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ATTRACTIVE POINT AND NONLINEAR ERGODIC THEOREMS FOR GENERIC 2-GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we define a new type of nonlinear mappings called generic 2-generalized hybrid mappings, which includes nonexpansive mappings, generalized hybrid mappings and normally 2-generalized hybrid mappings simultaneously. For that class of mappings, we establish a nonlinear ergodic theorem of finding an attractive point in a Hilbert space. An averaged sequence converges weakly to an attractive point of a generic 2-generalized hybrid mapping. The main theorem is proved without assuming that the domain of the mapping is closed or convex. Our results extend many existing results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and let C be a nonempty subset of H. For a mapping $T : C \to H$, the set of fixed points is denoted by $F(T) = \{u \in C : Tu = u\}$. The set of *attractive points*, which is introduced by Takahashi and Takeuchi [20], of T is denoted by

$$A(T) = \{u \in H : \|Ty - u\| \leq \|y - u\| \text{ for all } y \in C\}.$$

A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. It is well-known that if C is a bounded, closed and convex subset of H and $T : C \to C$ is nonexpansive, then $F(T)$ is nonempty. Furthermore, from Baillon [2], we know the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space:

Theorem 1.1 ([2]). *Let C be a nonempty, closed, convex and bounded subset of a real Hilbert space H and let $T : C \to C$ be a nonexpansive mapping. Then, for any $x \in C$, the sequence*

$$S_n z \equiv \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to a fixed point of T.

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In 2010, Kocourek, Takahashi and Yao [8] proposed a broad class of nonlinear mappings containing nonexpansive mappings. A mapping T from C into H is called a *generalized hybrid mapping* if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. Such a mapping T is also called (α, β)-*generalized hybrid*. It is obvious that a $(1,0)$-generalized hybrid mapping is nonexpansive. In addition to nonexpansive mappings, the class of generalized hybrid mappings contains other types of well-known nonlinear mappings. For example, a $(2,1)$-generalized hybrid mapping is *nonspreading* [9,10], and a $(3/2, 1/2)$-generalized hybrid mapping is *hybrid* [17]. It is known that nonspreading and hybrid mappings are not necessarily continuous (see [6] or [24]). For these types of mappings, see also Takahashi and Yao [22]. We also know that the class of generalized hybrid mappings covers λ-*hybrid mappings* [1]. For this point, see Hojo, Takahashi and Yao [5]. Kocourek, Takahashi and Yao [8] established a nonlinear ergodic theorem, which generalized Theorem 1.1, and another type of weak convergence result.

Generalized hybrid mappings are further extended. Maruyama, Takahashi and Yao [13] defined a wide class of nonlinear mappings containing generalized hybrid mappings. A mapping $T : C \to C$ is called 2-*generalized hybrid* if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1\|T^2x - Ty\|^2 + \alpha_2\|Tx - Ty\|^2 + (1-\alpha_1-\alpha_2)\|x - Ty\|^2&\\ \leq \beta_1\|T^2x - y\|^2 + \beta_2\|Tx - y\|^2 + (1-\beta_1-\beta_2)\|x - y\|^2&\end{aligned}$$

for all $x, y \in C$. If $\alpha_1 = \beta_1 = 0$, the class of 2-generalized hybrid mappings coincides with the class of generalized hybrid mappings. Kondo and Takahashi [11] introduced the following class of nonlinear mappings which covers 2-generalized hybrid mappings. A mapping $T : C \to C$ is called *normally* 2-*generalized hybrid* if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$ and

$$\begin{aligned}\alpha_2\|T^2x - Ty\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2&\\ + \beta_2\|T^2x - y\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \leq 0&\end{aligned}$$

for all $x, y \in C$. It is also known that the type of normally 2-generalized hybrid mappings contains *normally generalized hybrid mappings* [23]. Lin and Takahashi [12] and Kondo and Takahashi [11] proved nonlinear ergodic theorems that weakly approximate attractive points of 2-generalized hybrid mappings and normally 2-generalized hybrid mappings, respectively.

