Linear and Nonlinear Analysis
Volume 5, Number 1, 2019, 67–71
Yokohama Publishers
ISSN 2188-8167

AN ASPECT OF SPERNER'S LEMMA AS A FIXED POINT THEOREM

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Dedicated to Professor Jong Soo Jung for his 65th birthday

Abstract. Sperner's lemma is a combinatorial version of Brouwer's fixed point theorem. In this paper we present a discrete fixed point theorem by combining Sperner's lemma and a simplicial variant of the direction preserving condition. Our claim is that at least one of the vertices of any completely labeled simplex is a fixed point for a suitable labeling. Therefore we conclude that Sperner's lemma is a kind of combinatorial fixed point theorem.

1. Introduction

Let $\Delta = |a^0\, a^1 \cdots a^n|$ be an n-simplex, Σ be a subdivision of Δ, and V be the set of vertices of Σ. A *labeling* is a mapping from V to $\{0, 1, \ldots, n\}$. The *carrier* of a vertex $v \in V$ is the lowest-dimensional face $|a^{i_0}\, a^{i_1}\, \ldots\, a^{i_s}|$ of Δ that contains v. We denote by $I(v)$ the corresponding index set $\{i_0, i_1, \ldots, i_s\}$. A labeling is said to be *proper* if it assigns to each vertex $v \in V$ one of the numbers in $I(v)$. Given a proper labeling of Σ, an n-simplex in Σ is said to be *completely labeled* if its vertices are labeled $0, 1, \ldots, n$.

Theorem 1.1 (Sperner's lemma, [4]). *Given a proper labeling of Σ, the number of completely labeled simplices is odd.*

As is well-known, Sperner's lemma implies Brouwer's fixed point theorem, and vice versa, see e.g. Border [1]. Therefore Sperner's lemma is a discrete version of Brouwer's fixed point theorem. Although Sperner's lemma does not take the form of a fixed point theorem, we make clear the aspect of Sperner's lemma as a fixed point theorem by a simplicial variant of the direction preserving condition in this paper.

2. Vertices of completely labeled simplices

We first deal with the standard n-simplex

$$\Delta^n = |e^0\, e^1 \cdots e^n| = \{(x_0, \ldots, x_n) \mid x_0 + \cdots + x_n = 1,\ x_i \geq 0\ (i = 0, \ldots, n)\}.$$

U denotes the set of vertices of a given simplicial subdivision Σ^n of Δ^n. We say $u,\, u' \in U$ are *adjacent* if they are vertices of the same simplex in Σ^n, and denote

1991 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Sperner's lemma, proper labeling, discrete fixed point theorem.

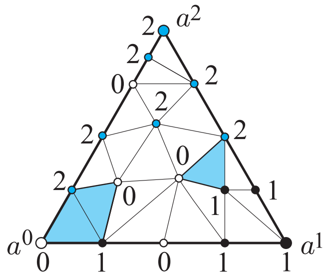

FIGURE 1. Proper labeling and completely labeled simplices.

this relation by $u \sim u'$. A mapping $g : U \to U$ is said to be *simplicially direction preserving* [1] if

$$u \sim u' \;\Rightarrow\; (g_i(u) - u_i)(g_i(u') - u'_i) \geq 0 \quad (i = 0, 1, \ldots, n). \tag{2.1}$$

Labeling (2.2) below was used in the proof of "Sperner's lemma implies Brouwer's fixed point theorem", see Border [1, 6.1].

Theorem 2.1. Let $g : U \to U$ be simplicially direction preserving. Define a labeling L_g by

$$L_g(u) := \min\{i \in I(u) \mid g_i(u) - u_i \leq g_j(u) - u_j \;\forall j \in I(u)\}. \tag{2.2}$$

Then for any completely labeled n-simplex in Σ^n one of its vertices is a fixed point of g.

Proof. Let $\sigma := |u^0 u^1 \cdots u^n|$ be a completely labeled simplex. We may assume that $L_g(u^i) = i$ $(i = 0, 1, \ldots, n)$ without loss of generalization by renumbering the vertices. Put

$$d^i = (d^i_0, d^i_1, \ldots, d^i_n) := g(u^i) - u^i.$$

Since both $g(u^i)$ and u^i belong to Δ^n, the component sum of d^i is 0 for any i.

In the case of there exists k s.t. $d^k_k \geq 0$, d^k is a nonnegative vector by the definition of $L_g(u^k)$. Hence d^k is a zero-vector, that is, u^k is a fixed point of g.

