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A NONLINEAR MEAN CONVERGENCE THEOREM FOR GENERIC 2-GENERALIZED NONSPREADING MAPPINGS IN A BANACH SPACE

MAYUMI HOJO AND WATARU TAKAHASHI

Abstract. In this paper, we prove a nonlinear mean convergence theorem for generic 2-generalized nonspreading mappings in a Banach space. Using this result, we prove well-known and new nonlinear mean convergence theorems in a Banach space. In particular, we apply this theorem to prove mean convergence theorems for generic nonspreading mappings, and 2-generalized nonspreading mappings in a Banach space, and a mean convergence theorem by Hojo [3] for nomally 2-generalized hybrid mappings in a Hilbert sace.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is called *nonexpansive* if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$. Kocourek, Takahashi and Yao [8] defined a broad class of nonlinear mappings which covers nonexpansive mappings. A mapping $T : C \to H$ is called *generalized hybrid* [8] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$. They called such a mapping (α, β)-*generalized hybrid*. Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is *nonspreading* [11, 12] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [22] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Kocourek, Takahashi and Yao [8] extended Baillon's nonlinear mean convergence theorem [2] for nonexpansive mappings to that of generalized hybrid mappings in a Hilbert space.

Theorem 1.1 ([8]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping with $F(T) \neq \emptyset$.*

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Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to a fixed point of T.

Takahashi and Takeuchi [24] proved this theorem without convexity by introducing the concept of attractive points [24] in a Hilbert space; see also [15]. A nonspreading mapping in a Hilbert space was generalized in a Banach space by Kohsaka and Takahashi [12]. Let C be a nonempty subset of a smooth Banach space E and let J be the duality mapping from E into E^*. A mapping $T : C \to E$ is called *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$, where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. Such a nonspreading mapping in a Banach space is deduced from a resolvent of a maximal monotone mapping; see [12]. Kocourek, Takahashi and Yao [9] introduced a class of nonlinear mappings in a Banach space which covers generalized hybrid mappings in a Hilbert space and nonspreading mappings in a Banach space. A mapping $T : C \to E$ is called *generalized nonspreading* if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\phi(Tx, Ty) + (1 - \alpha)\phi(x, Ty) + \gamma\{\phi(Ty, Tx) - \phi(Ty, x)\} \\ \leq \beta\phi(Tx, y) + (1 - \beta)\phi(x, y) + \delta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{1.2}$$

for all $x, y \in C$. Takahashi, Wong and Yao [27] generalized the concept of generalized nonspreading mappings as follows: A mapping $T : C \to E$ is called *generic generalized nonspreading* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that $\alpha+\beta+\gamma+\delta \geq 0$, $\alpha+\beta > 0$ and

$$\begin{aligned} \alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ \leq \varepsilon\{\phi(Ty, Tx) - \phi(Ty, x)\} + \zeta\{\phi(y, Tx) - \phi(y, x)\} \end{aligned} \tag{1.3}$$

for all $x, y \in C$; see also [26]. Takahashi, Wong and Yao [25] also extended the concept of generalized nonspreading mapping as follows: A mapping $T : C \to C$ is called 2-*generalized nonspreading* if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1\phi(T^2x, Ty) + \alpha_2\phi(Tx, Ty) + (1 - \alpha_1 - \alpha_2)\phi(x, Ty) \\ &\quad + \gamma_1\big\{\phi(Ty, T^2x) - \phi(Ty, x)\big\} + \gamma_2\big\{\phi(Ty, Tx) - \phi(Ty, x)\big\} \\ &\leq \beta_1\phi(T^2x, y) + \beta_2\phi(Tx, y) + (1 - \beta_1 - \beta_2)\phi(x, y) \\ &\quad + \delta_1\big\{\phi(y, T^2x) - \phi(y, x)\big\} + \delta_2\big\{\phi(y, Tx) - \phi(y, x)\big\} \end{aligned} \tag{1.4}$$

for all $x, y \in C$. Motivated by the definitions of generic generalized nonspreading mappings and 2-generalized nonspreading mappings, Takahashi [23] introduced a new class of nonlinear mappings in a Banach space which simultaneously generalizes these two mappings. A mapping $T : C \to C$ is called *generic 2-generalized nonspreading* [23] if there exist $\alpha_2, \alpha_1, \alpha_0, \beta_2, \beta_1, \beta_0, \gamma_2, \gamma_1, \gamma_0, \delta_2, \delta_1, \delta_0 \in \mathbb{R}$ such that $\alpha_2 + \alpha_1 + \alpha_0 + \beta_2 + \beta_1 + \beta_0 \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\begin{aligned}
&\alpha_2\phi(T^2x,Ty) + \alpha_1\phi(Tx,Ty) + \alpha_0\phi(x,Ty) \\
&\quad + \beta_2\phi(T^2x,y) + \beta_1\phi(Tx,y) + \beta_0\phi(x,y) \\
&\quad \leq \gamma_2\{\phi(Ty,T^2x) - \phi(Ty,Tx)\} + \gamma_1\{\phi(Ty,Tx) - \phi(Ty,x)\} \\
&\qquad + \gamma_0\{\phi(Ty,x) - \phi(Ty,T^2x)\} + \delta_2\{\phi(y,T^2x) - \phi(y,Tx)\} \\
&\qquad + \delta_1\{\phi(y,Tx) - \phi(y,x)\} + \delta_0\{\phi(y,x) - \phi(y,T^2x)\}
\end{aligned} \tag{1.5}$$

for all $x, y \in C$.