Very recently, Takahashi [18, 19] defined a broad class of nonlinear mappings. A mapping $T : C \to H$ is called a *demigeneric generalized hybrid mapping* if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta > 0$ and

$$\begin{aligned}\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2&\\ + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 \leq 0&\end{aligned}$$

for all $x, y \in C$. Such a mapping T is also called $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-*demigeneric generalized hybrid*. The class of mappings covers generalized hybrid mappings and *strict pseude-contractions* [3].

In this paper, motivated by these researches, we propose a new class of nonlinear mappings that includes nonexpansive mappings, generalized hybrid mappings and normally 2-generalized hybrid mappings as special cases, and establish a nonlinear ergodic theorem of Baillon's type for such mappings in a Hilbert space (Theorem 5.3). The existence of attractive points is also demonstrated (Theorem 4.1). These results are proved without supposing that the domain of the mapping is convex or closed. A fixed point theorem is also established with additionally supposing that the domain is closed and convex (Theorem 4.3).

2. Preliminaries

This section presents preliminary information and lemmas. Let H be a real Hilbert space. Strong and weak convergence of a sequence $\{x_n\}$ in H to $x \in H$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We know that for a bounded sequence $\{x_n\}$ in H, $\{x_n\}$ is weakly convergent if and only if every weakly convergent subsequence of $\{x_n\}$ has a same weak limit, that is,

$$x_n \rightharpoonup v \Longleftrightarrow [x_{n_i} \rightharpoonup u \text{ implies that } u = v], \tag{2.1}$$

where $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ (see Takahashi [16]). We also know that a closed and convex subset of H is weakly closed. For a mapping $T : C \to H$, an attractive point $u \in H$ of T is characterized as follows:

$$u \in A(T) \Longleftrightarrow \|Ty - y\| + 2\langle Ty - y,\ y - u\rangle \leq 0 \text{ for all } y \in C. \tag{2.2}$$

We know from Takahashi and Takeuchi [20] that the set of attractive points $A(T)$ is closed and convex in a Hilbert space. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx - u\| \leq \|x - u\|$$

for all $x \in C$ and $u \in F(T)$. We know that the set of fixed points $F(T)$ of a quasi-nonexpansive mapping is closed and convex (see Itoh and Takahashi [7]).

Let l^∞ be the Banach space of bounded sequences of real numbers with the supremum norm and let $(l^\infty)^*$ be its dual space. For $\mu \in (l^\infty)^*$, we denote $\mu(\{x_n\})$ by $\mu_n x_n$. A linear continuous functional $\mu \in (l^\infty)^*$ that satisfies the condition $\mu(\{1, 1, 1, \cdots\}) = \|\mu\| = 1$ is called a *mean* on l^∞. We know that a mean μ preserves order relations, that is,

$$x_n \leq y_n \ (\forall n \in \mathbb{N}) \Longrightarrow \mu_n x_n \leq \mu_n y_n,$$

where $\mathbb{N}$ is the set of natural numbers. When a mean additionally satisfies $\mu_n x_n = \mu_n x_{n+1}$, it is called a *Banach limit* on l^∞. It is well-known that a Banach limit exists, which is proved by using the Hahn–Banach theorem. For any $\{x_n\} \in l^\infty$, it holds that

$$\liminf_{n\to\infty} x_n \leq \mu_n x_n \leq \limsup_{n\to\infty} x_n. \tag{2.3}$$

As a direct result from (2.3), if $x_n \to a\,(\in \mathbb{R})$, then $\mu_n x_n = a$. For more details, see Takahashi [15].

Let A be a nonempty, closed and convex subset of H. We know that for any $x \in H$, there exists a unique nearest point $z \in A$, that is, $\|x - z\| \leq \|x - u\|$ for all $u \in A$. This correspondence is called the *metric projection* from H onto A and is denoted by P_A. We know that if P_A is the metric projection from H onto A, then $\langle x - P_A x,\ P_A x - u\rangle \geq 0$ for all $x \in H$ and $u \in A$.

The following lemma is utilized to show the existence of attractive and fixed points (Theorem 4.1 and 4.3). See Lin and Takahashi [12] and Takahashi [14].