Otherwise $d^i_i < 0$ for all i. Since $u^i \sim u^j$ for any $j \neq i$, by the simplicial direction preserving condition $(g_i(u^i) - u^i_i)(g_i(u^j) - u^j_i) \geq 0$, we have $d^j_i = g_i(u^j) - u^j_i \leq 0$. Since i is arbitrary, d^j is a non-trivial nonpositive vector, which contradicts that the component sum of d^j is zero. □

For any n-simplex $\Delta = |a^0\, a^1 \cdots a^n|$, Theorem 2.1 holds true by modifying the labeling and the simplicial direction preserving condition. In Theorem 2.2 below, we assume that $a^0, \ldots, a^n \in \mathbb{R}^{n+1}$ are linearly independent, and A denotes the square matrix of order $n + 1$ whose ith column is a^i.

Theorem 2.2. Let V denote the set of vertices of a given subdivision Σ of Δ, and f be a mapping from V into itself. For any $v \in V$, $d_i(v)$ denotes the ith component

[1] This is a simplicial variant of the direction preserving condition introduced in Iimura [2]

of $A^{-1}(f(v)-v)$. If f satisfies

$$v \sim v' \;\Rightarrow\; d_i(v)d_i(v') \geq 0 \quad (i=0,1,\ldots,n), \tag{2.3}$$

then there exists a proper labeling such that for any completely labeled n-simplex in Σ one of its vertices is a fixed point of f.

Proof. Since linear transformation A^{-1} maps vertex a^i of Δ to vertex e^i of Δ^n, $\Sigma^n := \{A^{-1}(\sigma) \mid \sigma \in \Sigma\}$ is a subdivision of the standard n-simplex Δ^n. Also for any $v \in V$ and $u := A^{-1}v$, $I(u)$ coincides with $I(v)$. Indeed, since v is uniquely expressed as a convex combination

$$v = \sum_{i \in I(v)} \lambda_i a^i,$$

so is u as

$$u = \sum_{i \in I(v)} \lambda_i A^{-1} a^i = \sum_{i \in I(v)} \lambda_i e^i.$$

Define $g(u) := A^{-1}f(Au)$, then g is a bijection from $U := A^{-1}(V)$ into itself and

$$g(u) - u = A^{-1}f(Au) - u = A^{-1}(f(v)-v) = d(v). \tag{2.4}$$

Hence

$$g(u) = u \;\Leftrightarrow\; f(v) = v. \tag{2.5}$$

Since $u \sim u'$ is equivalent to $v \sim Au'$, the simplicial direction preserving condition on g:

$$u \sim u' \;\Rightarrow\; (g_i(u)-u_i)(g_i(u')-u'_i) \geq 0 \; (i=0,1,\ldots,n)$$

reduces to

$$v \sim v' \;\Rightarrow\; d_i(v)d_i(v') \geq 0 \; (i=0,1,\ldots,n).$$

Applying Theorem 2.2 to g, we see that for any completely labeled n-simplex in Σ^n one of its vertices is a fixed point of g. Desired proper labeling is given by

$$L_f(v) := \min\{i \in I(v) \mid d_i(v) \leq d_j(v) \; \forall j \in I(v)\} = L_g(u),$$

where the last equality follows from (2.4). We get the conclusion by (2.5). □

Finally we remove the assumption that $a^0, \ldots, a^n$ are linearly independent. When $a^0, \ldots, a^n$ are linearly dependent, we take a vector b independent to them, and denote by B the square matrix of order $n+1$ whose ith column is $a^i - b$. Then B is nonsingular.

Theorem 2.3. Let V denote the set of vertices of a given subdivision Σ of Δ, and f be a mapping from V into itself. For any $v \in V$, $d_i(v)$ denotes the ith component of $B^{-1}(f(v)-v)$. If f satisfies

$$v \sim v' \;\Rightarrow\; d_i(v)d_i(v') \geq 0 \quad (i=0,1,\ldots,n), \tag{2.6}$$

then there exists a proper labeling such that for any completely labeled n-simplex in Σ one of its vertices is a fixed point of f.

Proof. Define an affine mapping by $\varphi(u) := Bu+b$, then $\varphi^{-1}(v) = B^{-1}(v-b)$ maps vertex a^i of Δ to vertex e^i of Δ^n, and $\Sigma^n := \{\varphi^{-1}(\sigma) \mid \sigma \in \Sigma\}$ is a subdivision of Δ^n. Also for any $v \in V$ and $u := \varphi^{-1}(v)$, $I(u)$ coincides with $I(v)$. Define

$$g(u) := (\varphi^{-1} \circ f \circ \varphi)(u) = B^{-1}(f(Bu+b) - b),$$

then g is a bijection from $U := \varphi^{-1}(V) = \{B^{-1}(v-b) \mid v \in V\}$ into itself and

$$g(u) - u = B^{-1}(f(Bu+b) - b - Bu) = B^{-1}(f(v) - v) = d(v). \tag{2.7}$$

Hence $g(u) = u$ is equivalent to $f(v) = v$. Since $u \sim u'$ is equivalent to $v \sim \varphi(u')$, the simplicial direction preserving condition on g reduces to

$$v \sim v' \;\Rightarrow\; d_i(v)d_i(v') \geq 0 \; (i = 0, 1, \ldots, n).$$

Desired proper labeling is given by

$$L_f(v) := \min\{i \in I(v) \mid d_i(v) \leq d_j(v) \; \forall j \in I(v)\} = L_g(u).$$