In this paper, we prove a nonlinear mean convergence theorem for generic 2-generalized nonspreading mappings in a Banach space. Using this result, we prove well-known and new nonlinear mean convergence theorems in a Banach space. In particular, we apply this theorem to prove mean convergence theorems for generic nonspreading mappings and 2-generalized nonspreading mappings in a Banach space, and a mean convergence theorem by Hojo [3] for nomally 2-generalized hybrid mappings in a Hilbert space.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for all ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let E be a Banach space. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U$, the limit (2.1) is attained uniformly for $x \in U$. It is also said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. A

Banach space E is called uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. It is known that if the norm of E is uniformly Gâteaux differentiable, then J is uniformly norm-to-weak* continuous on each bounded subset of E, and if the norm of E is Fréchet differentiable, then J is norm-to-norm continuous. If E is uniformly smooth, J is uniformly norm-to-norm continuous on each bounded subset of E. For more details, see [18, 20, 21]. The following result is well-known.

Lemma 2.1 ([20]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy\rangle \geq 0$ for all $x, y \in E$. Further, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let E be a smooth Banach space. The function $\phi\colon E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for $x, y \in E$, where J is the duality mapping of E; see [1] and [7]. We have from the definition of ϕ that

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.2}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x, y)$ for all $x, y \in E$, we can see that $\phi(x, y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.3}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Longleftrightarrow x = y. \tag{2.4}$$

The following lemmas are in Xu [28] and Kamimura and Takahashi [7].

Lemma 2.2 ([28]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.3 ([7]). *Let E be smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and*

$$g(\|x - y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space and let C be a nonempty subset of E. Then a mapping $T : C \to E$ is called generalized nonexpansive [5] if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y)$$

for all $x \in C$ and $y \in F(T)$. Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be sunny if

$$R(Rx + t(x - Rx)) = Rx$$

for all $x \in E$ and $t \geq 0$. A mapping $R : E \to D$ is said to be a retraction or a projection if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto D; see [4, 5] for more details. The following results are in Ibaraki and Takahashi [5].

Lemma 2.4 ([5]). *Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([5]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x, z) \in E \times C$. Then the following hold:*

(i) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;*
(ii) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

In 2007, Kohsaka and Takahashi [10] proved the following results:

Lemma 2.6 ([10]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed subset of E. Then the following are equivalent:*

(a) *C is a sunny generalized nonexpansive retract of E;*
(b) *C is a generalized nonexpansive retract of E;*
(c) *JC is closed and convex.*

Lemma 2.7 ([10]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E onto C and let $(x, z) \in E \times C$. Then the following are equivalent:*

(i) *$z = Rx$;*
(ii) *$\phi(x, z) = \min_{y \in C} \phi(x, y)$.*

Ibaraki and Takahashi [6] also obtained the following result concerning the set of fixed points of a generalized nonexpansive mapping.

Lemma 2.8 ([6]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following lemma by Ibaraki and Takahashi [6] is a direct consequence of Lemmas 2.6 and 2.8.

Lemma 2.9 ([6]). *Let E be a smooth, strictly convex and reflexive Banach space and let T be a generalized nonexpansive mapping from E into itself. Then, $F(T)$ is a sunny generalized nonexpansive retract of E.*

Using Lemma 2.6, we have the following result.

Lemma 2.10. *Let E be a smooth, strictly convex and reflexive Banach space and let $\{C_i : i \in I\}$ be a family of sunny generalized nonexpansive retracts of E such that $\cap_{i\in I} C_i$ is nonempty. Then $\cap_{i\in I} C_i$ is a sunny generalized nonexpansive retract of E.*

Proof. It is obvious that $J \cap_{i\in I} C_i = \cap_{i\in I} JC_i$. In fact, we have that

$$\begin{aligned} x \in J \cap_{i\in I} C_i &\iff J^{-1}x \in \cap_{i\in I} C_i \\ &\iff J^{-1}x \in C_i, \quad \forall i \in I \\ &\iff x \in JC_i, \quad \forall i \in I \\ &\iff x \in \cap_{i\in I} JC_i. \end{aligned}$$

From Lemma 2.6, JC_i is closed and convex for each $i \in I$ and hence $\cap_{i\in I} JC_i$ is closed and convex. Thus we have that $J \cap_{i\in I} C_i$ is closed and convex. Therefore, from Lemma 2.6, we have that $\cap_{i\in I} C_i$ is a sunny generalized nonexpansive retract of E. □

Let E be a smooth Banach space. Let C be a nonempty subset of E and let T be a mapping of C into E. We denote by $A(T)$ the set of *attractive points* [16] of T, i.e.,

$$A(T) = \{z \in E : \phi(z, Tx) \le \phi(z, x),\ \forall x \in C\}.$$

We also denote by $B(T)$ the set of *skew-attractive points* [16] of T, i.e.,

$$B(T) = \{z \in E : \phi(Tx, z) \le \phi(x, z),\ \forall x \in C\}.$$

The following results are crucial in our paper.

