Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167
Volume 5, Number 1, 2019, 51–65

EXISTENCE AND CONVERGENCE THEOREMS FOR SPLIT GENERAL RANDOM VARIATIONAL INCLUSIONS WITH RANDOM FUZZY MAPPINGS

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Dedicated to Professor Jong Soo Jung on the occasions of his 65th birthday

Abstract. In this paper, we introduce and study a new class of split general random variational inclusions with random fuzzy mappings in Hilbert spaces. The sufficient conditions for the existence of solutions of such a problem are provided. Further, by using the resolvent operator method, we construct the iterative algorithms for solving this class of problems and its special cases. We also consider the convergence criteria of iterative sequences generated by these proposed algorithms. The results presented in this paperare new and extend the previously known results in the literature.

1. Introduction

In 1965, Zadeh [32] introduced the fuzzy set theory which was applied in control engineering and optimization problems as an attractive way. A class of variational inequalities for fuzzy mappings was introduced and studied by Chang and Zhu [13] in 1989. After that, several kinds of variational inequalities and complementarity problems have been extended and generalized in various directions using techniques of fuzzy theory. For more details on this topic, we refer the readers to [9–11, 27].

The concept of random fuzzy mapping was introduced in 1998 by Huang [20] for studying a new class of random completely generalized strongly nonlinear quasi-complementarity problems. Huang [21] extended to the random generalized nonlinear variational inclusions for random fuzzy mappings. It is well known that the variational inclusion problems are regarded as one of the most important and useful generalization of the variational inequalities, which have wide applications in optimization and control theory, economics and transportation equilibrium problems, engineering science, see [15] and the references therein. In [21], Huang discussed the existence of random solutions for a class of random fuzzy variational inclusions and the convergence of random iterative sequences generated by the algorithm based on the resolvent operator technique. After that, the applications of resolvent operator technique for solving the various kinds of variational inclusions and random fuzzy variational inclusions are considered by many authors such as Kazmi and Bhat [25], Ahmad and Bazán [3], Lan et al. [26], (see also Ahmad and Farajzadeh [4], Lee et al. [28], Ahmad et al. [2], Balooee and Cho [5]).

2010 *Mathematics Subject Classification.* 47H40; 47H10; 60H25; 47S40.
Key words and phrases. Split general random variational inclusion; Iterative method; Iterative algorithms; Resolvent operator; Random fuzzy mapping.

In 2011, Moudafi [29] introduced the following split monotone variational inclusion problems (for short, (SMVIP)) stated as follows:

$$\text{finding } x^* \in H_1 \text{ such that } 0 \in f_1(x^*) + B_1(x^*), \tag{1.1}$$

and such that

$$y^* = A(x^*) \in H_2 \text{ solves } 0 \in f_2(y^*) + B_2(y^*). \tag{1.2}$$

where for each $i \in \{1,2\}$, H_i is a real Hilbert space, $A : H_1 \to H_2$ is a bounded linear operator, $f_i : H_i \to H_i$ is a given operator and $B_i : H_i \to 2^{H_i}$ is a multi-valued maximal monotone mapping. Then the (SMVIP) constitutes a pair of variational inclusion problems (1.1) and (1.2) which have to be solved so that the image $y^* = A(x^*)$ under a given bounded linear operator A, of the solution x^* of the problem (1.1) on H_1 is the solution of another problem (1.2) on another space H_2.

The split monotone variational inclusion problem includes as special cases: the split variational inequality problems, split convex minimization problems, split feasibility problems, split common fixed point problems, split zero problems and variational inclusion problems. In recent years, these problem models were interested by many authors in different topics as the existence conditions and iterative algorithms in Anh and Hung [1], Hung [22, 23], Byrne [6], Moudafi [29], Censor et al. [7], Kazmi [24], Tangkhawiwetkul and Petrot [30], (see also Chuang [14], Chang et al. [12] and Hieu [16]) and well-posedness in Hu and Fang [18, 19]. However, to the best of our knowledge, up to now, there are no works on the existence conditions and iterative algorithms for split general random variational inclusions with random fuzzy mappings.

Motivated by the above works, the aim of this paper is to introduce and study a new class of split general random variational inclusions with random fuzzy mappings in Hilbert spaces. The new iterative algorithms are proposed to compute the approximate solutions of these problems by using the resolvent operator method. Further, we also prove the existence of solution and convergence of iterative algorithms under suitable assumptions for these problems. Our results are new and extend the previously known results in the literature.

2. Preliminaries

Throughout the paper, unless otherwise stated, for each $i \in \{1,2\}$, let H_i be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, respectively; $d(x, B) = \inf_{b \in B} \|x - b\|$ is the distance from an element x to a subset B. We suppose that $(\Omega, \mathcal{A})$ is a measurable space, where Ω is a set and $\mathcal{A}$ is a σ-algebra of subsets of Ω. Let $\mathrm{CB}(H_i)$ be a family of all nonempty bounded closed subsets of H_i. We denote by $\mathcal{B}(H_i)$ the class of Borel σ-fields on H_i.

Let H is a real Hilbert space. The following definitions and concepts are needed in the sequel.

Definition 2.1. (See [21]) (a) A mapping $x : \Omega \to H$ is said to be measurable if for any $B \in \mathcal{B}(H)$, $\{t \in \Omega : x(t) \in B\} \in \mathcal{A}$.

(b) A mapping $S : \Omega \times H \to H$ is said to be a random operator if for any $x \in H, S(t, x) = x(t)$ is measurable. The mapping S is said to be Lipschitz continuous (resp., convex, monotone, linear, bounded, surjective) if for any $t \in \Omega$, the mapping $S(t, \cdot) : H \to H$ is Lipschitz continuous (resp., convex, monotone, linear, bounded, surjective).

Definition 2.2. (See [21]) A multi-valued mapping $\Gamma : \Omega \to 2^H$ is said to be measurable if for any $B \in \mathcal{B}(H)$, $\Gamma^{-1}(B) = \{t \in \Omega : \Gamma(t) \cap B \neq \emptyset\} \in \mathcal{A}$. A mapping $u : \Omega \to K$ is called a measurable selection of a multi-valued measurable mapping $\Gamma : \Omega \to 2^H$ if u is measurable and for any $t \in \Omega, u(t) \in \Gamma(t)$.

Definition 2.3. (See [21]) A random multi-valued mapping $T : \Omega \times H \to 2^H$ is said to be measurable, if for any $x \in K$, $T(\cdot, x)$ is measurable. T is said to be $\mathcal{H}$-continuous, if for any $t \in \Omega$, $T(t, \cdot)$ is continuous in the Hausdorff metric.

Let $\mathcal{F}(H)$ be a collection of all fuzzy sets over H, i.e. $\mathcal{F}(H) = \{\mu | \mu : H \to [0, 1]\}$. A mapping $T : H \to \mathcal{F}(H)$ is called a fuzzy mapping on H. If T is a fuzzy mapping on H, then $T(x)$ (denoted by T_x, in the sequel) is a fuzzy set on H and $T_x(y)$ is the membership function of y in T_x. Let $M \in \mathcal{F}(H)$, $\alpha \in [0, 1]$. Then the set $(M)_\alpha = \{x \in H \mid M(x) \geq \alpha\}$ ia called an α-cut set of M.

