_i : \exists \bar{v}_i \in \mathcal{V}_i(\bar{x}) \text{ such that } \xi_i \in \partial_x^\circ g_i(\bar{x}, \bar{v}_i)\}$, we conclude that $\bar{x} \in F$ is an optimal solution of (RCP) if and only if there exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i(\bar{x})$, $i = 1, \ldots, m$, such that (3.2) and (3.3) hold. □

3.2. **Illustrative examples.** We now give an example to show that Theorem 3.2 may not hold if the non-degeneracy condition fails.

Example 3.3. Consider the following convex optimization problem with data uncertainty:

$$\begin{aligned} (\mathrm{RCP})^1 \quad &\min \quad -x \\ &\text{s.t.} \quad g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i,\ i = 1, 2, \end{aligned}$$

where g_1 and g_2 are given by

$$g_1(x, v_1) = \begin{cases} -v_1 x - 1, & \text{if } x \leq 0, \\ -1, & \text{if } x > 0 \end{cases} \quad \text{and} \quad g_2(x, v_2) = v_2 x^3,$$

and $\mathcal{V}_1 = \mathcal{V}_2 = [1, 2]$. Then $F^1 := \{x \in \mathbb{R} : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i,\ i = 1, 2\} = [-\frac{1}{2}, 0]$ is the robust feasible set of $(\mathrm{RCP})^1$. So, $\bar{x} = 0$ is an optimal solution of $(\mathrm{RCP})^1$. We can easily see that $I(\bar{x}) = \{2\}$, $\mathcal{V}_2(\bar{x}) = [1, 2]$, $\partial f(\bar{x}) = \{-1\}$, $\partial_x^\circ g_1(\bar{x}, \bar{v}_1) = [-\bar{v}_1, 0]$ and $\partial_x^\circ g_2(\bar{x}, \bar{v}_2) = \{0\}$. Since $\partial_x^\circ g_2(\bar{x}, \bar{v}_2) = \{0\}$ for $\bar{v}_2 \in \mathcal{V}_2(\bar{x})$, the non-degeneracy condition fails. Moreover, we can easily see that g_1 and g_2 satisfy the Slater condition and the assumption (A1)–(A3). On the other hand, there do not exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i(\bar{x})$, $i = 1, 2$, such that

$$0 \in \partial f(\bar{x}) + \bar{\lambda}_1 \partial_x^\circ g_1(\bar{x}, \bar{v}_1) + \bar{\lambda}_2 \partial_x^\circ g_2(\bar{x}, \bar{v}_2).$$

The following example examines the validness of our main results whenever non-degeneracy condition is satisfied.

Example 3.4. Consider the following convex optimization problem with data uncertainty:

$$\begin{aligned} (\mathrm{RCP})^2 \quad &\min \quad -x \\ &\text{s.t.} \quad x \in F^2 := \{x \in \mathbb{R} : \max\{v x^3, v x\} - 2 \leq 0,\ \forall v \in \mathcal{V}\}, \end{aligned}$$

where $\mathcal{V} := [1, 2]$. Let $f(x) = -x$ and $g(x, v) = \max\{v x^3, v x\} - 2$. Then we can easily see that $F^2 = (-\infty, 1]$ is the robust feasible set of $(\mathrm{RCP})^2$ and $\bar{x} = 1$ is an optimal solution of $(\mathrm{RCP})^2$. Clearly, g satisfies the assumptions (A1)–(A3), and the Slater condition holds for $(\mathrm{RCP})^2$. Moreover, $\mathcal{V}(\bar{x}) = \{2\}$, and so for $\bar{v} := 2 \in \mathcal{V}(\bar{x})$,

$0 \notin \partial^\circ g(\bar{x}, \bar{v}) = [2, 6]$, i.e., the non-degeneracy condition holds. Let $0 \leq \bar{\lambda} \leq \frac{1}{2}$. Then we have

$$\begin{aligned} &0 \in \partial f(\bar{x}) + \bar{\lambda}\partial_x^\circ g(\bar{x}, \bar{v}) = \{-1\} + \bar{\lambda}[2, 6], \\ &0 = \bar{\lambda} g(\bar{x}, \bar{v}). \end{aligned}$$

So, Theorem 3.2 holds.

3.3. **An application to quasi ϵ-solutions for (RCP).** In this subsection, we employ Theorem 3.2 to examine the KKT optimality condition for a quasi ϵ-solution for (RCP). First of all, let us give the notion of a quasi ϵ-solution.

Definition 3.5. Given $\epsilon \geq 0$, a point $\bar{x} \in F$ is said to be a quasi ϵ-solution of problem (RCP), if

$$f(\bar{x}) \leq f(x) + \sqrt{\epsilon}\|x - \bar{x}\|, \quad \forall x \in F.$$

It is worth mentioning that the notion of a quasi ϵ-solution is motivated by the well-known Ekeland's Variational Principle [12]. Recently, Lee and Jiao [15] and Jiao and Lee [14] explored some characterizations of a quasi ϵ-solution in robust convex optimization problems and robust semidefinite convex optimization problems, respectively.

By employing Theorem 3.2, we give the following robust KKT optimality theorem for a quasi ϵ-solution in (RCP).