Lemma 2.1 ([12, 14])**.** *Let μ be a mean on l^∞ and let H be a real Hilbert space. Then, for any bounded sequence $\{x_n\}$ in H, there is a unique element $u \in \overline{co}\,\{x_n\}$ such that*

$$\mu_n \langle x_n,\ v\rangle = \langle u,\ v\rangle$$

for all $v \in H$, where $\overline{co}\,\{x_n\}$ is the closure of the convex hull of $\{x_n : n \in \mathbb{N}\}$.

The following lemma [20] is also useful to derive convergence results to fixed points.

Lemma 2.2 ([20])**.** *Let C be a nonempty subset of a real Hilbert space H and let T be a mapping from C into H. Then, $A\,(T) \cap C \subset F\,(T)$.*

Takahashi and Toyoda [21] proved the following lemma by using the parallelogram law.

Lemma 2.3 ([21])**.** *Let A be a nonempty, closed and convex subset of a real Hilbert space H, let P_A be the metric projection from H onto A and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - q\| \leq \|x_n - q\|$ for all $q \in A$ and $n \in \mathbb{N}$, then $\{P_A x_n\}$ is convergent in A.*

3. Generic 2-generalized hybrid mappings

In this section, we define a new class of nonlinear mappings, and show that the type of mapping is quasi-nonexpansive if it has a fixed point. Let C be a nonempty subset of a real Hilbert space H. We call a mapping $T : C \to C$ *generic* 2*-generalized hybrid* if there exist $\alpha_{ij}, \beta_i, \gamma_i \in \mathbb{R}$ $(i, j = 0, 1, 2)$ such that

$$\begin{aligned}
&\alpha_{00}\|x-y\|^2+\alpha_{01}\|x-Ty\|^2+\alpha_{02}\left\|x-T^2y\right\|^2\\
&+\alpha_{10}\|Tx-y\|^2+\alpha_{11}\|Tx-Ty\|^2+\alpha_{12}\left\|Tx-T^2y\right\|^2\\
&+\alpha_{20}\left\|T^2x-y\right\|^2+\alpha_{21}\left\|T^2x-Ty\right\|^2+\alpha_{22}\left\|T^2x-T^2y\right\|^2\\
&+\beta_0\|x-Tx\|^2+\beta_1\left\|Tx-T^2x\right\|^2+\beta_2\left\|T^2x-x\right\|^2\\
&+\gamma_0\|y-Ty\|^2+\gamma_1\left\|Ty-T^2y\right\|^2+\gamma_2\left\|T^2y-y\right\|^2\leq 0
\end{aligned}\tag{3.1}$$

for all $x, y \in C$. We also refer such a mapping as $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$*-generic* 2*-generalized hybrid.* For the main theorem (Theorem 5.3) of this paper, we will assume that T satisfies one of the following conditions:

$$(3.2)\qquad \begin{array}{ll} (1) & \alpha_{0\bullet}+\alpha_{1\bullet}\geq 0,\ \alpha_{20},\alpha_{21},\alpha_{22}\geq 0,\ \alpha_{1\bullet}>0, \\ & \beta_0,\beta_1,\beta_2\geq 0,\ \gamma_0+\gamma_1\geq 0,\ \gamma_2\geq 0; \\ (2) & \alpha_{\bullet 0}+\alpha_{\bullet 1}\geq 0,\ \alpha_{02},\alpha_{12},\alpha_{22}\geq 0,\ \alpha_{\bullet 1}>0, \\ & \beta_0+\beta_1\geq 0,\ \beta_2\geq 0,\ \gamma_0,\gamma_1,\gamma_2\geq 0, \end{array}$$

where

$$(3.3)\qquad \alpha_{i\bullet}\equiv \alpha_{i0}+\alpha_{i1}+\alpha_{i2}\ \text{ and }\ \alpha_{\bullet i}\equiv \alpha_{0i}+\alpha_{1i}+\alpha_{2i}$$

for $i=0,1,2$.