Definition 2.4. (See [21]) A fuzzy mapping $T : \Omega \to \mathcal{F}(H)$ is called measurable, if for any $\alpha \in (0, 1]$, $(T(\cdot))_\alpha : \Omega \to 2^H$ is a measurable multi-valued mapping. T is called a random fuzzy mapping, if for any $x \in H$, $T(\cdot, x) : \Omega \to \mathcal{F}(H)$ is a measurable fuzzy mapping.

Clearly, the random fuzzy mappings include multi-valued mappings, random multi-valued mappings, and fuzzy mappings as special cases.

Definition 2.5. (See [31]) Let $K : H_1 \to H_2$ be a linear and bounded operator. A mapping $K^* : H_2 \to H_1$ is called the adjoint operator of K, if

$$\langle K(x_1), x_2 \rangle = \langle x_1, K^*(x_2) \rangle, \quad \forall t \in \Omega, x_i \in H_i, i \in \{1, 2\}.$$

Lemma 2.6. (See [31]) *Suppose that $K : H_1 \to H_2$ is a linear and bounded operator. Then adjoint operator K^* is linear and bounded and $\|K\| = \|K^*\|$.*

Throughout the paper, given mappings $a_1, a_2 : H_i \to [0, 1]$, random fuzzy mappings $S : \Omega \times H_1 \to \mathcal{F}(H_1)$ and $T : \Omega \times H_2 \to \mathcal{F}(H_2)$ satisfy the following condition (Δ):

(Δ): There exist mappings $a_i : H_i \to [0, 1]$ such that

$$(S_{t,x_1})_{a_1(x_1)} \in \mathrm{CB}(H_1), \quad \forall t \in \Omega, x_1 \in H_1$$
$$(T_{t,x_2})_{a_2(x_2)} \in \mathrm{CB}(H_2), \quad \forall t \in \Omega, x_2 \in H_2.$$

By using the random fuzzy mappings S and T, we can define the random multi-valued mappings $\widetilde{S}$ and $\widetilde{T}$ as follows:

$$\widetilde{S} : \Omega \times H_1 \to \mathrm{CB}(H_1), \quad (t, x_1) \mapsto (S_{t,x_1})_{a_1(x)}, \quad \forall (t, x_1) \in \Omega \times H_1.$$
$$\widetilde{T} : \Omega \times H_2 \to \mathrm{CB}(H_2), \quad (t, x_2) \mapsto (T_{t,x_2})_{a_2(x)}, \quad \forall (t, x_2) \in \Omega \times H_2.$$

$\widetilde{S}$ and $\widetilde{T}$ are called the random multi-valued mappings induced by the random fuzzy mappings S and T, respectively.

For each $i \in \{1, 2\}$, let $f_i, g_i : \Omega \times H_i \to H_i$ and $R_i : \Omega \times H_i \to 2^{H_i}$ be random mappings with $\mathrm{Im}(g_i) \cap \mathrm{dom}(R_i(t, \cdot)) \neq \emptyset$, for $t \in \Omega$. Let $K : H_1 \to H_2$ be a bounded linear operator with its adjoint operator K^*. We consider the following split general random variational inclusions with random fuzzy mappings (for short, (SpGRVI)):

(SpGRVI) Find measurable mappings $x_1^*, w_1^* : \Omega \to H_1$ such that for all $t \in \Omega$, $g_1(t, x_1^*(t))$, $w_1^*(t) \in H_1$ satisfying $S_{t,x_1^*(t)}(w_1^*(t)) \geq a_1(x_1^*(t))$ such that

$$0 \in f_1(t, w_1^*(t)) + R_1(t, g_1(t, x_1^*(t))) \tag{2.1}$$

and $x_2^*(t) = K(x_1^*(t)), w_2^*(t) \in H_2$ satisfying $g_2(t, x_2^*(t)) \in H_2$, $T_{t,x_2^*(t)}(w_2^*(t)) \geq a_2(x_2^*(t))$ solve

$$0 \in f_2(t, w_2^*(t)) + R_2(t, g_2(t, x_2^*(t))). \tag{2.2}$$

The set of measurable mappings $\{x_1^*, w_1^*\}$ is called a solution of (SpGRVI) (2.1)–(2.2).

If $g_i \equiv I_i$, where I_i is an identity operator on H_i, then (SpGRVI) (2.1)–(2.2) reduces to the following split random variational inclusions with random fuzzy mappings (for short, (SpRVI)):

(SpRVI) Find measurable mappings $x_1^*, w_1^* : \Omega \to H_1$ such that for all $t \in \Omega$, $w_1^*(t) \in H_1$ satisfying $S_{t,x_1^*(t)}(w_1^*(t)) \geq a_1(x_1^*(t))$, for $t \in \Omega$ such that

$$0 \in f_1(t, w_1^*(t)) + R_1(t, x_1^*(t)) \tag{2.3}$$

and such that $x_2^*(t) = K(x_1^*(t)), w_2^*(t) \in H_2$ satisfying $T_{t,x_2^*(t)}(w_2^*(t)) \geq a_2(x_2^*(t))$ solve

$$0 \in f_2(t, w_2^*(t)) + R_2(t, x_2^*(t)). \tag{2.4}$$

Remark 2.7. $g_i \equiv I_i$, $a_i(x_i(t)) = 1$ and $w_i(t) = x_i(t)$, for all $t \in \Omega, x_i(t), w_i(t) \in H_i$, $x_i(\cdot) \equiv x_i$ is a element of H_i and $f_i(\cdot, x_i(\cdot)) \equiv f_i(x_i)$, $R_i(\cdot, x_i(\cdot)) \equiv R_i(x_i)$, then the (SpRVI) (2.3)–(2.4) is reduced to the (SpVI) in the determined environment consider in Chuang [14] and Moudafi [29].

3. Random iterative algorithms

In this section, by using the fuzzy resolvent operator method associated with A-monotone operator, we construct the iterative algorithms for solving (SpGRVI) (2.1)–(2.2) and its special cases.

Definition 3.1. (See [21]) Let $q : \Omega \times H \to H$ be a random mapping, $G : \Omega \times H \to \mathrm{CB}(H)$ be a multi-valued measurable mapping. Then

(a) q is said to be Lipschitz continuous with constant $l_q(t)$, if there exists a measurable function $l_q : \Omega \to (0, +\infty)$ such that, for any $t \in \Omega$ and $x(t), y(t) \in H$,

$$\|q(t, x(t)) - q(t, y(t))\| \leq l_q(t)\|x(t) - y(t)\|;$$

(b) G is said to be $\mathcal{H}$-Lipschitz continuous with constant $l_G(t)$, if there exists a measurable function $l_G : \Omega \to (0, +\infty)$ such that for any $t \in \Omega$ and $x(t), y(t) \in H$,

$$\mathcal{H}(T(t, x(t)), T(t, y(t))) \leq l_G(t)\|x(t) - y(t)\|,$$

where $\mathcal{H}(\cdot, \cdot)$ is the Hausdorff metric on $\mathrm{CB}(H)$ defined as follows: for any given $A, B \in \mathrm{CB}(H)$,

$$\mathcal{H}(A, B) = \max\left\{\sup_{x\in A}\inf_{y\in B}\|x - y\|, \sup_{y\in B}\inf_{x\in A}\|x - y\|\right\}.$$

