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LARGE TIME BEHAVIOR OF SMALL SOLUTIONS TO MULTI-COMPONENT NONLINEAR SCHRÖDINGER EQUATIONS RELATED WITH SPINOR BOSE-EINSTEIN CONDENSATE

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Dedicated to Professor Jong Soo Jung for his 65th happy birthday

Abstract. We study the asymptotic behavior of the solutions to the Cauchy problem of nonlinear Schrödinger equations, in which the unknown function takes matrix values. The nonlinearity is of gauge-invariant cubic type i.e., it is called long-range type in one space dimension. We present a result on the decay estimate of small data solutions, which tells us the non-existence of small soliton. In addition, we observe that the solution asymptotically tends to a modified free evolution. This modification is typically generated by the nonlinearity.

1. Introduction

We study the large time behavior of solutions to the initial value problem of the following multi-component nonlinear Schrödinger equation (NLS) :

$$\begin{cases} i\partial_t Q + \frac{1}{2}\partial_x^2 Q = \mathcal{N}(Q), \\ Q(0,x) = Q_0(x), \end{cases} \tag{1.1}$$

where $i = \sqrt{-1}$, $(t,x) \in \mathbb{R}\times\mathbb{R}$, $Q = Q(t,x) = (\varphi_{jk}(t,x))_{1\le j,k\le N}$ is an $N\times N$ matrix-valued unknown function whose entries take complex values, and $\partial_t Q$, $\partial_x^2 Q$ denotes $(\partial\varphi_{jk}/\partial t)_{1\le j,k\le N}$, $(\partial^2\varphi_{jk}/\partial x^2)_{1\le j,k\le N}$ respectively. The matrix-valued nonlinearity $\mathcal{N}(Q)$ is of gauge invariant cubic type described as

$$\mathcal{N}(Q) = \lambda_1 Q^* Q^2 + \lambda_2 Q Q^* Q + \lambda_3 Q^2 Q^*, \tag{1.2}$$

where λ_j's $(j = 1,2,3)$ are real values and Q^* denotes the Hermite conjugate of Q, i.e., $(\varphi_{jk})^*_{1\le j,k\le N} = (\overline{\varphi_{kj}})_{1\le j,k\le N}$. We are going to solve (1.1) under given matrix-valued initial data $Q_0(x)$.

The study of (1.1) is motivated by the physical model describing the Bose-Einstein condensate of alkali-metal atoms with spin $F = 1$, in which the temporal evolution of the state is represented by three order parameters ϕ_1, ϕ_0 and ϕ_{-1} – the subcripts 1, 0, -1 denote spin quantum numbers of the atom. T.L.Ho [7] and T.Ohmi-K.Machida [11] proposed that these functions obay a system of coupled

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Gross-Pitaevskii equations,

$$
(1.3)\qquad \begin{cases} i\partial_t\phi_1 + \frac{1}{2}\partial_x^2\phi_1 \\ = (c_0+c_2)(|\phi_1|^2+|\phi_0|^2)\phi_1 + (c_0-c_2)|\phi_{-1}|^2\phi_1 + c_2\overline{\phi}_{-1}\phi_0^2, \\ i\partial_t\phi_0 + \frac{1}{2}\partial_x^2\phi_0 \\ = (c_0+c_2)(|\phi_1|^2+|\phi_{-1}|^2)\phi_0 + c_0|\phi_0|^2\phi_0 + 2c_2\phi_1\phi_{-1}\overline{\phi}_0, \\ i\partial_t\phi_{-1} + \frac{1}{2}\partial_x^2\phi_{-1} \\ = (c_0+c_2)(|\phi_{-1}|^2+|\phi_0|^2)\phi_1 + (c_0-c_2)|\phi_1|^2\phi_{-1} + c_2\overline{\phi}_1\phi_0^2, \end{cases}
$$

where c_0 and c_2 are real constants – the expression in (1.3) is simplified through the appropriate scale-transformation. In particular, when $c_0 = c_2$, (1.3) becomes an integrable system, and J.Ieda-T.Miyakawa-M.Wadati [8] seek for solitary wave solutions by applying the inverse scattering method, where (1.3) is transformed into the matrix-valued NLS after arranging the entries $\phi_{\pm 1}$ and ϕ_0 into a 2×2 matrix Q,

$$
Q = \begin{pmatrix} \phi_1 & \phi_0 \\ \phi_0 & \phi_{-1} \end{pmatrix}.
$$

In this reduction, we see that (1.3) is equivalent to (1.1) with $\lambda_1 = \lambda_3 = 0$. In the present paper, however, we treat more general version of matrix-valued NLS rather than (1.3) — not only the number of matrix-components is increased and the symmetricity of Q is excluded, but also λ_j-terms ($j = 1, 3$) are newly contained. Note that, if $\lambda_1 = \lambda_3$, (1.1) is endowed with the Hamilton structure, i.e., it is described like $i\partial_t Q = \delta\mathcal{H}/\delta Q^*$ with $\mathcal{H} = \int \mathrm{Tr}(\partial_x Q^*\partial_x Q + \lambda_1 Q^*Q^*QQ + (\lambda_2/2)Q^*QQ^*Q)dx$, where "Tr" denotes the trace of matrices. Thus this kind of generalization is not so imaginary one, but it might be an acceptable physical model.