The class of generic 2-generalized hybrid mappings that satisfies (1) or (2) of (3.2) contains nonexpansive mappings, generalized hybrid mappings and normally 2-generalized hybrid mappings as special cases. Let T be $(\alpha_{ij},\beta_i,\gamma_i;\ i,j=0,1,2)$-generic 2-generalized hybrid. First, substitute $\alpha_{00}=-1$ and $\alpha_{11}=1$ into (3.1) and put the other coefficients are all 0. Then, both conditions (1) and (2) of (3.2) are satisfied. It is obvious that T is nonexpansive in this case. Secondly, if $\alpha_{11}=\alpha$, $\alpha_{01}=1-\alpha$, $\alpha_{10}=-\beta$, $\alpha_{00}=-(1-\beta)$ and the other coefficients are all 0, then the condition (2) of (3.2) is satisfied. In this case, T is (α,β)-generalized hybrid. Finally, substitute $(1)'$ $\alpha_{20}=\alpha_{21}=\alpha_{22}=0$ (resp. $(2)'$ $\alpha_{02}=\alpha_{12}=\alpha_{22}=0$) and $\beta_i=\gamma_i=0$ $(i=0,1,2)$ into (3.1). Then, it is easy to verify that the mapping T with the condition (1) (resp. (2)) of (3.2) is normally 2-generalized hybrid.

The following theorem asserts that a generic 2-generalized hybrid mapping with $F(T)\neq\emptyset$ is quasi-nonexpansive.

Theorem 3.1. *Let C be a nonempty subset of a real Hilbert space H and let T be an $(\alpha_{ij},\beta_i,\gamma_i;\ i,j=0,1,2)$-generic 2-generalized hybrid mapping from C into itself with $F(T)\neq\emptyset$. Suppose that T satisfies one of the following conditions:*

$$(1)\ \alpha_{0\bullet}+\alpha_{1\bullet}\geq 0,\ \alpha_{2\bullet}\geq 0,\ \alpha_{1\bullet}>0,\ \beta_i\geq 0;$$

$$(2)\ \alpha_{\bullet 0}+\alpha_{\bullet 1}\geq 0,\ \alpha_{\bullet 2}\geq 0,\ \alpha_{\bullet 1}>0,\ \gamma_i\geq 0;$$

for $i=0,1,2$, where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (3.3). *Then, T is quasi-nonexpansive.*

Proof. Let $x\in C$ and let $u\in F(T)\left(\subset F\left(T^2\right)\right)$. We will show that $\|Tx-u\|\leq\|x-u\|$.

Case (1). Suppose that $\alpha_{0\bullet}+\alpha_{1\bullet}\geq 0$, $\alpha_{2\bullet}\geq 0$, $\alpha_{1\bullet}>0$ and $\beta_i\geq 0$ for $i=0,1,2$. Since T is an $(\alpha_{ij},\beta_i,\gamma_i;\ i,j=0,1,2)$-generic 2-generalized hybrid mapping, it holds that

$$(3.4)\qquad \begin{aligned} &\alpha_{00}\|x-u\|^2+\alpha_{01}\|x-Tu\|^2+\alpha_{02}\left\|x-T^2u\right\|^2 \\ &+\alpha_{10}\|Tx-u\|^2+\alpha_{11}\|Tx-Tu\|^2+\alpha_{12}\left\|Tx-T^2u\right\|^2 \\ &+\alpha_{20}\left\|T^2x-u\right\|^2+\alpha_{21}\left\|T^2x-Tu\right\|^2+\alpha_{22}\left\|T^2x-T^2u\right\|^2 \\ &+\beta_0\|x-Tx\|^2+\beta_1\left\|Tx-T^2x\right\|^2+\beta_2\left\|T^2x-x\right\|^2 \\ &+\gamma_0\|u-Tu\|^2+\gamma_1\left\|Tu-T^2u\right\|^2+\gamma_2\left\|T^2u-u\right\|^2\leq 0. \end{aligned}$$

Since $u = Tu = T^2u$, we have that

$$\alpha_{0\bullet} \|x-u\|^2 + \alpha_{1\bullet} \|Tx-u\|^2 + \alpha_{2\bullet} \left\|T^2x-u\right\|^2 \\ + \beta_0 \|x-Tx\|^2 + \beta_1 \left\|Tx-T^2x\right\|^2 + \beta_2 \left\|T^2x-x\right\|^2 \le 0.$$