Lemma 2.11 ([16])**.** *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $A(T)$ is a closed and convex subset of E.*

Lemma 2.12 ([16])**.** *Let E be a smooth Banach space and let C be a nonempty subset of E. Let T be a mapping from C into E. Then $B(T)$ is closed and $JB(T)$ is closed and convex.*

Let E be a smooth Banach space, let C be a nonempty subset of E and let J be the duality mapping from E into E^*. A mapping $T : C \to C$ is called generic 2-generalized nonspreading if it satisfies (1.5). Such a mapping is called generic $(\alpha_2, \alpha_1, \alpha_0, \beta_2, \beta_1, \beta_0, \gamma_2, \gamma_1, \gamma_0, \delta_2, \delta_1, \delta_0)$-2-generalized nonspreading. This mapping $T : C \to C$ is generic generalized nonspreading in the sense of Takahashi, Wong and Yao [27] if $\alpha_2 = \beta_2 = \gamma_2 = \delta_2 = \gamma_0 = \delta_0 = 0$. Furthermore, putting $\alpha_1 = 1, \beta_1 = -1, \gamma_1 = -1$ and $\delta_1 = 0$, we obtain that

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x)$$

for all $x, y \in C$. This is nonspreading in the sense of Kohsaka and Takahashi [12]. Putting $\alpha_0 = 1 - \alpha_2 - \alpha_1,\ \beta_0 = 1 - \beta_2 - \beta_1$ and $\gamma_2 = \delta_2 = 0$ in (1.5), we can also see that the mapping T is 2-generalized nonspreading in the sense of Takahashi, Wong and Yao [25]. If E is a Hilbert space, then we have that

$$\phi(x,y) = \|x-y\|^2, \quad \forall x,y \in E.$$

In a Hilbert space, we obtain the following from (1.5):

$$\begin{aligned}
&\alpha_2\|T^2x-Ty\|^2+\alpha_1\|Tx-Ty\|^2+\alpha_0\|x-Ty\|^2\\
&\quad+\beta_2\|T^2x-y\|^2+\beta_1\|Tx-y\|^2+\beta_0\|x-y\|^2\\
&\qquad\le \gamma_2\{\|Ty-T^2x\|^2-\|Ty-Tx\|^2\}+\gamma_1\{\|Ty-Tx\|^2-\|Ty-x\|^2\}\\
&\qquad\quad+\gamma_0\{\|Ty-x\|^2-\|Ty-T^2x\|^2\}+\delta_2\{\|y-T^2x\|^2-\|y-Tx\|^2\}\\
&\qquad\quad+\delta_1\{\|y-Tx\|^2-\|y-x\|^2\}+\delta_0\{\|y-x\|^2-\|y-T^2x\|^2\}
\end{aligned}$$

and then

$$\begin{aligned}
&(\alpha_2-\gamma_2+\gamma_0)\|T^2x-Ty\|^2+(\alpha_1+\gamma_2-\gamma_1)\|Tx-Ty\|^2\\
&\quad+(\alpha_0+\gamma_1-\gamma_0)\|x-Ty\|^2+(\beta_2-\delta_2+\delta_0)\|T^2x-y\|^2\\
&\quad\quad+(\beta_1+\delta_2-\delta_1)\|Tx-y\|^2+(\beta_0+\delta_1-\delta_0)\|x-y\|^2\le 0.
\end{aligned}$$

Since

$$\begin{aligned}
&(\alpha_2-\gamma_2+\gamma_0)+(\alpha_1+\gamma_2-\gamma_1)+(\alpha_0+\gamma_1-\gamma_0)\\
&\quad+(\beta_2-\delta_2+\delta_0)+(\beta_1+\delta_2-\delta_1)+(\beta_0+\delta_1-\delta_0)\\
&\quad=\alpha_2+\alpha_1+\alpha_0+\beta_2+\beta_1+\beta_0\ge 0
\end{aligned}$$

and $(\alpha_2-\gamma_2+\gamma_0)+(\alpha_1+\gamma_2-\gamma_1)+(\alpha_0+\gamma_1-\gamma_0)=\alpha_2+\alpha_1+\alpha_0>0$, this implies that T is a normally 2-generalized hybrid mapping in the sense of Kondo and Takahashi [13]. A mapping $T: C \to C$ is *normally* 2-*generalized hybrid* [13] if there exist $\alpha_0,\beta_0,\alpha_1,\beta_1,\alpha_2,\beta_2 \in \mathbb{R}$ such that $\sum_{n=0}^{2}(\alpha_n+\beta_n)\ge 0$, $\alpha_2+\alpha_1+\alpha_0>0$ and

$$\begin{aligned}
\alpha_2\|T^2x-Ty\|^2+\alpha_1\|Tx-Ty\|^2+\alpha_0\|x-Ty\|^2&\\
+\beta_2\|T^2x-y\|^2+\beta_1\|Tx-y\|^2+\beta_0\|x-y\|^2\le 0&
\end{aligned}\tag{2.5}$$

for all $x,y \in C$.

Using an idea of [19], Takahashi [23] proved the following attractive and fixed point theorem for generic 2-generalized nonspreading mappings in a Banach space.