The rest of the proof is same with that of Theorem 2.2. □

We note that Kawasaki-Hashiyama [3] gave a characterization of simplicial direction preserving condition (2.1) for any subdivision of an integral interval

$$\{z_1, z_1+1, \ldots, z_1+w_1\} \times \cdots \times \{z_n, z_n+1, \ldots, z_n+w_n\}.$$

3. Weak simplicial direction preserving condition

Yang [5] presented a discrete fixed point theorem below under the assumption [2]

$$v \sim v' \;\Rightarrow\; (f(v) - v) \cdot (f(v') - v') \geq 0, \tag{3.1}$$

where $u \cdot v$ stands for the inner product of u and v. It is evident that (3.1) is weaker than the simplicial direction preserving condition (2.1).

Theorem 3.1 (Yang [5, Theorem 5.2]). *Let V be a finite set in $\mathbb{R}^n$ and f be a mapping from V into itself. Given a simplicial subdivision for the convex hull of V. If f satisfies (3.1), then f has a fixed point.*

In this section we show that theorems in Section 2 hold true under assumption (3.1) in two-dimensional case. First we deal with the standard 2-simple $\Delta^2 = |e^0\, e^1\, e^2|$. Next we consider arbitrary 2-simplices as well as in Section 2.

Proposition 3.2. Given a subdivision Σ^2 of Δ^2. Let U^2 denote the set of vertices of Σ^2 and $g : U^2 \to U^2$ be a mapping. Let L_g be defined by (2.2). If g satisfies (3.1), then for any completely labeled 2-simplex in Σ^2 one of its vertices is a fixed point of g.

Proof. Let $|u^0\, u^1\, u^2|$ be a completely labeled 2-simplex in Σ^2. We may assume that $L_g(u^i) = i$ $(i = 0, 1, 2)$ without loss of generalization. Put $d^i := g(u^i) - u^i$ and $d^i = (d^i_0, d^i_1, d^i_2)$. Then (3.1) implies

$$d^i \cdot d^j \geq 0 \; (i \neq j). \tag{3.2}$$

[2] Such a function f is said to be *simplicially local gross direction preserving by $-x$* in [5].

Since both $g(u^i)$ and u^i belong to Δ^2, the component sum of d^i is 0. Suppose that none of u^0, u^1 , u^2 is a fixed point, then by definition of L_g, d^0_0, d^1_1 and d^2_2 are negative. Since (3.2) holds true by positive scaling, we may put

$$d^0 = (-1, a, 1-a),\ d^1 = (1-p, -1, p),\ d^2 = (s, 1-s, -1) \tag{3.3}$$

for some $a, p, s \in \mathbb{R}$. Then by (3.2) and a, p, $s \geq -1$,

$$\begin{aligned} 0 &\leq d^1 \cdot d^2 = 2s - (sp + p + 1) = 3s - (s+1)(p+1) \leq 3s \\ 0 &\leq d^0 \cdot d^2 = 2a - (as + s + 1) = 3a - (a+1)(s+1) \leq 3a \\ 0 &\leq d^0 \cdot d^1 = 2p - (pa + a + 1) = 3p - (p+1)(a+1) \leq 3p, \end{aligned} \tag{3.4}$$

we have $a, p, s \geq 0$. Here if $s = 0$, then $p + 1 = 0$, which contradicts $p \geq 0$. Hence s is positive. Similarly, p and a are positive. Hence we have from the first inequality of (3.4) that

$$0 \leq 2s - 1 - (ps + p) < 2s - 1,$$

so that $s > 1/2$.

By putting $b = 1 - a$, $q = 1 - p$, and $t = 1 - s$ in (3.3), (3.4) reduces to

$$\begin{aligned} 0 &\leq d^1 \cdot d^2 = 3b - (b+1)(q+1) \leq 3b \\ 0 &\leq d^0 \cdot d^2 = 3q - (q+1)(t+1) \leq 3q \\ 0 &\leq d^0 \cdot d^1 = 3t - (t+1)(b+1) \leq 3t. \end{aligned} \tag{3.5}$$

As well as a, p and s, we see from (3.5) that b, q and t are positive. Similarly we get from the third inequality of (3.5) that $t > 1/2$, which contradicts that $s + t = 1$. $\square$

Acknowledgements

This research was supported by JSPS KAKENHI Grant Number 16K05278.

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Manuscript received 19 February 2019
revised 6 April 2019

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