Lemma 3.2. (See [8]) *Let $T : \Omega \times H \to \mathrm{CB}(H)$ be an $\mathcal{H}$-continuous random multi-valued mapping. Then for any measurable mapping $x : \Omega \to H$, the multi-valued mapping $T(\cdot, x) : \Omega \to \mathrm{CB}(H)$ is measurable.*

Lemma 3.3. (See [8]) *Let $T, Q : \Omega \times H \to \mathrm{CB}(H)$ be two measurable multi-valued mappings, let $\varepsilon > 0$ be a constant and let $u : \Omega \to H$ be a measurable selection of T. Then there exists a measurable selection $v : \Omega \to H$ of G such that*

$$\|u(t) - v(t)\| \leq (1 + \epsilon)\mathcal{H}(T(t, \cdot), Q(t, \cdot)), \quad \forall t \in \Omega.$$

Definition 3.4. (See [26]) A single-valued mapping $A : \Omega \times H \to H$ is said to be strongly monotone with constant $m(t)$, if there exists a measurable function $m : \Omega \to (0, +\infty)$ such that

$$\langle A(t, x(t)) - A(t, y(t)), x(t) - y(t)\rangle \geq m(t)\|x(t) - y(t)\|^2, \quad \forall x(t), y(t) \in H, t \in \Omega.$$

Definition 3.5. (See [26]) A multi-valued mapping $R : \Omega \times H \to 2^H$ is said to be relaxed monotone with constant $r(t)$, if there exists a measurable function $r : \Omega \to (0, +\infty)$ such that

$$\langle u(t) - v(t), x(t) - y(t)\rangle \geq -r(t)\|x(t) - y(t)\|^2$$

$\forall u(t) \in R(t, x(t)), v(t) \in R(t, y(t)), x(t), y(t) \in H, t \in \Omega.$

Definition 3.6. (See [26]) Let $A : \Omega \times H \to H$ be a single-valued mapping. Then a random multi-valued mapping $R : \Omega \times H \to 2^H$ is said to be A-monotone if:

(a) R is relaxed monotone with constant $r(t)$;
(b) $[A_t(x) + \rho(t)R_t(x)](H) = H, \forall x(t) \in H$, $t \in \Omega$ and $\rho(t) > 0$ is a real valued random variable,

where $A_t(x) = A(t, x(t))$ and $R_t(x) = R(t, x(t))$.

Definition 3.7. (See [26]) Let $A : \Omega \times H \to H$ be a strongly monotone mapping with constant $m(t)$ and $R : \Omega \times H \to 2^H$ be A-monotone. The A-resolvent operator $J_{R_t}^{\rho(t),A_t} : H \to H$ associated with A and R is defined by

$$J_{R_t}^{\rho(t),A_t}(x) = (A_t + \rho(t)R_t)^{-1}(x), \quad \forall x(t) \in H, \rho(t) > 0, t \in \Omega.$$

Lemma 3.8. (See [26]) *Let $A : \Omega \times H \to H$ be strongly monotone with constant $m(t)$ and $R : \Omega \times H \to 2^H$ be A-monotone. Then the A-resolvent operator $J_{R_t}^{\rho(t),A_t} : \Omega \times H \to H$ is $(m(t) - \rho(t)r(t))^{-1}$-Lipschitz continuous for $\rho(t) \in (0, m(t)/r(t))$.*

Lemma 3.9. *The set of measurable mappings* $\{x_1^*, w_1^* : \Omega \to H_1\}$ *is a random solution of the* (RVI) (2.1) *if and only if for all* $t \in \Omega, g_1(t, x_1^*(t)) \in H_1, w_1^*(t) \in \widetilde{S}(t, x_1^*(t))$ *and*

$$g_1(t, x_1^*(t)) = J_{R_1(t,g_1(t,x_1^*(t)))}^{\rho_1(t),A_{1,t}} \left(A_1(t, g_1(t, x_1^*(t))) - \rho_1(t)f_1(t, w_1^*(t))\right),$$

where $\rho_1 : \Omega \to (0, +\infty)$ *is a measurable function.*

Proof. Suppose that the set of measurable mappings $\{x_1^*, w_1^* : \Omega \to H_1\}$ is a random solution of problem (RVI) (2.1). Then for all $t \in \Omega, g_{1,t}(x_1^*) \in H_1, w_1^*(t) \in \widetilde{S}(t, x_1^*(t))$ and

$$0 \in f_1(t, w_1^*(t)) + R_1(t, g_1(t, x_1^*(t))).$$

This implies

$$\begin{aligned}
&0 \in \rho_1(t)f_1(t, w_1^*(t)) + \rho_1(t)R_1(t, g_1(t, x_1^*(t)))\\
\Rightarrow\ &0 \in -\left(A_1(t, g_1(t, x_1^*(t))) - \rho_1(t)f_1(t, w_1^*(t))\right) + A_1(t, g_1(t, x_1^*(t)))\\
&\qquad + \rho_1(t)R_1(t, g_1(t, x_1^*(t)))\\
\Rightarrow\ &0 \in -\left(A_1(t, g_1(t, x_1^*(t))) - \rho_1(t)f_1(t, w_1^*(t))\right)\\
&\qquad + \left(A_1(t, \cdot) + \rho_1(t)R_1(t, \cdot)\right)\left(g_1(t, x_1^*(t))\right)
\end{aligned}$$

where $\rho_1 : \Omega \to (0, +\infty)$ is a measurable function. Hence

$$g_1(t, x_1^*(t)) = J_{R_1(t,g_1(t,x_1^*(t)))}^{\rho_1(t),A_{1,t}} \left(A_1(t, g_1(t, x_1^*(t))) - \rho_1(t)f_1(t, w_1^*(t))\right).$$

Conversely, suppose that for all $t \in \Omega, g_{1,t}(x_1^*) \in H_1, w_1^*(t) \in \widetilde{S}(t, x_1^*(t))$ and

$$g_1(t, x_1^*(t)) = J_{R_1(t,g_1(t,x_1^*(t)))}^{\rho_1(t),A_{1,t}} \left(A_1(t, g_1(t, x_1^*(t))) - \rho_1(t)f_1(t, w_1^*(t))\right),$$

where $\rho_1 : \Omega \to (0, +\infty)$ is a measurable function, i.e.,

$$g_1(t, x_1^*(t)) = \left(A_1(t, \cdot) + \rho_1(t)R_1(t, \cdot)\right)^{-1}\left(A_1(t, g_1(t, x_1^*(t))) - \rho_1(t)f_1(t, w_1^*(t))\right)$$

so

$$A_1(t, g_1(t, x_1^*(t))) - \rho_1(t)f_1(t, w_1^*(t)) \in \left(A_1(t, \cdot) + \rho_1(t)R_1(t, \cdot)\right)\left(g_1(t, x_1^*(t))\right).$$

Hence,

$$0 \in f_1(t, w_1^*(t)) + R_1(t, g_1(t, x_1^*(t))).$$

Thus, the set of measurable mappings $\{x_1^*, w_1^* : \Omega \to H_1\}$ is a random solution of problem (RVI) (2.1). □

Remark 3.10. From the result of Lemma 3.9, the problem (SpGRVI) (2.1)–(2.2) can be reformulated as follows: Finding the measurable mappings $x_1^*, w_1^* : \Omega \to H_1$ with $x_2^*(t) = K(x_1^*(t))$, $w_2^*(t) \in H_2$ such that for all $t \in \Omega$, $g_i(t, x_i^*(t)) \in H_i$, $w_1^*(t) \in \widetilde{S}(t, x_1^*(t)), w_2^*(t) \in \widetilde{T}(t, x_2^*(t))$ and

$$g_i(t, x_i^*(t)) = J_{R_i(t,g_i(t,x_i^*(t)))}^{\rho_i(t),A_{i,t}} \left(A_i(t, g_i(t, x_i^*(t))) - \rho_i(t)f_i(t, w_i^*(t))\right).$$

where $\rho_i : \Omega \to (0, +\infty)$ is measurable function, for all $i \in \{1, 2\}$.