From the mathematical point of view, there are various kinds of works concerning the asymptotic analysis on the solution to NLS (see e.g. [3–6, 10, 12], and refer to [1, 2, 9] for the instructive text on NLS). The one-dimensional NLS containing a cubic nonlinearity usually brings an interesting asymptotic profile of the solution, which is away from any solutions of associated linear Schrödinger equation, because of the slow decay of the nonlinearity. We want to discuss this point in detail with the help of scalar-valued case: $i\partial_t u + (1/2)u = |u|^2 u$. In order to observe what is the matter on the slow decay of the nonlinearity, let $v(t) = U(-t)u(t)$ where $U(t) = \exp(it\partial_x^2/2)$ denotes the solution operator of the linear Schrödinger equation. Then $v(t)$ satisfies $i\partial_t v = U(-t)(|U(t)v|^2U(t)v)$. Note here that $U(t)$ is factorized like

$$
U(t) = MD\mathcal{F}M,
$$

where M is the multiplication of $\exp(ix^2/2t)$, $Df(x) = (it)^{-1/2}f(x/t)$ and $\mathcal{F}$ denotes the Fourier transform defined by $(2\pi)^{-1/2}\int e^{-i\xi x}f(x)dx$. Applying this factorization and $U(-t) = M^{-1}\mathcal{F}^{-1}D^{-1}M^{-1}$ to the equation of v and taking the Fourier transform on both hand sides, we see that

$$
(1.4)\qquad \begin{aligned} \partial_t\mathcal{F}v &= -i|t|^{-1}\mathcal{F}M^{-1}\mathcal{F}^{-1}(|\mathcal{F}Mv|^2\mathcal{F}Mv) \\ &= -i|t|^{-1}|\mathcal{F}v|^2\mathcal{F}v + (\text{error}), \end{aligned}
$$

since $M \to 1$ as $t \to \infty$. Then the dominant of the slow decay appears on the right hand side of (1.4). The idea used in [3–6, 10, 12] to control this term is applying certain gauge transform. Namely, let $\eta(t,\xi) = \int_1^t |\tau|^{-1}|\mathcal{F}v(\tau,\xi)|^2 d\tau$, and the simple computation leads to

$$\partial_t(e^{i\eta}\mathcal{F}v) = e^{i\eta} \times (\text{error}),$$

from which we can show that there exists a limit of $e^{i\eta}\mathcal{F}v$ since the error term expectedly decays so rapidly that it is integrable around $t = \infty$. Therefore the reverses of $\mathcal{F}$ and $U(-t)$ etc. yield the desired asymptotic profile of $u(t)$. We also see that $u(t)$ approaches to the free solution modified in phase. This modification associated with η arises typically from the nonlinearity of long range type.

When it comes to the multi-component case (of our interest in this paper), the situation changes. Of course, the analogy to derive (1.4) specifies the dominant slowly decaying factor in the nonlinearity. However, unlike the scalar-valued case, there are many components complicatedly included in one equation and so we can not control the dominant term by using such a scalar-valued modification in phase. To overcome this difficulty, we want to introduce a suggestive idea in T.Wada's works [14, 15]. In his works, he considers Hartree-Fock type equation, i.e., $i\partial_t\vec{u} + (1/2)\Delta\vec{u} = F(\vec{u})\vec{u}$, where $\vec{u}(t,x) = (u_1, u_2, \cdots, u_N)^t$ is a $\mathbf{C}^N$-valued function on $\mathbb{R} \times \mathbb{R}^n$ $(n \geq 3)$, $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \cdots + \partial^2/\partial x_n^2$, and $F(\vec{u}) = (F_{jk}(\vec{u}))_{1\leq j,k\leq N}$ is the Hermite-symmetric matrix defined by

$$F_{jk}(\vec{u}) = \lambda \int |x-y|^{-1}(|\vec{u}(t,y)|^2\delta_{jk} - u_j\bar{u}_k(t,y))\, dy,$$

with $\lambda \in \mathbf{R}$ and δ_{jk} is Kronecker's delta. Analogously in (1.4), we arrive at

$$\partial_t\mathcal{F}\vec{v} = -i|t|^{-1}F(\mathcal{F}\vec{v})\mathcal{F}\vec{v} + (\text{error}), \tag{1.5}$$

where $\vec{v}(t) = \exp(-it\Delta/2)\vec{u}(t)$. To control the slowly decaying term on the right hand side of (1.5), T.Wada employed a matrix-valued gauge transform, i.e., an $N \times N$ unitary matrix $\tilde{A}(t,\xi)$ satisfying the ODE,

$$\begin{cases} \partial_t\tilde{A} = i|t|^{-1}\tilde{A}F(\mathcal{F}\vec{v}), \\ \tilde{A}(1,\xi) = I_N, \end{cases}$$

where I_N is the $N \times N$ unit matrix, so that (1.5) is rewritten as $\partial_t(\tilde{A}\mathcal{F}\vec{v}) = \tilde{A} \times (\text{error})$. Hence the leading term of asymptotics is determined.

Let us return to our equation (1.1). If $\lambda_3 = 0$, then T.Wada's idea is applicable to the study of asymptotic analysis of Q in parallel. However, the presence of λ_3-term makes the problem slightly complicated. In fact, one can not expect any more to rewrite (1.1) as $\partial_t(A'\mathcal{F}V) = A' \times (\text{error})$ for any unitary matrix A' with $V(t) = u(-t)Q(t)$, since Q^* in λ_3-term interrupts such calculation. Our idea to estimate the slowly decaying term is to use a pair of unitary matrices A and B, the detail of which will be shown in § 3. Shortly speaking, we rewrite (1.1) like $\partial_t(A(\mathcal{F}V)B) = A \times (\text{error}) \times B$ and determine the asymptotic profile of Q. Our goal consists of two theorems — the function spaces stated in these results will be defined at the end of this section.