Theorem 2.13 ([23]). *Let E be a smooth and reflexive Banach space and let C be a nonempty subset of E. Let T be a generic 2-generalized nonspreading mapping of C into itself. Then the following are equivalent:*

(a) $A(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

Additionally, if E is strictly convex and C is closed and convex, then the following are equivalent:

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ *is bounded for some* $x \in C$.

3. Nonlinear ergodic theorems

In this section, we first prove a nonlinear mean convergence theorem for commutative generic 2-generalized nonspreading mappings in a Banach space. Before proving the theorem, we need the following lemma.

Lemma 3.1. *Let C be a nonempty subset of a smooth, strictly convex and reflexive Banach space E and let S and T be commutative generic 2-generalized nonspreading mappings of C into itself. Suppose that $\{S^kT^lx : k,l \in \mathbb{N}\cup\{0\}\}$ for some $x \in C$ is bounded and define*

$$S_nx = \frac{1}{(1+n)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^kT^lx$$

for all $n \in \mathbb{N}\cup\{0\}$. Then every weak cluster point of $\{S_nx\}$ is a point of $A(S)\cap A(T)$. Additionally, if C is closed and convex, then every weak cluster point of $\{S_nx\}$ is a point of $F(S)\cap F(T)$.

Proof. Since S is a generic 2-generalized nonspreading mapping of C into itself, there exist $\alpha_2,\alpha_1,\alpha_0,\beta_2,\beta_1,\beta_0,\gamma_2,\gamma_1,\gamma_0,\delta_2,\delta_1,\delta_0 \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_2\phi(S^2x,Sy) + \alpha_1\phi(Sx,Sy) + \alpha_0\phi(x,Sy)\\
&\quad + \beta_2\phi(S^2x,y) + \beta_1\phi(Sx,y) + \beta_0\phi(x,y)\\
&\le \gamma_2\{\phi(Sy,S^2x) - \phi(Sy,Sx)\} + \gamma_1\{\phi(Sy,Sx) - \phi(Sy,x)\}\\
&\quad + \gamma_0\{\phi(Sy,x) - \phi(Sy,S^2x)\} + \delta_2\{\phi(y,S^2x) - \phi(y,Sx)\}\\
&\quad + \delta_1\{\phi(y,Sx) - \phi(y,x)\} + \delta_0\{\phi(y,x) - \phi(y,S^2x)\}
\end{aligned} \tag{3.1}$$

for all $y \in C$. Replacing x by S^kT^lx in (3.1), we have that, for any $y \in C$ and $k,l \in \mathbb{N}\cup\{0\}$,

$$\begin{aligned}
&\alpha_2\phi(S^{k+2}T^lx,Sy) + \alpha_1\phi(S^{k+1}T^lx,Sy) + \alpha_0\phi(S^kT^lx,Sy)\\
&\quad + \beta_2\phi(S^{k+2}T^lx,y) + \beta_1\phi(S^{k+1}T^lx,y) + \beta_0\phi(S^kT^lx,y)\\
&\le \gamma_2\{\phi(Sy,S^{k+2}T^lx) - \phi(Sy,S^{k+1}T^lx)\}\\
&\quad + \gamma_1\{\phi(Sy,S^{k+1}T^lx) - \phi(Sy,S^kT^lx)\}\\
&\quad + \gamma_0\{\phi(Sy,S^kT^lx) - \phi(Sy,S^{k+2}T^lx)\}\\
&\quad + \delta_2\{\phi(y,S^{k+2}T^lx) - \phi(y,S^{k+1}T^lx)\}\\
&\quad + \delta_1\{\phi(y,S^{k+1}T^lx) - \phi(y,S^kT^lx)\}\\
&\quad + \delta_0\{\phi(y,S^kT^lx) - \phi(y,S^{k+2}T^lx)\}
\end{aligned}$$

and hence

$$\begin{aligned}
&\alpha_2\{\phi(S^{k+2}T^lx,y) + \phi(y,Sy) + 2\langle S^{k+2}T^lx - y, Jy - JSy\rangle\}\\
&\quad + \alpha_1\{\phi(S^{k+1}T^lx,y) + \phi(y,Sy) + 2\langle S^{k+1}T^lx - y, Jy - JSy\rangle\}\\
&\quad + \alpha_0\{\phi(S^kT^lx,y) + \phi(y,Sy) + 2\langle S^kT^lx - y, Jy - JSy\rangle\}\\
&\quad + \beta_2\phi(S^{k+2}T^lx,y) + \beta_1\phi(S^{k+1}T^lx,y) + \beta_0\phi(S^kT^lx,y)
\end{aligned}$$