Based on the above discussion, an iterative algorithm for solving (SpGRVI) (2.1)–(2.2) is proposed as follows:

Algorithm 3.1. Suppose that $S : \Omega\times H_1 \to \mathcal{F}(H_1)$ and $T : \Omega\times H_2 \to \mathcal{F}(H_2)$ be two random fuzzy mappings satisfying the condition (Δ). Let $\widetilde{S} : \Omega\times H_1 \to CB(H_1)$ and $\widetilde{T} : \Omega \times H_2 \to CB(H_2)$ be $\mathcal{H}$-continuous random multi-valued mappings induced by S and T, respectively. Assume that $\alpha : \Omega \to (0, 1]$ is a measurable step size function. For each $i \in \{1, 2\}$, we assume that g_i is surjective. Let $A_i, f_i, g_i : \Omega\times H_i \to H_i$ be the single-valued random mappings and $R_i : \Omega \times H \to 2^{H_i}$ be a multi-valued random mapping such that for each fixed $t \in \Omega, R_i(t, \cdot) : H_i \to 2^{H_i}$ is an A_i-monotone mapping with $\mathrm{Im}(g_i) \cap \mathrm{dom}(R_i(t, \cdot)) \neq \emptyset$.

Given a measurable mapping $x_1^0 : \Omega \to H_1$, then the multi-valued mapping $\widetilde{S}(\cdot, x_1^0(\cdot)) : \Omega \to CB(H_1)$ is measurable by Lemma 3.2. Hence there exists a measurable selection $w_1^0 : \Omega \to H_1$ of $\widetilde{S}(\cdot, x_1^0(\cdot))$, by Himmelberg [17]. Since g_1 is surjective, there exists a measurable mapping $y^0 : \Omega \to H_1$ such that

$$g_1(t, y^0(t)) = J^{\rho_1(t),A_{1,t}}_{R_1(t,g_1(t,x_1^0(t)))}\left(A_1(t, g_1(t, x_1^0(t))) - \rho_1(t)f_1(t, w_1^0(t))\right),$$

where $\rho_1 : \Omega \to (0, +\infty)$ is a measurable function. Let $K : H_1 \to H_2$ be a bounded linear operator and let K^* be its adjoint operator. Then, $x_2^0(t) = K(x_1^0(t))$ is measurable. Hence there exists a measurable selection $w_2^0 : \Omega \to H_2$ of $\widetilde{T}(\cdot, K(x_1^0(\cdot)))$. Since g_2 is surjective, there exists a measurable mapping $z^0 : \Omega \to H_2$ such that

$$g_2(t, z^0(t)) = J^{\rho_2(t),A_{2,t}}_{R_2(t,g_2(t,K(y^0(t))))}\left(A_2(t, g_2(t, K(y^0(t)))) - \rho_2(t)f_2(t, w_2^0(t))\right),$$

where $\rho_2 : \Omega \to (0, +\infty)$ is a measurable function. We consider

$$x_1^1(t) = (1 - \alpha(t))x_1^0(t) + \alpha(t)\left[y^0(t) + \gamma(t)K^*(z^0(t) - K(y^0(t))\right],$$

where $\gamma : \Omega \to (0, +\infty)$ is a measurable function. It is easy to see that $x_1^1 : \Omega \to H_1$ is measurable. By Lemma 3.3, there exist measurable selections $w_1^1 : \Omega \to H_1$ of $\widetilde{S}(\cdot, x_1^1(\cdot))$ and $w_2^1 : \Omega \to H_2$ of $\widetilde{T}(\cdot, K(x_1^1(\cdot)))$ such that $\forall t \in \Omega$,

$$\|w_1^0(t) - w_1^1(t)\| \le (1 + 1)\,\mathcal{H}_1\left(\widetilde{S}(t, x_1^0(t)), \widetilde{S}(t, x_1^1(t))\right),$$

$$\|w_2^0(t) - w_2^1(t)\| \le (1 + 1)\,\mathcal{H}_2\left(\widetilde{T}(t, K(x_1^0(t))), \widetilde{T}(t, K(x_1^1(t)))\right).$$

Let

$$g_1(t, y^1(t)) = J^{\rho_1(t),A_{1,t}}_{R_1(t,g_1(t,x_1^1(t))}\left(A_1(t, g_1(t, x_1^1(t))) - \rho_1(t)f_1(t, w_1^1(t))\right),$$

$$g_2(t, z^1(t)) = J^{\rho_2(t),A_{2,t}}_{R_2(t,g_2(t,K(y^1(t)))))}\left(A_2(t, g_2(t, K(y^1(t)))) - \rho_2(t)f_2(t, w_2^1(t))\right),$$

$$x_1^2(t) = (1 - \alpha(t))x_1^1(t) + \alpha(t)\left[y^1(t) + \gamma(t)K^*(z^1(t) - K(y^1(t))\right],$$

where $\gamma : \Omega \to (0, +\infty)$ is a measurable function. Then $x_1^2 : \Omega \to H_1$ is measurable. Then, there exist measurable selections $w_1^2 : \Omega \to H_1$ of $\widetilde{S}(\cdot, x_1^2(\cdot))$ and $w_2^2 : \Omega \to H_2$ of $\widetilde{T}(\cdot, K(x_1^2(\cdot)))$ such that $\forall t \in \Omega$,

$$\|w_1^1(t) - w_1^2(t)\| \le \left(1 + \frac{1}{2}\right)\mathcal{H}_1\left(\widetilde{S}(t, x_1^1(t)), \widetilde{S}(t, x_1^2(t))\right),$$

$$\|w_2^1(t) - w_2^2(t)\| \leq \left(1 + \frac{1}{2}\right) \mathcal{H}_2 \left(\widetilde{T}(t, K(x_1^1(t))), \widetilde{T}(t, K(x_1^2(t)))\right).$$