Theorem 1.1 (Global Existence and Decay Estimates). *Let* $\|Q_0\|_{\Sigma^1_x} < \varepsilon$ *with* $\varepsilon > 0$ *sufficiently small. Then, there exists a unique global solution to* (1.1) *such that*

$$Q(t,x) \in C(\mathbb{R};\Sigma^1_x(\mathbb{R})) \cap C^1(\mathbb{R};H^{-1}_x(\mathbb{R})). \tag{1.6}$$

Furthermore, there exists some $C > 0$ *such that, for* $p \in [2,\infty]$, *we have*

$$\|Q(t,\cdot)\|_{L^p_x} \leq C\varepsilon(1+|t|)^{-(1/2-1/p)}. \tag{1.7}$$

In Theorem 1.1 (1.7), we can not see any nonlinear effect in the decay rate. Namely it is similar to the decay estimate of the free evolution. However, when we observe the asymptotic profile of $Q(t,x)$, the nonlinear effect is visible in phase modification.

Theorem 1.2 (Asymptotic Profile). *Let* $Q(t,x)$ *be the solution as in Theorem 1.1. Then there exist some matrices* $\hat{\Phi}(x) \in L^2_x(\mathbb{R}) \cap L^\infty_x(\mathbb{R})$ *and Hermite-symmetric* $\Theta(x) \in L^2_x(\mathbb{R}) \cap L^\infty_x(\mathbb{R})$ *such that*

$$\begin{aligned} Q(t,x) \quad &= \quad U(t)\mathcal{F}^{-1} e^{i\log t\Theta} e^{i\log t\Lambda}\ \hat{\Phi}\ e^{-i\log t\Theta} \\ &\quad + o(1) \quad \text{as } t\to\infty \text{ in } L^2_x(\mathbb{R}), \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} Q(t,x) \quad &= \quad MDe^{i\log t\Theta} e^{i\log t\Lambda}\ \hat{\Phi}\ e^{-i\log t\Theta} \\ &\quad + o(t^{-1/2}) \quad \text{as } t\to\infty \text{ in } L^\infty_x(\mathbb{R}), \end{aligned} \tag{1.9}$$

where $\Lambda(x) = -(\lambda_1+\lambda_2+\lambda_3)\hat{\Phi}\hat{\Phi}^*(x)$.

Let us close this section by introducing several notations and conventions frequently used in this paper. For $Q = (\psi_{jk})_{1\leq j,k\leq N}$, $|Q|$ denotes the absolute value of Q defined by

$$|Q| = \sqrt{\mathrm{Tr}(QQ^*)},$$

which is equivalent to $(\sum_{1\leq j,k\leq N} |\psi_{jk}|^2)^{1/2}$. For this absolute value, we see that $|M_1M_2| \leq |M_1||M_2|$ holds for any $N\times N$ matrices M_1 and M_2. For any linear operator P, PQ denotes a matrix each entry of which is described as $P\psi_{jk}$, i.e., $PQ = (P\psi_{jk})_{1\leq j,k\leq N}$. When $Q(x)$ is a matrix-valued function of x, the $L^p_x(\mathbb{R})$-norm of Q is given by

$$\|Q\|_{L^p_x} = \begin{cases} \left(\int |Q(x)|^p\ dx\right)^{1/p} & \text{if } 1\leq p<\infty, \\ \operatorname{ess.sup}_{x\in\mathbb{R}} |Q(x)| & \text{if } p=\infty. \end{cases}$$

When $Q(t,x)$ is a function of t and x, the time-space norm of Q is defined by

$$\|Q\|_{L^q_t(I;L^p_x)} = (\int_I \|Q(t,\cdot)\|^q_{L^p_x}\ dt)^{1/q}.$$

The Sobolev space $H^1_x(\mathbb{R})$ and weighted Sobolev space $\Sigma^1_x(\mathbb{R})$ for matrix-valued functions are respectively defined by

$$\begin{aligned} H^1_x(\mathbb{R}) &= \{Q(x) \in L^2_x(\mathbb{R})\ ;\ \|Q\|_{H^1_x} < \infty\}, \\ \Sigma^1_x(\mathbb{R}) &= \{Q(x) \in L^2_x(\mathbb{R})\ ;\ \|Q\|_{\Sigma^1_x} < \infty\}, \end{aligned}$$

where $\|Q\|_{H^1_x} = \|Q\|_{L^2_x} + \|\partial_x Q\|_{L^2_x}$ and $\|Q\|_{\Sigma^1_x} = \|Q\|_{H^1_x} + \|xQ\|_{L^2_x}$. The Sobolev space of negative index $H^{-1}_x(\mathbb{R})$ denotes the dual space of $H^1_x(\mathbb{R})$. In the proof of Theorem 1.1 and 1.2, we will often use the operator $J = U(t)xU(-t)$ for the estimate of error terms. Note that J has two kinds of another expression like $J = x + it\partial_x$ or $J = M(it\partial_x)\bar{M}$ with $M = \exp(ix^2/2t)$. Actually we need to estimate the error terms in use of $\|\partial_x \mathcal{F}V\|_{L^2_x} = \|JQ\|_{L^2_x}$. Also, the operator J is more convenient in the derivation of several a priori estimates than the multiplication of x itself, since J commutes with $i\partial_t + \frac{1}{2}\partial_x^2$.