$$
\begin{aligned}
&\le \gamma_2\{\phi(Sy, S^{k+2}T^lx) - \phi(Sy, S^{k+1}T^lx)\} \\
&\qquad + \gamma_1\{\phi(Sy, S^{k+1}T^lx) - \phi(Sy, S^kT^lx)\} \\
&\quad + \gamma_0\{\phi(Sy, S^kT^lx) - \phi(Sy, S^{k+2}T^lx)\} \\
&\qquad + \delta_2\{\phi(y, S^{k+2}T^lx) - \phi(y, S^{k+1}T^lx)\} \\
&\quad + \delta_1\{\phi(y, S^{k+1}T^lx) - \phi(y, S^kT^lx\} \\
&\qquad + \delta_0\{\phi(y, S^kT^lx) - \phi(y, S^{k+2}T^lx)\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&(\alpha_2 + \alpha_1 + \alpha_0 + \beta_2 + \beta_1 + \beta_0)\phi(S^{k+2}T^lx, y) \\
&\quad + \alpha_1\{\phi(S^{k+1}T^lx, y) - \phi(S^{k+2}T^lx, y)\} \\
&\quad + \alpha_0\{\phi(S^kT^lx, y) - \phi(S^{k+2}T^lx, y)\} \\
&\quad + \beta_1\{\phi(S^{k+1}T^lx, y) - \phi(S^{k+2}T^lx, y)\} \\
&\quad + \beta_0\{\phi(S^kT^lx, y) - \phi(S^{k+2}T^lx, y)\} + (\alpha_2 + \alpha_1 + \alpha_0)\phi(y, Sy) \\
&\quad + 2\Big\langle \alpha_2S^{k+2}T^lx + \alpha_1S^{k+1}T^lx + \alpha_0S^kT^lx \\
&\qquad - (\alpha_2 + \alpha_1 + \alpha_0)y, Jy - JSy\Big\rangle \\
&\le \gamma_2\{\phi(Sy, S^{k+2}T^lx) - \phi(Sy, S^{k+1}T^lx)\} \\
&\qquad + \gamma_1\{\phi(Sy, S^{k+1}T^lx) - \phi(Ty, S^kT^lx)\} \\
&\quad + \gamma_0\{\phi(Sy, S^kT^lx) - \phi(Sy, S^{k+2}T^lx)\} \\
&\qquad + \delta_2\{\phi(y, S^{k+2}T^lx) - \phi(y, S^{k+1}T^lx)\} \\
&\quad + \delta_1\{\phi(y, S^{k+1}T^lx_n) - \phi(y, S^kT^lx_n)\} \\
&\qquad + \delta_0\{\phi(y, S^kT^lx) - \phi(y, S^{k+2}T^lx)\}.
\end{aligned}
$$

Since $\alpha_2 + \alpha_1 + \alpha_0 + \beta_2 + \beta_1 + \beta_0 \ge 0$, we have that

$$
\begin{aligned}
&\alpha_1\{\phi(S^{k+1}T^lx, y) - \phi(S^{k+2}T^lx, y)\} \\
&\quad + \alpha_0\{\phi(S^kT^lx, y) - \phi(S^{k+2}T^lx, y)\} \\
&\quad + \beta_1\{\phi(S^{k+1}T^lx, y) - \phi(S^{k+2}T^lx, y)\} \\
&\quad + \beta_0\{\phi(S^kT^lx, y) - \phi(S^{k+2}T^lx, y)\} + (\alpha_2 + \alpha_1 + \alpha_0)\phi(y, Sy) \\
&\quad + 2\Big\langle \alpha_2S^{k+2}T^lx + \alpha_1S^{k+1}T^lx + \alpha_0S^kT^lx \\
&\qquad - (\alpha_2 + \alpha_1 + \alpha_0)y, Jy - JSy\Big\rangle \\
&\le \gamma_2\{\phi(Sy, S^{k+2}T^lx) - \phi(Sy, S^{k+1}T^lx)\} \\
&\qquad + \gamma_1\{\phi(Sy, S^{k+1}T^lx) - \phi(Sy, S^kT^lx)\} \\
&\quad + \gamma_0\{\phi(Sy, S^kT^lx) - \phi(Sy, S^{k+2}T^lx)\}
\end{aligned}
$$

$$
\begin{aligned}
&+ \delta_2\{\phi(y, S^{k+2}T^l x) - \phi(y, S^{k+1}T^l x)\}\\
&+ \delta_1\{\phi(y, S^{k+1}T^l x) - \phi(y, S^k T^l x)\}\\
&+ \delta_0\{\phi(y, S^k T^l x) - \phi(y, S^{k+2}T^l x)\}.
\end{aligned}
$$

Summing up these inequalities with respect to $k = 0, 1, \dots, n$, we have

$$
\begin{aligned}
&\alpha_1\{\phi(ST^l x, y) - \phi(S^{n+2}T^l x, y)\}\\
&+ \alpha_0\{\phi(T^l x, y) + \phi(ST^l x, y) - \phi(S^{n+1}T^l x, y) - \phi(S^{n+2}T^l x, y)\}\\
&+ \beta_1\{\phi(ST^l x, y) - \phi(S^{n+2}T^l x, y)\}\\
&+ \beta_0\{\phi(T^l x, y) + \phi(ST^l x, y) - \phi(S^{n+1}T^l x, y) - \phi(S^{n+2}T^l x, y)\}\\
&+ (\alpha_2 + \alpha_1 + \alpha_0)(n+1)\phi(y, Sy)\\
&+ 2\Big\langle (\alpha_2 + \alpha_1 + \alpha_0)\sum_{k=0}^{n} S^k T^l x\\
&+ S^{n+2}T^l x + S^{n+1}T^l x - ST^l x - T^l x + S^{n+1}T^l x - T^l x\\
&- (\alpha_2 + \alpha_1 + \alpha_0)(n+1)y, Jy - JSy\Big\rangle\\
&\le \gamma_2\{\phi(Sy, S^{n+2}T^l x) - \phi(Sy, ST^l x)\} + \gamma_1\{\phi(Sy, S^{n+1}T^l x) - \phi(Sy, T^l x)\}\\
&+ \gamma_0\{\phi(Sy, T^l x) + \phi(Sy, ST^l x) - \phi(Sy, S^{n+1}T^l x) - \phi(Sy, S^{n+2}T^l x)\}\\
&+ \delta_2\{\phi(y, S^{n+2}T^l x) - \phi(y, ST^l x)\}\\
&+ \delta_1\{\phi(y, S^{n+1}T^l x) - \phi(y, T^l x)\}\\
&+ \delta_0\{\phi(y, T^l x_n) + \phi(y, ST^l x) - \phi(y, S^{n+1}T^l x) - \phi(y, S^{n+2}T^l x)\}.
\end{aligned}
$$