Continuing the above process inductively, we can propose the following random iterative sequences $\{x_1^n(t)\}$, and $\{w_1^n(t)\}$ for solving (SpGRVI) (2.1)–(2.2) as follows:

$$g_1(t, y^n(t)) = J_{R_1(t,g_1(t,x_1^n(t)))}^{\rho_1(t),A_{1,t}} \left(A_1(t, g_1(t, x_1^n(t))) - \rho_1(t) f_1(t, w_1^n(t))\right), \tag{3.1}$$

$$g_2(t, z^n(t)) = J_{R_2(t,g_2(t,K(y^n(t))))}^{\rho_2(t),A_{2,t}} \left(A_2(t, g_2(t, K(y^n(t)))) - \rho_2(t) f_2(t, w_2^n(t))\right), \tag{3.2}$$

$$x_1^{n+1}(t) = (1 - \alpha(t))x^n(t) + \alpha(t) \left[y^n(t) + \gamma(t)K^*(z^n(t) - K(y^n(t))\right], \tag{3.3}$$

$$w_1^{n+1}(t) \in \widetilde{S}(t, x_1^{n+1}(t)), w_2^{n+1}(t) \in \widetilde{T}(t, K(x_1^{n+1}(t))),$$

$$\|w_1^n(t) - w_1^{n+1}(t)\| \leq (1 + (1+n)^{-1})\mathcal{H}_1 \left(\widetilde{S}(t, x_1^n(t)), \widetilde{S}(t, x_1^{n+1}(t))\right), \tag{3.4}$$

$$\|w_2^n(t) - w_2^{n+1}(t)\| \leq (1 + (1+n)^{-1})\mathcal{H}_2 \left(\widetilde{T}(t, K(x_1^n(t))), \widetilde{T}(t, K(x_1^{n+1}(t)))\right), \tag{3.5}$$

for any $t \in \Omega$ and $n = 0, 1, 2....$

Next, we propose an iterative algorithm for solving (SpRVI) (2.3)–(2.4) as a special case of Algorithm 3.1.

Algorithm 3.2. For each $i \in \{1, 2\}$, let f_i, R_i be the same as in the $(\text{SpRVI})_1$ (2.3)–(2.4). Suppose that $S : \Omega \times H_1 \to \mathcal{F}(H_1)$ and $T : \Omega \times H_2 \to \mathcal{F}(H_2)$ be two random fuzzy mappings satisfying the condition (Δ). Let $\widetilde{S} : \Omega \times H_1 \to CB(H_1)$ and $\widetilde{T} : \Omega \times H_2 \to CB(H_2)$ be $\mathcal{H}$-continuous random multi-valued mappings induced by S and T, respectively. Let $A_i : \Omega \times H_i \to H_i$ be a single-valued random mapping and for each fixed $t \in \Omega, R_i(t, \cdot) : H_i \to 2^{H_i}$ be an A_i-monotone mapping. Further, let $\alpha : \Omega \to (0, 1]$ be a measurable step size function. In similar to Algorithm 3.1, for any measurable mapping $x_1^0 : \Omega \to H_1$, we can define sequences $\{x_1^n(t)\}$ and $\{w_1^n(t)\}$ for solving the $(\text{SpRVI})_1$ (2.3)–(2.4) as follows:

$$\begin{cases} y^n(t) = J_{R_1(t,x_1^n(t))}^{\rho_1(t),A_{1,t}} \left(A_1(t, x_1^n(t)) - \rho_1(t) f_1(t, w_1^n(t))\right), \\ z^n(t) = J_{R_2(t,K(y^n(t)))}^{\rho_2(t),A_{2,t}} \left(A_2(t, K(y^n(t))) - \rho_2(t) f_2(t, w_2^n(t))\right), \\ x_1^{n+1}(t) = (1 - \alpha(t))x^n(t) + \alpha(t) \left[y^n(t) + \gamma(t)K^*(z^n(t) - K(y^n(t))\right], \\ w_1^{n+1}(t) \in \widetilde{S}(t, x_1^{n+1}(t)), w_2^{n+1}(t) \in \widetilde{T}(t, K(x_2^{n+1}(t))), \\ \|w_1^n(t) - w_1^{n+1}(t)\| \leq (1 + (1+n)^{-1})\mathcal{H}_1 \left(\widetilde{S}(t, x_1^n(t)), \widetilde{S}(t, x_1^{n+1}(t))\right), \\ \|w_2^n(t) - w_2^{n+1}(t)\| \leq (1 + (1+n)^{-1})\mathcal{H}_2 \left(\widetilde{T}(t, K(x_1^n(t))), \widetilde{T}(t, K(x_1^{n+1}(t)))\right), \end{cases}$$

where $\gamma, \rho_i : \Omega \to (0, +\infty)$ are measurable functions, for all $t \in \Omega$ and $n = 0, 1, 2....$

4. Existence and convergence results

In this section, using the sufficient conditions, we now establish the existence and convergence of iterative Algorithm 3.1 for (SpGRVI) (2.1)–(2.2). Further, the existence of solution and convergences of iterative Algorithm 3.2 is also proposed for (SpRVI) (2.3)–(2.4).

Theorem 4.1. *For each $i \in \{1,2\}$, let H_i be a real Hilbert space, $R_i : \Omega \times H_i \to 2^{H_i}$ be a random multi-valued mapping such that for each fixed $t \in \Omega, R_i(t,\cdot) : H_i \to 2^{H_i}$ be an A_i-monotone mapping and $A_i : \Omega \times H_i \to H_i$ be strongly monotone with constant $m_i(t)$ and Lipschitz continuous with constant $l_{A_i}(t)$. Let $f_i, g_i : \Omega \times H_i \to H_i$ be Lipschitz continuous random mappings with constants $l_{f_i}(t)$ and $l_{g_i}(t)$, respectively. Suppose that g_i is strongly monotone with constant $\kappa_i(t)$. Let $S : \Omega \times H_1 \to \mathcal{F}(H_1)$ and $T : \Omega \times H_2 \to \mathcal{F}(H_2)$ be two random fuzzy mappings satisfying the condition (Δ). Let $\widetilde{S} : \Omega \times H_1 \to CB(H_1)$ and $\widetilde{T} : \Omega \times H_2 \to CB(H_2)$ be random multi-valued mappings induced by S and T, respectively. Suppose that $\widetilde{S}$ and $\widetilde{T}$ are $\mathcal{H}$-continuous with constants $l_{\widetilde{S}}(t)$ and $l_{\widetilde{T}}(t)$, respectively. Let $K : H_1 \to H_2$ be bounded linear operator with K^* be its adjoint operator. If there exist measurable functions $\mu_i, \rho_i, \gamma : \Omega \to (0, +\infty)$ with $\rho_i(t) \in (0, m_i(t)/r_i(t))$, for all $t \in \Omega$, such that*

(i) $\left\| J^{\rho_i(t),A_{i,t}}_{R_i(t,x_i(t))}(z_i(t)) - J^{\rho_i(t),A_{i,t}}_{R_i(t,y_i(t))}(z_i(t)) \right\| \le \mu_i(t)\|x_i(t) - y_i(t)\|$, $\forall x_i(t), y_i(t), z_i(t) \in H_i$;

(ii) $\gamma(t) \in \left(0, \frac{2}{\|K\|^2}\right)$ *and* $\alpha(t)\left(1 - \lambda_1(t) - \gamma(t)\lambda_2(t)\|K\|^2\right) \in (0,1)$, *where*

$$\lambda_1(t) = \frac{1}{\kappa_1(t)}\left[\frac{1}{m_1(t) - \rho_1(t)r_1(t)}\left(l_{A_1}(t)l_{g_1}(t) + \rho_1(t)l_{f_1}(t)l_{\widetilde{S}}(t)\right) + \mu_1(t)l_{g_1}(t)\right],$$

$$\lambda_2(t) = \frac{1}{\kappa_2(t)}\left[\frac{1}{m_2(t) - \rho_2(t)r_2(t)}\left(l_{A_2}(t)l_{g_2}(t)\lambda_1(t) + \rho_2(t)l_{f_2}(t)l_{\widetilde{T}}(t)\right) + \mu_2(t)l_{g_2}(t)\lambda_1(t)\right],$$

then the random iterative sequences $\{x_1^n(t)\}, \{y^n(t)\}, \{z^n(t)\}, \{w_1^n(t)\}$ and $\{w_2^n(t)\}$ constructed by Algorithm 3.1 are convergent sequences.