2. Proof of Theorem 1.1

It is easy to show the existence and uniqueness of the local solution. In fact, it is accomplished due to the well-known contraction mapping principle applied to the associated integral equation, i.e.,

$$\begin{aligned} Q(t) &= \Phi(Q(t)) \\ &\equiv U(t)Q_0 - i\int_0^t U(t-\tau)\mathcal{N}(Q(\tau))\, d\tau, \end{aligned} \tag{2.1}$$

where the integral with respect to τ is defined as the Riemannian integral for the $H^{-1}(\mathbb{R})$-valued functions. Let $I = [-T, T]$ with $T > 1$ and

$$\overline{B}_{4\varepsilon} = \{Q(t,x) \in C(I; H^1_x(\mathbb{R})); \|\!|Q|\!\|_X \le 4\varepsilon\},$$

where $\|\!|Q|\!\|_X = \|Q\|_{L^\infty_t(I;H^1_x)} + \|JQ\|_{L^\infty_t(I;L^2_x)}$. Then, Φ is a contraction map on $\overline{B}_{4\varepsilon}$ for small ϵ. In fact, the embedding $H^1_x(\mathbb{R}) \subset L^\infty_x$ yields

$$\begin{aligned} \|\Phi(Q)\|_{L^\infty_t(I;H^1_x)} &\le \varepsilon + CT\|Q\|^2_{L^\infty_t(I;L^\infty_x)}\|Q\|_{L^\infty_t(I;H^1_x)} \\ &\le \varepsilon + CT\|Q\|^3_{L^\infty_t(I;H^1_x)}. \end{aligned} \tag{2.2}$$

In addition, noting that

$$\begin{aligned} J\mathcal{N}(Q) &= \lambda_1(-(JQ)^*Q^2 + Q^*(JQ)Q + Q^*Q(JQ)) \\ &\quad +\lambda_2(JQQ^*Q - Q(JQ)^*Q + QQ^*(JQ)) \\ &\quad +\lambda_3((JQ)QQ^* + Q(JQ)Q^* - Q^2(JQ)^*), \end{aligned}$$

we have

$$\begin{aligned} \|J\Phi(Q)\|_{L^\infty_t(I;L^2_x)} &\le \varepsilon + CT\|Q\|^2_{L^\infty_t(I;L^\infty_x)}\|JQ\|_{L^\infty_t(I;L^2_x)} \\ &\le \varepsilon + CT\|Q\|^2_{L^\infty_t(I;H^1_x)}\|JQ\|_{L^\infty_t(I;L^2_x)}. \end{aligned} \tag{2.3}$$

Combining (2.2) and (2.3) and taking $\varepsilon > 0$ sufficiently small, we see that

$$\begin{aligned} \|\Phi(Q)\|_X &\le 2\varepsilon + CT\|Q\|_X^3 \\ &\le 2\varepsilon + CT(4\varepsilon)^3 \\ &\le 4\varepsilon. \end{aligned}$$

Analogously, taking $\varepsilon > 0$ further small if needed, we also see that

$$\begin{aligned} \|\Phi(Q_1) - \Phi(Q_2)\|_X &\le CT(\|Q_1\|_X^2 + \|Q_2\|_X^2)\|Q_1 - Q_2\|_X \\ &\le CT\varepsilon^2\|Q_1 - Q_2\|_X \\ &\le \frac{1}{2}\|Q_1 - Q_2\|_X. \end{aligned}$$

Thus, Φ is the contraction map on $\overline{B}_{4\varepsilon}$, and so there exists a solution to (2.1) in $C(I; \Sigma_x^1(\mathbb{R}))$. The solution belongs to $C^1(I; H_x^{-1}(\mathbb{R}))$. The uniqueness follows from the routine work.

To continue the local solution to the global one, we need constructing the a priori estimate which denies the blow-up in finite time. The former part of Theorem 1.1 is the direct consequence of the proposition below.

Proposition 2.1 (Global Existence and Estimate of JQ)**.** *Let* $\|Q_0\|_{\Sigma_x^1} < \varepsilon$ *with* $\varepsilon > 0$ *sufficiently small. Then, there exists a unique global solution to* (1.1) *such that*

$$Q(t,x) \in C(\mathbb{R}; H_x^1(\mathbb{R})) \cap C^1(\mathbb{R}; H^{-1}(\mathbb{R})) \tag{2.4}$$

$$JQ(t,x) \in C(\mathbf{R}; L_x^2(\mathbb{R})). \tag{2.5}$$

Furthermore, for some $C, K > 0$, $Q(t,x)$ *satisfies*

$$\|Q(t,\cdot)\|_{H_x^1} + \|JQ(t,\cdot)\|_{L_x^2} \le C\varepsilon(1+|t|)^{K\varepsilon^2}. \tag{2.6}$$

In the proof of Proposition 2.1, we will only consider the case $t > 0$, since the negative time version follows analogously. Let

$$T' = \sup\{T > 0; \sup_{0 \le t < T} (1+t)^{-K\varepsilon^2}\|Q(t)\|_Y < 10\varepsilon\},$$

where

$$\|Q(t)\|_Y = \|Q(t,\cdot)\|_{H_x^1} + \|JQ(t,\cdot)\|_{L_x^2}.$$

The set $\{T > 0; \sup_{0 \le t < T}(1+t)^{-K\varepsilon^2}\|Q(t)\|_Y < 10\varepsilon\}$ is not empty, since there exist a local solution for $T > 1$ as shown at the beginning of this section. Therefore T' is taken as a number greater than one. In order to prove Proposition 2.2, it suffices to show that T' can reach infinity by choosing appropriate $K > 0$. This will be accomplished by the contradiction argument together with the a priori estimate, which follows from the three estimates below.