Furthermore, summing up these inequalities with respect to $l = 0, 1, \dots, n$, we have

$$
\begin{aligned}
&\alpha_1 \sum_{l=0}^{n}\{\phi(ST^l x, y) - \phi(S^{n+2}T^l x, y)\}\\
&+ \alpha_0 \sum_{l=0}^{n}\{\phi(T^l x, y) + \phi(ST^l x, y) - \phi(S^{n+1}T^l x, y) - \phi(S^{n+2}T^l x, y)\}\\
&+ \beta_1 \sum_{l=0}^{n}\{\phi(ST^l x, y) - \phi(S^{n+2}T^l x, y)\}\\
&+ \beta_0 \sum_{l=0}^{n}\{\phi(T^l x, y) + \phi(ST^l x, y) - \phi(S^{n+1}T^l x_n, y) - \phi(S^{n+2}T^l x, y)\}\\
&+ (\alpha_2 + \alpha_1 + \alpha_0)(n+1)^2\phi(y, Sy)\\
&+ 2\Big\langle (\alpha_2 + \alpha_1 + \alpha_0)\sum_{l=0}^{n}\sum_{k=0}^{n} S^k T^l x\\
&+ \sum_{l=0}^{n}(S^{n+2}T^l x + S^{n+1}T^l x - ST^l x - T^l x + S^{n+1}T^l x - T^l x)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - (\alpha_2 + \alpha_1 + \alpha_0)(n+1)^2 y, Jy - JSy\Big\rangle \\
&\le \gamma_2 \sum_{l=0}^{n} \{\phi(Sy, S^{n+2}T^l x) - \phi(Sy, ST^l x)\} \\
&\quad + \gamma_1 \sum_{l=0}^{n} \{\phi(Sy, S^{n+1}T^l x) - \phi(Sy, T^l x)\} \\
&+ \gamma_0 \sum_{l=0}^{n} \{\phi(Sy, T^l x) + \phi(Sy, ST^l x) - \phi(Sy, S^{n+1}T^l x) - \phi(Sy, S^{n+2}T^l x)\} \\
&+ \delta_2 \sum_{l=0}^{n} \{\phi(y, S^{n+2}T^l x) - \phi(y, ST^l x)\} \\
&\quad + \delta_1 \sum_{l=0}^{n} \{\phi(y, S^{n+1}T^l x) - \phi(y, T^l x)\} \\
&+ \delta_0 \sum_{l=0}^{n} \{\phi(y, T^l x) + \phi(y, ST^l x) - \phi(y, S^{n+1}T^l x) - \phi(y, S^{n+2}T^l x)\}.
\end{aligned}
$$

Dividing by $(n+1)^2$, we have

$$
\begin{aligned}
&\alpha_1 \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(ST^l x, y) - \phi(S^{n+2}T^l x, y)\} \\
&\quad + \alpha_0 \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(T^l x, y) + \phi(ST^l x, y) \\
&\qquad - \phi(S^{n+1}T^l x, y) - \phi(S^{n+2}T^l x, y)\} \\
&\quad + \beta_1 \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(ST^l x, y) - \phi(S^{n+2}T^l x, y)\} \\
&\quad + \beta_0 \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(T^l x, y) + \phi(ST^l x, y) \\
&\qquad - \phi(S^{n+1}T^l x, y) - \phi(S^{n+2}T^l x, y)\} \\
&\quad + (\alpha_2 + \alpha_1 + \alpha_0)\phi(y, Sy) + 2\Big\langle (\alpha_2 + \alpha_1 + \alpha_0) S_n x \\
&\qquad + \frac{1}{(n+1)^2} \sum_{l=0}^{n} \Big(S^{n+2}T^l x + 2S^{n+1}T^l x - ST^l x - 2T^l x \Big) \\
&\qquad - (\alpha_2 + \alpha_1 + \alpha_0) y, Jy - JSy \Big\rangle \\
&\quad \le \gamma_2 \frac{1}{(n+1)^2} \sum_{l=0}^{n} \{\phi(Sy, S^{n+2}T^l x) - \phi(Sy, ST^l x)\}
\end{aligned}
$$

$$+\gamma_1 \frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(Sy, S^{n+1}T^l x) - \phi(Sy, T^l x)\}$$

$$+\gamma_0 \frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(Sy, T^l x) + \phi(Sy, ST^l x)$$

$$- \phi(Sy, S^{n+1}T^l x) - \phi(Sy, S^{n+2}T^l x)\}$$

$$+\delta_2 \frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(y, S^{n+2}T^l x) - \phi(y, ST^l x)\}$$