Theorem 3.6. *Let us consider the problem* (RCP). *Assume that each g_i satisfies the assumptions* (A1)–(A3). *Moreover assume that the non-degeneracy condition holds at $\bar{x} \in F$, and the Slater constraint qualification also holds. Then $\bar{x} \in F$ is a quasi ϵ-solution of* (RCP) *if and only if there exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i(\bar{x})$, $i = 1, \ldots, m$, such that*

$$\begin{aligned} &0 \in \partial f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial_x^\circ g_i(\bar{x}, \bar{v}_i) + \sqrt{\epsilon}\mathbb{B}, \\ &0 = \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i), \ i = 1, \ldots, m, \end{aligned} \tag{3.4}$$

where $\mathbb{B}$ stands for the unit ball.

Proof. A quasi ϵ-solution $\bar{x} \in F$ of (RCP) can be considered as a minimizer of the following problem:

$$\min f(x) + \sqrt{\epsilon}\|x - \bar{x}\| \ \text{ subject to } \ x \in F.$$

Set $\phi(x) = f(x) + \sqrt{\epsilon}\|x - \bar{x}\|$, observe that ϕ is a convex function. By the KKT optimality conditions (see Theorem 3.2), we have

$$\begin{aligned} &0 \in \partial \phi(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \partial_x^\circ g_i(\bar{x}, \bar{v}_i), \\ &0 = \bar{\lambda}_i g_i(\bar{x}, \bar{v}_i), \ i = 1, \ldots, m. \end{aligned} \tag{3.5}$$

As $\operatorname{dom} f \cap \operatorname{dom} \| \cdot - \bar{x} \| = \mathbb{R}^n$, invoking the Sum Rule, along with the fact that $\partial \| \cdot - \bar{x} \| = \mathbb{B}$, the above inclusion (3.5) becomes (3.4).

For sufficiency of the above conditions we proceed as follows. Assume on the contrary that $\bar{x}$ is not a quasi ϵ-solution, and hence there exists $\hat{x} \in F$ such that

$$f(\bar{x}) > f(\hat{x}) + \sqrt{\epsilon} \parallel \hat{x} - \bar{x} \parallel . \tag{3.6}$$

On the other hand, since $0 \in \partial f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \partial_x^\circ g_i(\bar{x}, \bar{v}_i) + \sqrt{\epsilon}\mathbb{B}$, there exist $\xi_0 \in \partial f(\bar{x})$, $\xi_i \in \partial_x^\circ g_i(\bar{x}, \bar{v}_i)$, $\bar{v}_i \in \mathcal{V}_i(\bar{x})$, $i = 1, \dots, m$, and $b \in \mathbb{B}$ such that

$$0 = \xi_0 + \sum_{i=1}^{m} \bar{\lambda}_i \xi_i + \sqrt{\epsilon} b.$$

Furthermore, one has

$$0 = \langle \xi_0, \hat{x} - \bar{x} \rangle + \langle \sum_{i=1}^{m} \bar{\lambda}_i \xi_i, \hat{x} - \bar{x} \rangle + \langle \sqrt{\epsilon} b, \hat{x} - \bar{x} \rangle. \tag{3.7}$$

Along with (3.6) and the convexity of f, it follows from (3.7) that

$$\begin{aligned}
0 =& \langle\ \xi_0, \hat{x} - \bar{x} \rangle + \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \hat{x} - \bar{x} \rangle + \sqrt{\epsilon} \langle b, \hat{x} - \bar{x} \rangle \\
\leq& \ f(\hat{x}) - f(\bar{x}) + \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \hat{x} - \bar{x} \rangle + \sqrt{\epsilon} \| \langle b, \hat{x} - \bar{x} \rangle \| \\
<& \ - \sqrt{\epsilon} \| \hat{x} - \bar{x} \| + \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \hat{x} - \bar{x} \rangle + \sqrt{\epsilon} \| \hat{x} - \bar{x} \| \\
=& \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \hat{x} - \bar{x} \rangle.
\end{aligned}$$

Thus, we have

$$0 < \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \hat{x} - \bar{x} \rangle = \sum_{i \in I} \bar{\lambda}_i \langle \xi_i, \hat{x} - \bar{x} \rangle,$$

where $I := \{ i \in \{1, \dots, m\} : \bar{\lambda}_i > 0 \}$. Note that $g_i(\bar{x}, \bar{v}_i) = 0$ for all $i \in I$. Since the non-degeneracy condition holds, from (2.1) and Theorem 3.1, we see that

$$0 < \sum_{i=1}^{m} \bar{\lambda}_i \langle \xi_i, \hat{x} - \bar{x} \rangle \leq \sum_{i=1}^{m} \bar{\lambda}_i {g_i}_x^\circ \big(\bar{x}, \bar{v}_i; \hat{x} - \bar{x} \big) \leq 0,$$

which arrives at a contradiction. Hence, $\bar{x}$ is a quasi ϵ-solution of (RCP). □

4. Conclusions

In this paper, we studied the representation of the feasible set of a robust convex optimization problem, and a robust version of KKT optimality conditions (Theorem 3.2) was explored. As an application, the robust KKT optimality conditions (Theorem 3.6) for a quasi ϵ-solution of (RCP) were investigated.

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