Lemma 2.2. *Let $\varepsilon > 0$ be sufficiently small. Then there exists some $C_0 > 0$ such that, for $t \in [0, T')$,*

$$\|Q(t)\|_Y \le \varepsilon + C_0 \int_0^t \|Q(\tau)\|_{L_x^\infty}^2 \|Q(\tau)\|_Y \, d\tau, \tag{2.7}$$

$$\|Q(\tau)\|_{L_x^\infty} \le t^{-1/2} \|\mathcal{F}U(-\tau)Q(\tau)\|_{L_x^\infty} + C_0 \varepsilon t^{-3/4+K\varepsilon^2}, \tag{2.8}$$

$$\|\mathcal{F}U(-\tau)Q(\tau)\|_{L_x^\infty} \le C_0 \varepsilon. \tag{2.9}$$

Proof of Lemma 2.2. The estimate (2.7) follows directly from the integral equation (2.1). To prove (2.8), we use the factorization $U(\tau) = MD\mathcal{F}M$. Note that

$$\begin{aligned} \|Q(\tau)\|_{L_x^\infty} & \\ &= \|U(\tau)U(-\tau)Q(\tau)\|_{L_x^\infty} \\ &= \|MD\mathcal{F}MU(-\tau)Q(\tau)\|_{L_x^\infty} \\ &= t^{-1/2}\|\mathcal{F}U(-\tau)Q(\tau)\|_{L_x^\infty} + t^{-1/2}\|\mathcal{F}(M-1)U(-\tau)Q(\tau)\|_{L_x^\infty}. \end{aligned} \tag{2.10}$$

Gagliardo-Nirenberg's inequality and Prancherell's identity yield

$$\begin{aligned} \|\mathcal{F}(M-1)U(-\tau)Q(\tau)\|_{L_x^\infty} & \\ &\le C\|\mathcal{F}(M-1)U(-\tau)Q(\tau)\|_{L_x^2}^{1/2} \|\partial_x \mathcal{F}(M-1)U(-\tau)Q(\tau)\|_{L_x^2}^{1/2} \\ &\le C\|(x/\sqrt{\tau})U(-\tau)Q(\tau)\|_{L_x^2}^{1/2} \|xU(-\tau)Q(\tau)\|_{L_x^2}^{1/2} \\ &= C\tau^{-1/4}\|JQ(\tau)\|_{L_x^2} \\ &\le C\tau^{-1/4}\|Q(\tau)\|_Y \\ &\le C\varepsilon\tau^{-1/4+K\varepsilon^2}. \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11), we obtain (2.8).

To prove (2.9), let $V(\tau) = U(-\tau)Q(\tau)$. For the rigorous proof, we need to overcome the lack of regularity in $V(\tau)$ by considering $\eta_\nu * V(\tau) \in C^1([0, T' - \nu); L_x^2(\mathbb{R}) \cap L_x^\infty(\mathbb{R}))$, where $\eta_\nu(\tau) = \nu^{-1}\eta(\nu^{-1}\tau)$ with $\eta \in C_0^\infty(\mathbb{R})$, $\int \eta(\tau)d\tau = 1$ and $\nu \in (0, 1]$. However, for simplicity of the proof, we proceed in formal way. By taking advantage of $U(t) = MD\mathcal{F}M$ and $U(-t) = M^{-1}\mathcal{F}^{-1}D^{-1}M^{-1}$ together with the gauge invariance of $\mathcal{N}(Q)$, the original equation (1.1) can be transformed as

$$\partial_\tau \mathcal{F}V = -i\tau^{-1}\mathcal{N}(\mathcal{F}V) + R(\tau), \tag{2.12}$$

where

$$R(\tau) = -i\tau^{-1}(\mathcal{F}M^{-1}\mathcal{F}^{-1}\mathcal{N}(\mathcal{F}MV) - \mathcal{N}(\mathcal{F}V)).$$

Note that, similarly to (2.11), we have

$$\begin{aligned} \|R(\tau)\|_{L_x^2} &\le C\tau^{-3/2}\|Q(\tau)\|_Y^3 \\ &\le C\varepsilon^3\tau^{-3/2+3K\varepsilon^2}, \end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\|R(\tau)\|_{L^\infty_x} &\le C\tau^{-5/4}\|Q(\tau)\|_Y^3 \\ &\le C\varepsilon^3\tau^{-5/4+3K\varepsilon^2}.\end{aligned} \tag{2.14}$$

From (2.12), it follows that

$$\begin{aligned}&\partial_\tau|\mathcal{F}V|^2 \\ &= -i\tau^{-1}\mathrm{Tr}\{\mathcal{N}(\mathcal{F}V)(\mathcal{F}V)^*\} + i\tau^{-1}\mathrm{Tr}\{(\mathcal{F}V)(\mathcal{N}(\mathcal{F}V))^*\} \\ &\quad +\mathrm{Tr}\{R(\tau)(\mathcal{F}V)^* + (\mathcal{F}V)R(\tau)^*\} \\ &= \mathrm{Tr}\{R(\tau)(\mathcal{F}V)^* + (\mathcal{F}V)R(\tau)^*\},\end{aligned} \tag{2.15}$$

where the cancellation of the first and second terms on the top right hand side of (2.15) occurs due to $\mathrm{Tr}(M_1M_2) = \mathrm{Tr}(M_2M_1)$ for any $N \times N$ matrices M_1 and M_2. Combining (2.14) and (2.15), we see that

$$|\mathcal{F}V(\tau)|^2 \le |\mathcal{F}V(1)|^2 + C\varepsilon^3\int_1^\tau \sigma^{-5/4+3K\varepsilon^2}|\mathcal{F}V(\sigma)|d\sigma.$$