Proof. From Algorithm 3.1 (3.3), we have

$$\begin{aligned}
&\|x_1^{n+1}(t) - x_1^n(t)\| \\
&\quad= \|(1-\alpha(t))x^n(t) + \alpha(t)\left[y^n(t) + \gamma(t)K^*(z^n(t) - K(y^n(t))\right] \\
&\qquad - \left[(1-\alpha(t))x^{n-1}(t) + \alpha(t)\left[y^{n-1}(t) + \gamma(t)K^*(z^{n-1}(t) - K(y^{n-1}(t))\right]\right]\| \\
&\quad\le (1-\alpha(t))\|x_1^n(t) - x_1^{n-1}(t)\| + \alpha(t)\gamma(t)\|K^*(z^n(t) - z^{n-1}(t))\| \\
&\qquad + \alpha(t)\|y^n(t) - y^{n-1}(t) - \gamma(t)K^*(K(y^n(t)) - K(y^{n-1}(t)))\|.
\end{aligned} \tag{4.1}$$

It follows from iterative Algorithm 3.1 (3.1), Assumption (i) and Lemma 3.8 that

$$\begin{aligned}
&\|g_1(t, y^n(t)) - g_1(t, y^{n-1}(t))\| \\
&\quad= \Big\| J^{\rho_1(t),A_{1,t}}_{R_1(t,g_1(x_1^n(t)))}\left(A_1(t, g_1(t, x_1^n(t))) - \rho_1(t)f_1(t, w_1^n(t))\right) \\
&\qquad - J^{\rho_1(t),A_1}_{R_1(t,g_1(t,x_1^{n-1}(t)))}\left(A_1(t, g_1(t, x_1^{n-1}(t))) - \rho_1(t)f_1(t, w_1^{n-1}(t))\right)\Big\| \\
&\quad\le \Big\| J^{\rho_1(t),A_{1,t}}_{R_1(t,g_1(x_1^n(t)))}\left(A_1(t, g_1(t, x_1^n(t))) - \rho_1(t)f_1(t, w_1^n(t))\right) \\
&\qquad - J^{\rho_1(t),A_{1,t}}_{R_1(t,g_1(t,x_1^n(t)))}\left(A_1(t, g_1(t, x_1^{n-1}(t))) - \rho_1(t)f_1(t, w_1^{n-1}(t))\right)\Big\| \\
&\qquad + \Big\| J^{\rho_1(t),A_{1,t}}_{R_1(t,g_1(t,x_1^n(t)))}\left(A_1(t, g_1(t, x_1^{n-1}(t))) - \rho_1(t)f_1(t, w_1^{n-1}(t))\right)
\end{aligned}$$

$$-J^{\rho_1(t),A_1}_{R_1(t,g_1(t,x_1^{n-1}(t)))}\left(A_1(t,g_1(t,x_1^{n-1}(t)))-\rho_1(t)f_1(t,w_1^{n-1}(t))\right)\Big\|.$$

Hence,

$$\begin{aligned}\|g_1(t,y^n(t))-&g_1(t,y^{n-1}(t))\|\\ &\leq\frac{1}{m_1(t)-\rho_1(t)r_1(t)}\left(\|A_1(t,g_1(t,x_1^n(t)))-A_1(t,g_1(t,x_1^{n-1}(t)))\|\right.\\ &\quad\left.+\rho_1(t)\|f_1(t,w_1^n(t))-f_1(t,w_1^{n-1}(t))\|\right)\\ &\quad+\mu_1(t)\|g_1(t,x_1^n(t))-g_1(t,x_1^{n-1}(t))\|.\end{aligned}\tag{4.2}$$

Since A_1 is Lipschitz continuous with constant $l_{A_1}(t)$, we have

$$\|A_1(t,g_1(t,x_1^n(t)))-A_1(t,g_1(t,x_1^{n-1}(t)))\|\leq l_{A_1}(t)\|g_1(t,x_1^n(t))-g_1(t,x_1^{n-1}(t))\|.\tag{4.3}$$

Since f_1 is Lipschitz continuous with constant $l_{f_1}(t)$, $\widetilde{S}$ is $\mathcal{H}$-continuous with constant $l_{\widetilde{S}}(t)$ and by Algorithm 3.1 (3.4), we have

$$\begin{aligned}\|f_1(t,w_1^n(t))-f_1(t,w_1^{n-1}(t))\|&\leq l_{f_1}(t)\|w_1^n(t)-w_1^{n-1}(t)\|\\ &\leq l_{f_1}(t)l_{\widetilde{S}}(t)(1+n^{-1})\|x_1^n(t)-x_1^{n-1}(t)\|.\end{aligned}\tag{4.4}$$

As g_1 is Lipschitz continuous with constant $l_{g_1}(t)$,

$$\|g_1(t,x_1^n(t))-g_1(t,x_1^{n-1}(t))\|\leq l_{g_1}(t)\|x_1^n(t)-x_1^{n-1}(t)\|.\tag{4.5}$$

From (4.2)-(4.5), we obtain

$$\begin{aligned}&\|g_1(t,y_1^n(t))-g_1(t,y_1^{n-1}(t))\|\\ &\quad\leq\left[\frac{1}{m_1(t)-\rho_1(t)r_1(t)}\left(l_{A_1}(t)l_{g_1}(t)+\rho_1(t)l_{f_1}(t)l_{\widetilde{S}}(t)(1+n^{-1})\right)+\mu_1(t)l_{g_1}(t)\right]\\ &\qquad\times\|x_1^n(t)-x_1^{n-1}(t)\|.\end{aligned}\tag{4.6}$$

By Cauchy-Schwartz inequality and g_1 is strongly monotone with constant $\kappa_1(t)$, we have

$$\begin{aligned}\|g_1(t,y^n(t))-g_1(t,y^{n-1}(t))\|&\|y^n(t)-y^{n-1}(t)\|\\ &\geq\langle g_1(t,y^n(t))-g_1(t,y^{n-1}(t)),y^n(t)-y^{n-1}(t)\rangle\\ &\geq\kappa_1(t)\|y^n(t)-y^{n-1}(t)\|^2,\end{aligned}$$

which implies that

$$\|y^n(t)-y^{n-1}(t)\|\leq\frac{1}{\kappa_1(t)}\|g_1(t,y^n(t))-g_1(t,y^{n-1}(t))\|.\tag{4.7}$$

So, it follows from (4.6) and (4.7) that

$$\|y^n(t)-y^{n-1}(t)\|\leq\lambda_1^n(t)\|x_1^n(t)-x_1^{n-1}(t)\|,\tag{4.8}$$

where

$$\lambda_1^n(t) = \frac{1}{\kappa_1(t)}\left[\frac{1}{m_1(t)-\rho_1(t)r_1(t)}\left(l_{A_1}(t)l_{g_1}(t)+\rho_1(t)l_{f_1}(t)l_{\widetilde{S}}(t)(1+n^{-1})\right)+\mu_1(t)l_{g_1}(t)\right].$$