Since $\beta_i \ge 0$ and $\alpha_{2\bullet} \ge 0$, we obtain that

$$\alpha_{0\bullet} \|x-u\|^2 + \alpha_{1\bullet} \|Tx-u\|^2 \le 0.$$

We have from $\alpha_{0\bullet} + \alpha_{1\bullet} \ge 0$ that

$$\begin{aligned} \alpha_{1\bullet} \|Tx-u\|^2 &\le -\alpha_{0\bullet} \|x-u\|^2 \\ &\le \alpha_{1\bullet} \|x-u\|^2 . \end{aligned}$$

Since $\alpha_{1\bullet} > 0$, we obtain that $\|Tx-u\| \le \|x-u\|$.

Case (2). Suppose that $\alpha_{\bullet 0}+\alpha_{\bullet 1} \ge 0$, $\alpha_{\bullet 2} \ge 0$, $\alpha_{\bullet 1} > 0$ and $\gamma_i \ge 0$ for $i = 0, 1, 2$. Replacing the variables x and u in (3.4), we can derive the desired result in much the same way as the proof for Case (1). □

4. Attractive and fixed point theorems

This section presents attractive and fixed point theorems that guarantee the existence of attractive and fixed points of generic 2-generalized hybrid mappings in Hilbert spaces. Our proofs are generalization of those of many existing papers; see, for example, [8, 12, 13, 20, 22, 23] and [11].

Theorem 4.1. *Let C be a nonempty subset of a real Hilbert space H and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i; i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping. Suppose that T satisfies one of the following conditions:*

(1) $\alpha_{0\bullet} + \alpha_{1\bullet} \ge 0,\ \alpha_{2\bullet} \ge 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \ge 0,\ \gamma_0 + \gamma_1 \ge 0,\ \gamma_2 \ge 0;$

(2) $\alpha_{\bullet 0} + \alpha_{\bullet 1} \ge 0,\ \alpha_{\bullet 2} \ge 0,\ \alpha_{\bullet 1} > 0,\ \beta_0 + \beta_1 \ge 0,\ \beta_2 \ge 0,\ \gamma_0, \gamma_1, \gamma_2 \ge 0,$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (3.3). *If there exists an element $x \in C$ such that the sequence $\{T^nx\}$ in C is bounded, then $A(T)$ is nonempty.*

Proof. Let $\mu \in (l^\infty)^*$ be a Banach limit. For the bounded sequence $\{T^nx\}$, we obtain from Lemma 2.1 that there exists a unique element $u \in \overline{co}\{T^nx\} (\subset H)$ such that

$$\mu_n \langle T^nx,\ v\rangle = \langle u,\ v\rangle \tag{4.1}$$

for all $v \in H$. We will show that $u \in A(T)$.

Case (1). First, suppose that $\alpha_{0\bullet} + \alpha_{1\bullet} \ge 0$, $\alpha_{2\bullet} \ge 0$, $\alpha_{1\bullet} > 0$, $\beta_0, \beta_1, \beta_2 \ge 0$, $\gamma_0 + \gamma_1 \ge 0$ and $\gamma_2 \ge 0$. Let $y \in C$. From (2.2), it is enough to prove that $\|Ty-y\|^2 + 2\langle Ty-y,\ y-u\rangle \le 0$. Since T is $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid, it holds that

$$\alpha_{00} \|y-T^nx\|^2 + \alpha_{01} \left\|y-T^{n+1}x\right\|^2 + \alpha_{02} \left\|y-T^{n+2}x\right\|^2 \\ +\alpha_{10} \|Ty-T^nx\|^2 + \alpha_{11} \left\|Ty-T^{n+1}x\right\|^2 + \alpha_{12} \left\|Ty-T^{n+2}x\right\|^2 \tag{4.2}$$

$$+\alpha_{20}\left\|T^2y-T^nx\right\|^2+\alpha_{21}\left\|T^2y-T^{n+1}x\right\|^2+\alpha_{22}\left\|T^2y-T^{n+2}x\right\|^2$$
$$+\beta_0\left\|y-Ty\right\|^2+\beta_1\left\|Ty-T^2y\right\|^2+\beta_2\left\|T^2y-y\right\|^2$$
$$+\gamma_0\left\|T^nx-T^{n+1}x\right\|^2+\gamma_1\left\|T^{n+1}x-T^{n+2}x\right\|^2+\gamma_2\left\|T^{n+2}x-T^nx\right\|^2\le 0$$