Since $2\varepsilon|\mathcal{F}V(\tau)| \le \varepsilon^2 + |\mathcal{F}V(\tau)|^2$ and $|\mathcal{F}V(1)| \le C\varepsilon$, we have

$$|\mathcal{F}V(\tau)| \le C\varepsilon + C\varepsilon^2\int_1^\tau \sigma^{-5/4+3K\varepsilon^2}|\mathcal{F}V(\sigma)|d\sigma.$$

Then Gronwall's inequality yields

$$\begin{aligned}|\mathcal{F}V(\tau)| &\le C\varepsilon\exp\frac{C\varepsilon^2}{1/4-3K\varepsilon^2} \\ &\le C_0\varepsilon,\end{aligned}$$

if ε is taken small enough. □

Proof of Proposition 2.1. Combining (2.7)–(2.9) in Lemma 2.2, we have, for some positive C_1,

$$\begin{aligned}\|Q(t)\|_Y &\le \varepsilon + C_0\int_0^1 \|Q(\tau)\|_{L^\infty_x}^2\|Q(\tau)\|_Y d\tau + C_1\varepsilon^2\int_1^t \tau^{-1}\|Q(\tau)\|_Y d\tau \\ &\le 2\varepsilon + C_1\varepsilon^2\int_0^1 \tau^{-1}\|Q(\tau)\|_Y d\tau.\end{aligned}$$

Gronwall's inequality leads to

$$\|Q(t)\|_Y \le 2\varepsilon(1+|t|)^{C_1\varepsilon^2},$$

for $t \in [0, T')$. Regard here C_1 as K in the definition of T'. Then, if $T' < \infty$, it follows from the continuity of $\|Q(t)\|_Y$ that

$$10\varepsilon = \lim_{t\uparrow T'}(1+|t|)^{-K\varepsilon^2}\|Q(t)\|_Y \le 2\varepsilon.$$

This is the contradiction. Hence, T' must be ∞. □

We are now in the position to finish the proof of Theorem 1.1 except for the L^p estimate of $Q(t,x)$.

completion of the proof for Theorem 1.1 Since $T' = \infty$, the estimates (2.8) and (2.9) hold globally in time. Then $\|Q(t)\|_{L^\infty_x} \le C\varepsilon(1+|t|)^{-1/2}$. Additionally $\|Q(t)\|_{L^2_x} = \|Q_0\|_{L^2_x}$ since

$$\begin{aligned} \frac{d}{dt}\|Q(t)\|^2_{L^2_x} &= i^{-1}\int \{\mathrm{Tr}(\mathcal{N}(Q)Q^*) - \mathrm{Tr}(Q\mathcal{N}(Q)^*)\}\,dx \\ &= 0 \end{aligned}$$

holds due to the trace property : $\mathrm{Tr}(M_1M_2) = \mathrm{Tr}(M_2M_1)$.Hence, by the interpolation, we have $\|Q(t)\|_{L^p_x} \le C\varepsilon(1+|t|)^{-(1/2-1/p)}$. □

3. Proof of Theorem 1.2

To determine the asymptotic profile of $Q(t,x)$, we reuse the expression (2.12), i.e.,

$$\begin{aligned} &\partial_t\mathcal{F}V \\ &+it^{-1}(\lambda_1(\mathcal{F}V)^*(\mathcal{F}V)^2 + \lambda_2(\mathcal{F}V)(\mathcal{F}V)^*(\mathcal{F}V) + \lambda_3(\mathcal{F}V)^2(\mathcal{F}V)^*) \\ &= R(t), \end{aligned} \tag{3.1}$$

where $V = U(-t)Q(t)$. In (3.1), the decay rate of $it^{-1}\mathcal{N}(\mathcal{F}V)$ is not so enough, and hence it possibly affects the large time behavior of the solution. Now our interest is confined to finding how to control this badly decaying term. Our idea is to use two $N\times N$ matrix-valued functions $A(t,x)$ and $B(t,x)$ for which the following identity holds :

$$\partial_t\left(A(\mathcal{F}V)B\right) = AR(t)B. \tag{3.2}$$

The simple computation such as

$$\partial_t(A(\mathcal{F}V)B) = (\partial_tA)(\mathcal{F}V)B + A(\partial_t\mathcal{F}V)B + A(\mathcal{F}V)\partial_tB$$

presents a system of ordinary differential equations on A and B :

$$\begin{cases} \partial_tA = it^{-1}A\{\lambda_1(\mathcal{F}V)^*(\mathcal{F}V) + \lambda_2(\mathcal{F}V)(\mathcal{F}V)^*\}, \\ \partial_tB = it^{-1}\lambda_3(\mathcal{F}V)(\mathcal{F}V)^*B. \end{cases} \tag{3.3}$$

We solve (3.3) under the initial condition :

$$A(1,x) = I_N, \quad B(1,x) = I_N, \tag{3.4}$$

where I_N is the unit matrix. As we shall see, (3.4) makes $A(t,x)$ and $B(t,x)$ be unitary matrices.

Lemma 3.1. *There exists a unique pair of solutions to* (3.3) *and* (3.4) *such that*

$$A(t,x), B(t,x) \in C^1([1,\infty); L^\infty_x(\mathbb{R})). \tag{3.5}$$

Furthermore, both $A(t,x)$ and $B(t,x)$ are the unitary matrices for any $(t,x) \in [1,\infty)\times\mathbb{R}$.