$$+\delta_1 \frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(y, S^{n+1}T^l x) - \phi(y, T^l x)\}$$

$$+\delta_0 \frac{1}{(n+1)^2}\sum_{l=0}^{n}\{\phi(y, T^l x) + \phi(y, ST^l x)$$

$$- \phi(y, S^{n+1}T^l x) - \phi(y, S^{n+2}T^l x)\}.$$

where $S_n x_n = \frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x$. Since $\{S^k T^l x : k, l \in \mathbb{N}\cup\{0\}\}$ is bounded by assumption, there exists a subsequence $\{S_{n_i}x\}$ of $\{S_n x\}$ such that $\{S_{n_i}x\}$ converges weakly to a point $u \in E$. Letting $n_i \to \infty$ in the above inequality, we obtain

$$(\alpha_2 + \alpha_1 + \alpha_0)\Big(\phi(y, Sy) + 2\langle u - y, Jy - JSy\rangle\Big) \le 0$$

and hence

$$(\alpha_2 + \alpha_1 + \alpha_0)\Big(\phi(y, Sy) + \phi(u, Sy) + \phi(y, y) - \phi(u, y) - \phi(y, Sy)\Big) \le 0.$$

Since $\alpha_2 + \alpha_1 + \alpha_0 > 0$, we have

$$\phi(u, Sy) \le \phi(u, y). \tag{3.2}$$

Similarly, replacing S and T by T and S, respectively, we have

$$\phi(u, Ty) \le \phi(u, y). \tag{3.3}$$

Every weak cluster point of $\{x_n\}$ is a point of $A(S) \cap A(T)$. Additionally, if C is closed and convex, then $u \in C$. Putting $y = u$ in (3.2) and (3.3), we have $\phi(u, Su) \le \phi(u, u) = 0$ and $\phi(u, Tu) \le \phi(u, u) = 0$. Thus we get $u \in F(S) \cap F(T)$. Then every weak cluster point of $\{x_n\}$ is a point of $F(S) \cap F(T)$. □

Let $D = \{(k, l) : k, l \in \mathbb{N}\cup\{0\}\}$. Then D is a directed set by the binary relation:

$$(k, l) \le (i, j) \quad \text{if } k \le i \text{ and } l \le j.$$

Now, we can prove the following nonlinear ergodic theorem for generic 2-generalized nonspreading mappings in a Banach space.

Theorem 3.2. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let S and T be commutative generic 2-generalized nonspreading mappings of C into itself with $A(S) \cap A(T) \neq \emptyset$*

such that $A(S) = B(S)$ and $A(T) = B(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $B(S) \cap B(T)$. Then, for any $x \in C$,

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $A(S) \cap A(T)$, where $q = \lim_{(k,l)\in D} RS^k T^l x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to a point of $F(S) \cap F(T)$.

Proof. Since $A(S) \cap A(T) \neq \emptyset$, we have that from Theorem 2.13 that for any $x \in C$, $k, l \in \mathbb{N}$ and $z \in A(S) \cap A(T)$,

$$\phi(z, S^i T^l x) \leq \phi(z, x).$$

Thus $\{S^i T^l x\}$ is bounded for all $x \in C$ and then $S_n x$ is bounded.

We have from Lemma 2.10 that $B(S) \cap B(T)$ is a sunny generalized nonexpansive retract. Then there exists the sunny generalized nonexpansive retraction R of E onto $B(S) \cap B(T)$. From Lemma 2.7, this retraction R is characterized by

$$Rx = \arg \min_{u \in B(S) \cap B(T)} \phi(x, u)$$

for all $x \in E$. We also know from Lemma 2.5 that

$$0 \leq \langle v - Rv, JRv - Ju \rangle, \quad \forall u \in B(S) \cap B(T),\ v \in C.$$

Adding up $\phi(Rv, u)$ to both sides of this inequality, we have

$$\begin{aligned} \phi(Rv, u) &\leq \phi(Rv, u) + 2\langle v - Rv, JRv - Ju \rangle \\ &= \phi(Rv, u) + \phi(v, u) + \phi(Rv, Rv) - \phi(v, Rv) - \phi(Rv, u) \\ &= \phi(v, u) - \phi(v, Rv). \end{aligned} \tag{3.4}$$

Since $\phi(Sz, u) \leq \phi(z, u)$ and $\phi(Tz, u) \leq \phi(z, u)$ for any $u \in B(S) \cap B(T)$ and $z \in C$, it follows that for any $(k, l), (i, j) \in D$ with $(k, l) \leq (i, j)$,

$$\begin{aligned} \phi(S^i T^j x, RS^i T^j x) &\leq \phi(S^i T^j x, RS^k T^l x) \\ &\leq \phi(S^k T^l x, RS^k T^l x). \end{aligned}$$

Hence the net $\phi(S^k T^l x, RS^k T^l x)$ is nonincreasing. Putting $u = RS^k T^l x$ and $v = S^i T^j x$ with $(k, l) \leq (i, j)$ in (3.4), we have from Lemma 2.3 that

$$\begin{aligned} g(\|RS^i T^j x - RS^k T^l x\|) &\leq \phi(RS^i T^j x, RS^k T^l x) \\ &\leq \phi(S^i T^j x, RS^k T^l x) - \phi(S^i T^j x, RS^i T^j x) \\ &\leq \phi(S^k T^l x, RS^k T^l x) - \phi(S^i T^j x, RS^i T^j x), \end{aligned}$$

where g is a strictly increasing, continuous and convex real-valued function with $g(0) = 0$. From the properties of g, $\{RS^k T^l x\}$ is a Cauchy net; see [14]. Therefore $\{RS^k T^l x\}$ converges strongly to a point $q \in B(S) \cap B(T)$ since $B(S) \cap B(T)$ is closed from Lemma 2.12.