Similarly, from Algorithm 3.1 (3.2) and (3.5), Assumption (i) and Lemma 3.2 and using the facts that A_2, f_2 and g_2 are Lipschitz continuous with constants $l_{A_2}(t), l_{f_2}(t)$ and $l_{g_2}(t)$, respectively, $\widetilde{T}$ is $\mathcal{H}$-continuous with constan $l_{\widetilde{T}}(t)$ and g_2 is strongly monotone with constant $\kappa_2(t)$, we get

$$\begin{aligned}
\|z^n(t)-z^{n-1}(t)\| \leq & \frac{1}{\kappa_2(t)}\left[\frac{1}{m_2(t)-\rho_2(t)r_2(t)}\left(l_{A_2}(t)l_{g_2}(t)\|K(y^n(t))-K(y^{n-1}(t))\|\right.\right.\\
&\left. +\rho_2(t)l_{f_2}(t)l_{\widetilde{T}}(t)(1+n^{-1})\|K(x_1^n(t))-K(x_1^{n-1}(t))\|\right)\\
&\left. +\mu_2(t)l_{g_2}(t)\|K(y^n(t))-K(y^{n-1}(t))\|\right]\\
\leq & \frac{\|K\|}{\kappa_2(t)}\left[\frac{1}{m_2(t)-\rho_2(t)r_2(t)}\left(l_{A_2}(t)l_{g_2}(t)\|y^n(t)-y^{n-1}(t)\|\right.\right.\\
&\left. +\rho_2(t)l_{f_2}(t)l_{\widetilde{T}}(t)(1+n^{-1})\|x_1^n(t)-x_1^{n-1}(t)\|\right)\\
&\left. +\mu_2(t)l_{g_2}(t)\|y^n(t)-y^{n-1}(t)\|\right]. \qquad (4.9)
\end{aligned}$$

From (4.8) and (4.9), we have

$$\|z^n(t)-z^{n-1}(t)\| \leq \lambda_2^n(t)\|K\|\|x_1^n(t)-x_1^{n-1}(t)\|, \qquad (4.10)$$

where

$$\lambda_2^n(t) =$$

$$\frac{1}{\kappa_2(t)}\left[\frac{1}{m_2(t)-\rho_2(t)r_2(t)}\left(l_{A_2}(t)l_{g_2}(t)\lambda_1^n(t)+\rho_2(t)l_{f_2}(t)l_{\widetilde{T}}(t)(1+\frac{1}{n})\right)+\mu_2(t)l_{g_2}(t)\lambda_1^n(t)\right].$$

It should be noted that K^* is an adjoint operator of the bounded linear operator K with $\|K\|=\|K^*\|$. So

$$\begin{aligned}
\|y^n(t)-y^{n-1}(t)-\gamma(t)K^*(K(y^n(t))-K(y^{n-1}(t)))\|^2 &= \|y^n(t)-y^{n-1}(t)\|^2\\
&-2\gamma(t)\langle y^n(t)-y^{n-1}(t), K^*(K(y^n(t))-K(y^{n-1}(t)))\rangle\\
&+\gamma^2(t)\|K^*(K(y^n(t))-K(y^{n-1}(t)))\|^2\\
&\leq \|y^n(t)-y^{n-1}(t)\|^2-\gamma(t)(2-\gamma(t)\|K\|^2)\|K(y^n(t))-K(y^{n-1}(t))\|^2\\
&\leq \|y^n(t)-y^{n-1}(t)\|^2 \quad \left(\text{since } \gamma(t)\in\left(0,2/\|K\|^2\right)\right)\\
&\leq \left(\lambda_1^n(t)\|x_1^n(t)-x_1^{n-1}(t)\|\right)^2 \quad (\text{by } (4.7)),
\end{aligned}$$

which implies that

$$\|y^n(t)-y^{n-1}(t)-\gamma(t)K^*(K(y^n(t))-K(y^{n-1}(t)))\| \leq \lambda_1^n(t)\|x_1^n(t)-x_1^{n-1}(t)\|. \qquad (4.11)$$

From (4.1), (4.8), (4.10) and (4.11), we get

$$\|x_1^{n+1}(t)-x_1^n(t)\| \leq \lambda^n(t)\|x_1^n(t)-x_1^{n-1}(t)\|, \qquad (4.12)$$

where

$$\lambda^n(t) = 1-\alpha(t)\left(1-\lambda_1^n(t)-\gamma(t)\lambda_2^n(t)\|K\|^2\right), \forall t\in\Omega.$$

Letting

$$\lambda(t) = 1-\alpha(t)\left(1-\lambda_1(t)-\gamma(t)\lambda_2(t)\|K\|^2\right),$$

$$\lambda_1(t) = \frac{1}{\kappa_1(t)} \left[\frac{1}{m_1(t) - \rho_1(t) r_1(t)} \left(l_{A_1}(t) l_{g_1}(t) + \rho_1(t) l_{f_1}(t) l_{\widetilde{S}}(t) \right) + \mu_1(t) l_{g_1}(t) \right],$$
$$\lambda_2(t) = \frac{1}{\kappa_2(t)} \left[\frac{1}{m_2(t) - \rho_2(t) r_2(t)} \left(l_{A_2}(t) l_{g_2}(t) \lambda_1(t) + \rho_2(t) l_{f_2}(t) l_{\widetilde{T}}(t) \right) + \mu_2(t) l_{g_2}(t) \lambda_1(t) \right],$$
$$\forall t \in \Omega.$$

It follows from Assumption (ii) that $0 < \lambda(t) < 1$, for all $t \in \Omega$. Moreover, it is easily seen that for any $t \in \Omega$, $\lambda^n(t) \to \lambda(t)$. Hence, for any $t \in \Omega$, $0 < \lambda^n(t) < 1$, for n sufficiently large. Therefore, it follows from (4.12) that $\{x_1^n(t)\}$ is a Cauchy sequence in H_1. By the completeness of H_1, we get that $\{x_1^n(t)\}$ is a convergent sequence. By the convergence of the sequence $\{x_1^n(t)\}$ and from (4.8) and (4.10), we also obtain that $\{y^n(t)\}$ and $\{z^n(t)\}$ are convergent sequences.

From (3.4) and (3.5), we have

$$\|w_1^n(t) - w_1^{n+1}(t)\| \leq (1 + (1+n)^{-1}) l_{\widetilde{S}}(t) \|x_1^n(t) - x_1^{n+1}(t)\|,$$
$$\|w_2^n(t) - w_2^{n+1}(t)\| \leq (1 + (1+n)^{-1}) l_{\widetilde{T}}(t) \|K\| \|x_1^n(t) - x_1^{n+1}(t)\|, \forall t \in \Omega.$$

So $\{w_1^n(t)\}$ and $\{w_2^n(t)\}$ are also the convergent sequences. This completes the proof. $\square$

Theorem 4.2. *Impose the assumptions of Theorem 4.1 and the following additional condition:*

(Θ) $\lim_{n\to\infty} x_1^n(t) = \lim_{n\to\infty} y^n(t)$ *and* $\lim_{n\to\infty} K(x_1^n(t)) = \lim_{n\to\infty} z^n(t)$*, for all* $t \in \Omega$.
Then, the problem (SpGRVI) (2.1)–(2.2) *has a solution.*