Proof of Lemma 3.1. Let us consider the integral equations associated with (3.3) and (3.4):

$$
\begin{aligned}
(3.6)\qquad A(t) &= \Phi_1(A) \\
&\equiv I_N + i\int_1^t \tau^{-1}A\{\lambda_1(\mathcal{F}V)^*(\mathcal{F}V)+\lambda_2(\mathcal{F}V)(\mathcal{F}V)^*\}(\tau)d\tau, \\
(3.7)\qquad B(t) &= \Phi_2(B) \\
&\equiv I_N + i\lambda_3\int_1^t \tau^{-1}(\mathcal{F}V)(\mathcal{F}V)^*B(\tau)d\tau.
\end{aligned}
$$

The fixed points of Φ_1 and Φ_2 will be found in a closed ball $\overline{B}_{2\sqrt{N}}$, where

$$\overline{B}_{2\sqrt{N}} = \{M(t,x)\ ;\ \|M\|_{L_t^\infty([1,1+T);L_x^\infty)} \le 2\sqrt{N}\}.$$

Recall that $V = U(-t)Q(t)$ and apply Lemma 2.2 (2.9) to (3.6). Then we see that, for $A, A_1, A_2 \in \overline{B}_{2\sqrt{N}}$ and small $T > 0$,

$$
\begin{aligned}
\|\Phi_1(A)\|_{L_t^\infty([1,1+T);L_x^\infty)} &\le \sqrt{N} + C\varepsilon^2\sqrt{N}\log(1+T) \\
&\le 2\sqrt{N},
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\Phi_1(A_1)-\Phi_1(A_2)\|_{L_t^\infty([1,1+T);L_x^\infty)} \\
&\quad\le C\varepsilon^2\log(1+T)\|A_1-A_2\|_{L_t^\infty([1,1+T);L_x^\infty)} \\
&\quad\le \frac{1}{2}\|A_1-A_2\|_{L_t^\infty([1,1+T);L_x^\infty)}.
\end{aligned}
$$

Thus Φ_1 is the contraction map on $\overline{B}_{2\sqrt{N}}$, and we have a solution to (3.6). The solution also belongs to $C^1([1,1+T);L_x^\infty(\mathbb{R}))$ and it satisfies the first equation of (3.3). In order to continue this local solution to the global one, we differentiate $A(t,x)A(t,x)^*$ with respect to t so that

$$
\begin{aligned}
\partial_t(AA^*) &= it^{-1}\{A(\lambda_1(\mathcal{F}V)^*(\mathcal{F}V)+\lambda_2(\mathcal{F}V)(\mathcal{F}V)^*)A^*\} \\
&\quad -it^{-1}\{A(\lambda_1(\mathcal{F}V)^*(\mathcal{F}V)+\lambda_2(\mathcal{F}V)(\mathcal{F}V)^*)A^*\} \\
&= 0.
\end{aligned}
$$

This implies that $AA^* = I_N$, and so $A(t,x)$ is unitary. Of course, we have $\|A(t,\cdot)\|_{L_x^\infty} = \sqrt{N} < \infty$. Hence we obtain the global solution. The argument on $B(t,x)$ is similar. $\square$

Lemma 3.2. *Let $\|Q_0\|_{\Sigma_x^1} < \varepsilon$ with $\varepsilon > 0$ sufficiently small. Then, there exist $\hat{\Psi}(x) \in L_x^2(\mathbb{R}) \cap L_x^\infty(\mathbb{R})$ and $\Theta(x) \in L_x^\infty(\mathbb{R})$ such that*

$$
\begin{aligned}
(3.8)\qquad & A(\mathcal{F}V)B(t,x) \\
&= \hat{\Psi}(x) + O(t^{-1/4-1/2q+3K\varepsilon^2}) \quad \text{as } t\to\infty \text{ in } L_x^q(\mathbb{R})\ (q=2,\infty), \\
(3.9)\qquad & \lambda_3(A^*\hat{\Psi}\hat{\Psi}^*A)(t,x) - \lambda_1(B\hat{\Psi}^*\hat{\Psi}B^*)(t,x) \\
&= \Theta(x) + O(t^{-1/4+3K\varepsilon^2}) \quad \text{as } t\to\infty \text{ in } L_x^\infty(\mathbb{R}).
\end{aligned}
$$

Furthermore, $\Theta(x)$ is determined as a Hermite-symmetric matrix.

Proof of Lemma 3.2. As for $AR(t)B$ in (3.2), we have

$$\|AR(t)B\|_{L^q_x} \le Ct^{-5/4-1/2q+3K\varepsilon^2}$$

due to the estimates (2.13), (2.14) and the unitarity of A, B stated in Lemma 3.1. Then $AR(t)B$ is integrable around $t = \infty$, and so $\hat{\Psi}$ is determined as

$$\hat{\Psi} = A(\mathcal{F}V)B(1) + \int_1^\infty A(\tau)R(\tau)B(\tau)d\tau.$$

We next consider (3.9). Recall that $A(t,x)$ and $B(t,x)$ satisfy (3.3), and they are rewritten like

$$\partial_t A = it^{-1}A\{\lambda_1(B\hat{\Psi}^*\hat{\Psi}B^*) + \lambda_2(A^*\hat{\Psi}\hat{\Psi}^*A)\} + O(t^{-5/4+3K\varepsilon^2}), \tag{3.10}$$

$$\partial_t B = it^{-1}\lambda_3(A^*\hat{\Psi}\hat{\Psi}^*A)B + O(t^{-5/4+3K\varepsilon^2}), \tag{3.11}$$

as $t \to \infty$ in $L^\infty_x(\mathbb{R})$. Then it is easy to see that

$$\partial_t(A^*\hat{\Psi}\hat{\Psi}^*A) = it^{-1}\lambda_1[A^*\hat{\Psi}\hat{\Psi}^*A,\ B\hat{\Psi}^*\hat{\Psi}B^*] + O(t^{-5/4+3K\varepsilon^2}), \tag{3.12}$$

$$\partial_t(B\hat{\Psi}^*\hat{\Psi}B^*) = it^{-1}\lambda_3[A^*\hat{\Psi}\hat{\Psi}^*A,\ B\hat{\Psi}^*\hat{\Psi}B^*] + O(t^{-5/4+3K\varepsilon^2}), \tag{3.13}$$

where $[M_1, M_2] = M_1M_2 - M_2M_1$. By (3.12) and (3.13), we have

$$\partial_t\{\lambda_3(A^*\hat{\Psi}\hat{\Psi}^*A) - \lambda_1(B\hat{\Psi}^*\hat{\Psi}B)\} = O(t^{-5/4+3K\varepsilon^2}).$$