Next, consider a fixed $x \in C$ and an arbitrary subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$ convergent weakly to a point v. From Lemma 3.1, we know that $v \in A(S) \cap A(T)$.

Rewriting the characterization of the retraction R, we have that for any $u \in B(S) \cap B(T)$,

$$0 \leq \left\langle S^k T^l x - RS^k T^l x, JRS^k T^l x - Ju \right\rangle$$

and hence

$$\begin{aligned}\left\langle S^k T^l x - RS^k T^l x, Ju - Jq \right\rangle &\leq \left\langle S^k T^l x - RS^k T^l x, JRS^k T^l x - Jq \right\rangle \\ &\leq \|S^k T^l x - RS^k T^l x\| \cdot \|JRS^k T^l x - Jq\| \\ &\leq K\|JRS^k T^l x - Jq\|,\end{aligned}$$

where K is an upper bound for $\|S^k T^l x - RS^k T^l x\|$. Summing up these inequalities for $k = 0, 1, \ldots, n$ and $l = 0, 1, \ldots, n$ and dividing by $(n+1)^2$, we arrive to

$$\left\langle S_n x - \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} RS^k T^l x, Ju - Jq \right\rangle \leq K \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} \|JRS^k T^l x - Jq\|,$$

where $S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$. Letting $n_i \to \infty$ and remembering that J is continuous, we get

$$\langle v - q, Ju - Jq \rangle \leq 0.$$

This inequality holds for any $u \in B(S) \cap B(T)$. Therefore, we have $Rv = q$. But because $v \in B(S) \cap B(T)$, we have $v = q$. Thus the sequence $\{S_n x\}$ converges weakly to the point q. Additionally, if C is closed and convex, then $q \in C$ and hence $z_0 \in F(S) \cap F(T)$. $\{S_n x\}$ converges weakly to a point of $F(S) \cap F(T)$. □

Using Theorem 3.2, we obtain the following theorem.

Theorem 3.3. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let S and T be commutative generic generalized nonspreading mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$, $A(S) = B(S)$ and $A(T) = B(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $B(S) \cap B(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $A(S) \cap A(T)$, where $q = \lim_{(k,l) \in D} RS^k T^l x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to a point of $F(S) \cap F(T)$.

Proof. If S and T are generic generalized nonspreading, then the mappings are generic 2-generalized nonspreading. Therefore, we have the desired result from Theorem 3.2. □

We also heve the following nonlinear mean convergence theorem; see [17].

Theorem 3.4. *Let E be a uniformly convex Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let S and T be commutative 2-generalized nonspreading mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$,*

$A(S) = B(S)$ and $A(T) = B(T)$. Let R be the sunny generalized nonexpansive retraction of E onto $B(S) \cap B(T)$. Then, for any $x \in C$,

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $A(S) \cap A(T)$, where $q = \lim_{(k,l)\in D} RS^k T^l x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to a point of $F(S) \cap F(T)$.

Proof. If S and T are 2-generalized nonspreading, then the mappings are generic 2-generalized nonspreading. Therefore, we have the desired result from Theorem 3.2. □

Using Theorem 3.2, we have the following nonlinear mean convergence theorem by Hojo [3] in a Hilbert space.

Theorem 3.5 ([3]). *Let H be a Hilbert space and let C be a nonempty subset of H. Let S, T be commutative normally 2-generalized hybrid mappings of C into itself such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ for some $z \in C$ is bounded. Let P be the metric projection of H onto $A(S) \cap A(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converges weakly to an element q of $A(S) \cap A(T)$, where $q = \lim_{(k,l)\in D} PS^k T^l x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to a point of $F(S) \cap F(T)$.

Proof. It is obvious that normally 2-generalized nonspreading mappings in a Hilbert space are generic 2-generalized nonspreading mappings. Since $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ for some $z \in C$ is bounded, we have from Theorem 2.13 that $A(S) \cap A(T) \neq \emptyset$. In a Hilbert space, the metric projection of H onto $A(S) \cap A(T)$ is equivalent to the sunny generalized nonexpansive retraction of H onto $A(S) \cap A(T)$. Furthermore, we have $A(S) = B(S)$ and $A(T) = B(T)$. Thus, we have the desired result from Theorem 3.2. □

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Mayumi Hojo
Shibaura Institute of Technology, Tokyo 135-8548, Japan
E-mail address: mayumi-h@shibaura-it.ac.jp

Wataru Takahashi
Research Center for Interneural Computing, China Medical University Hospital, China Medical University, Taichung 40447, Taiwan; Keio Research and Education Center for Natural Sciences, Keio University, Kouhoku-ku, Yokohama 223-8521, Japan; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp; wataru@a00.itscom.net