Proof. From Theorem 4.1, we obtain that $\{x_1^n(t)\}, \{y^n(t)\}, \{z^n(t)\}, \{w_1^n(t)\}$ and $\{w_2^n(t)\}$ constructed by Algorithm 3.1 are convergent sequences. Then, let $\lim_{n\to\infty} x_1^n(t) = \lim_{n\to\infty} y^n(t) = x_1^*(t)$ and $\lim_{n\to\infty} w_1^n(t) = w_1^*(t)$. Now, we prove that $\{x_1^*(t), w_1^*(t)\}$ is a solution of (SpGRVI) (2.1)–(2.2). In fact, for any $t \in \Omega$, since $w_1^n(t) \in \widetilde{S}(t, x_1^n(t))$, we get

$$\begin{aligned} d(w_1^*(t), \widetilde{S}(t, x_1^*(t))) &= \inf\{\|w_1^*(t) - s(t)\| : s(t) \in \widetilde{S}(t, x_1^*(t))\} \\ &\leq \|w_1^*(t) - w_1^n(t)\| + d(w_1^n(t), \widetilde{S}(t, x_1^*(t))) \\ &\leq \|w_1^*(t) - w_1^n(t)\| + \mathcal{H}_1\left(\widetilde{S}(t, x_1^n(t)), \widetilde{S}(t, x_1^*(t))\right) \\ &\leq \|w_1^*(t) - w_1^n(t)\| + l_{\widetilde{S}}(t) \|x_1^n(t) - x_1^*(t)\| \to 0. \end{aligned} \tag{4.13}$$

Hence, $w_1^*(t) \in \widetilde{S}(t, x_1^*(t))$, for all $t \in \Omega$. Moreover, using the continuity of K, let $\lim_{n\to\infty} K(x_1^n(t)) = K(x_1^*(t)) = x_2^*(t)$. Then, $\lim_{n\to\infty} z^n(t) = x_2^*(t)$. Let $\lim_{n\to\infty} w_2^n(t) = w_2^*(t)$. Similarly to (4.13), we can also show that $w_2^*(t) \in \widetilde{T}_t(x_2^*)$, for all $t \in \Omega$. Noted that $\{x_1^n(t)\}$, $\{y^n(t)\}$, $\{z^n(t)\}$, $\{w_1^n(t)\}$ and $\{w_2^n(t)\}$ are the sequences of measurable mappings, we know that $x_1^*(t), x_2^*(t), w_1^*(t)$ and $w_2^*(t)$ are also measurable.

Since g_i is continuous, $g_1(t, y^n(t)) \to g_1(t, x_1^*(t))$, $g_2(t, z^n(t)) \to g_2(t, x_2^*(t))$. Then as g_i, A_i, f_i and $J_{R_{i,t}}^{\rho_{i,t}, A_{i,t}}$ are continuous, it follows from (3.1) and (3.2) that, for each $i \in \{1, 2\}$,

$$g_i(t, x_i^*(t)) = J_{R_i(t, g_i(t, x_i^*(t)))}^{\rho_i(t), A_{i,t}} \left(A_i(t, g_i(t, x_i^*(t))) - \rho_i(t) f_i(t, w_i^*(t)) \right), \text{ for all } t \in \Omega.$$

Thus by Remark 3.10, the set of measurable mappings $\{x_1^*(t), w_1^*(t)\}$ is a solution of (SpGRVI) (2.1)–(2.2). This completes the proof. $\square$

If $g_i \equiv I_i$, then the results of Theorems 4.1 and 4.2 reduce to the following result for the existence of solution and convergence of Algorithm 3.2 for (SpRVI) (2.3)–(2.4).

Corollary 4.3. *For each $i \in \{1, 2\}$, let H_i be a real Hilbert space, $R_i : \Omega \times H \to 2^{H_i}$ be a random multi-valued mapping such that for each fixed $t \in \Omega, R_i(t, \cdot) : H_i \to 2^{H_i}$ be an A_i-monotone mapping and $A_i : \Omega \times H_i \to H_i$ be strongly monotone with constant $m_i(t)$ and Lipschitz continuous with constant $l_{A_i}(t)$. Let $f_i : \Omega \times H_i \to H_i$ be a Lipschitz continuous random mapping with constant $l_{f_i}(t)$. Let $S : \Omega \times H_1 \to \mathcal{F}(H_1)$ and $T : \Omega \times H_2 \to \mathcal{F}(H_2)$ be two random fuzzy mappings satisfying the condition (Δ). Let $\widetilde{S} : \Omega \times H_1 \to CB(H_1)$ and $\widetilde{T} : \Omega \times H_2 \to CB(H_2)$ be random multi-valued mappings induced by S and T, respectively. Suppose that $\widetilde{S}$ and $\widetilde{T}$ are $\mathcal{H}$-continuous with constants $l_{\widetilde{S}}(t)$ and $l_{\widetilde{T}}(t)$, respectively. Let $K : H_1 \to H_2$ be a bounded linear operator with K^* be its adjoint operator. Moreover, suppose that the condition (Θ) holds. If there exist measurable functions $\mu_i, \rho_i, \gamma : \Omega \to (0, +\infty)$ with $\rho_i(t) \in (0, m_i(t)/r_i(t))$, for all $t \in \Omega$, such that*

$$\left\| J^{\rho_i(t),A_{i,t}}_{R_i(t,x_i(t))}(z_i(t)) - J^{\rho_i(t),A_{i,t}}_{R_i(t,y_i(t))}(z_i(t)) \right\| \le \mu_i(t)\|x_i(t) - y_i(t)\|, \forall x_i(t), y_i(t), z_i(t) \in H_i$$

and

$$\alpha(t)\left(1 - \lambda_1(t) - \gamma(t)\lambda_2(t)\|K\|^2\right) \in (0, 1), \quad \gamma(t) \in \left(0, \frac{2}{\|K\|^2}\right)$$

where

$$\lambda_1(t) = \left[\frac{1}{m_1(t) - \rho_1(t)r_1(t)}\left(l_{A_1}(t) + \rho_1(t)l_{f_1}(t)l_{\widetilde{S}}(t)\right) + \mu_1(t)\right],$$
$$\lambda_2(t) = \left[\frac{1}{m_2(t) - \rho_2(t)r_2(t)}\left(l_{A_2}(t)\lambda_1(t) + \rho_2(t)l_{f_2}(t)l_{\widetilde{T}}(t)\right) + \mu_2(t)\lambda_1(t)\right],$$

then there exists the set of measurable mappings $\{x_1^*, w_1^* : \Omega \to H_1\}$ be a solution of (SpRVI) (2.3)–(2.4). *Moreover, $x_1^n(t) \to x_1^*(t)$ and $w_1^n(t) \to w_1^*(t)$, where $\{x_1^n(t)\}$ and $\{w_1^n(t)\}$ are random iterative sequences obtained by Algorithm 3.2.*

Remark 4.4. Corollary 4.3 improves and extends Theorem 3.1 in [29], Theorems 3.1, 3.2 and 3.4 in [14] in the following aspects:

(a) The problem (SpRVI) is a generalization of the problems in [29] and [14].
(b) The problem (SpRVI) is established in random and fuzzy environments.
(c) The assumptions and our proof methods are very different from Theorem 3.1 in [29], Theorems 3.1, 3.2 and 3.4 in [14].

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Manuscript received 20 January 2019
revised 12 March 2019

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