Hence the proof of (3.9) is complete. □

Combining Lemma 3.1 and 3.2, we can prove Theorem 1.2.

Proof of Theorem 1.2. Applying Lemma 3.2 (3.9) to (3.10) and (3.11), we see that

$$\begin{aligned}\partial_t A &= it^{-1}\{(\lambda_2+\lambda_3)(\hat{\Psi}\hat{\Psi}^*)A - A\Theta\} + O(t^{-5/4+3K\varepsilon^2}) \quad \text{in } L^\infty_x(\mathbb{R}),\\ \partial_t B &= it^{-1}\{\lambda_1 B(\hat{\Psi}^*\hat{\Psi}) + \Theta B\} + O(t^{-5/4+3K\varepsilon^2}) \quad \text{in } L^\infty_x(\mathbb{R}).\end{aligned}$$

From these two identities, it follows that, in $L^\infty_x(\mathbb{R})$,

$$\partial_t\{\exp\left(-i(\lambda_2+\lambda_3)\log t\hat{\Psi}\hat{\Psi}^*\right)\ A\ \exp(i\log t\Theta)\} = O(t^{-5/4+3K\varepsilon^2})$$

$$\partial_t\{\exp(-i\log t\Theta)\ B\ \exp(-i\lambda_1\log t\hat{\Psi}^*\hat{\Psi})\} = O(t^{-5/4+3K\varepsilon^2}).$$

Hence there exist some unitary matrices $M_1(x), M_2(x) \in L^\infty_x(\mathbb{R})$ such that

$$\begin{aligned}A(t) &= \exp\left(i(\lambda_2+\lambda_3)\log t\hat{\Psi}\hat{\Psi}^*\right)\ M_1\ \exp(-i\log t\Theta)\\ &\quad +O(t^{-1/4+3K\varepsilon^2}) \quad \text{in } L^\infty_x(\mathbb{R}),\end{aligned} \tag{3.14}$$

$$\begin{aligned}B(t) &= \exp(i\log t\Theta)\ M_2\ \exp(i\lambda_1\log t\hat{\Psi}^*\hat{\Psi})\\ &\quad +O(t^{-1/4+3K\varepsilon^2}) \quad \text{in } L^\infty_x(\mathbb{R}).\end{aligned} \tag{3.15}$$

By (3.14), (3.15) and Lemma 3.2 (3.8), we see that, in $L^q(\mathbb{R})$ ($q = 2$ or ∞),

$$\begin{aligned}\mathcal{F}V(t) &= A^*\hat{\Psi}B^*(t) + O(t^{-1/4-1/2q+3K\varepsilon^2})\\ &= \exp(i\log t\Theta)M_1^*\exp\left(-i(\lambda_2+\lambda_3)\log t\hat{\Psi}\hat{\Psi}^*\right)\\ &\quad \times\hat{\Psi}\ \exp(-i\lambda_1\log t\hat{\Psi}^*\hat{\Psi})M_2^*\exp(-i\log t\Theta) + O(t^{-1/4+3K\varepsilon^2}).\end{aligned}$$

Note here that $\hat{\Psi}\exp(-i\lambda_1 \log t\hat{\Psi}^*\hat{\Psi}) = \exp(-i\lambda_1 \log t\hat{\Psi}\hat{\Psi}^*)\hat{\Psi}$, and let $\hat{\Phi}(x) = M_1^*\hat{\Psi}M_2^*$. Then we see that, in $L_x^q(\mathbb{R})$ ($q = 2$ or ∞),

$$\mathcal{F}V(t) = e^{i\log t\Theta}e^{i\log t\Lambda}\hat{\Phi}e^{-i\log t\Theta} + O(t^{-1/4+3K\varepsilon^2}),$$

where $\Lambda(x) = -(\lambda_1 + \lambda_2 + \lambda_3)\hat{\Phi}\hat{\Phi}^*$. Hence we observe that, in $L_x^2(\mathbb{R})$,

$$\begin{aligned} Q(t) &= U(t)\mathcal{F}^{-1}\mathcal{F}V(t) \\ &= U(t)\mathcal{F}^{-1}e^{i\log t\Theta}e^{i\log t\Lambda}\hat{\Phi}e^{-i\log t\Theta} + O(t^{-1/4+3K\varepsilon^2}). \end{aligned}$$

In addition, by the error estimate (2.11), we see that, in $L_x^\infty(\mathbb{R})$,

$$\begin{aligned} Q(t) &= U(t)V(t) \\ &= MD\mathcal{F}MV(t) \\ &= MD\mathcal{F}V(t) + O(t^{-3/4+K\varepsilon}) \\ &= MDe^{i\log t\Theta}e^{i\log t\Lambda}\hat{\Phi}e^{-i\log t\Theta} + O(t^{-3/4+3K\varepsilon^2}). \end{aligned}$$

This completes the proof. □

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