for all $n\in\mathbb{N}$. Since $\beta_0,\beta_1,\beta_2,\gamma_2\ge 0$, we obtain that

$$\begin{aligned}
&\alpha_{00}\left\|y-T^nx\right\|^2+\alpha_{01}\left\|y-T^{n+1}x\right\|^2+\alpha_{02}\left\|y-T^{n+2}x\right\|^2\\
&\quad+\alpha_{10}\left\|Ty-T^nx\right\|^2+\alpha_{11}\left\|Ty-T^{n+1}x\right\|^2+\alpha_{12}\left\|Ty-T^{n+2}x\right\|^2\\
&\quad+\alpha_{20}\left\|T^2y-T^nx\right\|^2+\alpha_{21}\left\|T^2y-T^{n+1}x\right\|^2+\alpha_{22}\left\|T^2y-T^{n+2}x\right\|^2\\
&\quad+\gamma_0\left\|T^nx-T^{n+1}x\right\|^2+\gamma_1\left\|T^{n+1}x-T^{n+2}x\right\|^2\le 0.
\end{aligned}$$

It holds that

$$\begin{aligned}
&\alpha_{0\bullet}\left\|y-T^nx\right\|^2+\alpha_{01}\left(\left\|y-T^{n+1}x\right\|^2-\left\|y-T^nx\right\|^2\right)\\
&\quad+\alpha_{02}\left(\left\|y-T^{n+2}x\right\|^2-\left\|y-T^nx\right\|^2\right)\\
&\quad+\alpha_{1\bullet}\left\|Ty-T^nx\right\|^2+\alpha_{11}\left(\left\|Ty-T^{n+1}x\right\|^2-\left\|Ty-T^nx\right\|^2\right)\\
&\quad+\alpha_{12}\left(\left\|Ty-T^{n+2}x\right\|^2-\left\|Ty-T^nx\right\|^2\right)\\
&\quad+\alpha_{2\bullet}\left\|T^2y-T^nx\right\|^2+\alpha_{21}\left(\left\|T^2y-T^{n+1}x\right\|^2-\left\|T^2y-T^nx\right\|^2\right)\\
&\quad+\alpha_{22}\left(\left\|T^2y-T^{n+2}x\right\|^2-\left\|T^2y-T^nx\right\|^2\right)\\
&\quad+\gamma_0\left\|T^nx-T^{n+1}x\right\|^2+\gamma_1\left\|T^{n+1}x-T^{n+2}x\right\|^2\le 0.
\end{aligned}$$

Applying the Banach limit μ, we obtain that

$$\begin{aligned}
&\alpha_{0\bullet}\mu_n\left\|y-T^nx\right\|^2+\alpha_{1\bullet}\mu_n\left\|Ty-T^nx\right\|^2+\alpha_{2\bullet}\mu_n\left\|T^2y-T^nx\right\|^2\\
&\quad+(\gamma_0+\gamma_1)\,\mu_n\left\|T^nx-T^{n+1}x\right\|^2\le 0.
\end{aligned}$$

It holds from $\alpha_{2\bullet}\ge 0$ and $\gamma_0+\gamma_1\ge 0$ that

$$\alpha_{0\bullet}\mu_n\left\|y-T^nx\right\|^2+\alpha_{1\bullet}\mu_n\left\|Ty-T^nx\right\|^2\le 0. \tag{4.3}$$

Using $\alpha_{0\bullet}+\alpha_{1\bullet}\ge 0$, we obtain that

$$\begin{aligned}
\alpha_{1\bullet}\mu_n\left\|Ty-T^nx\right\|^2&\le -\alpha_{0\bullet}\mu_n\left\|y-T^nx\right\|^2\\
&\le \alpha_{1\bullet}\mu_n\left\|y-T^nx\right\|^2.
\end{aligned}$$

Since $\alpha_{1\bullet}>0$, we have that

$$\mu_n\left\|Ty-T^nx\right\|^2\le\mu_n\left\|y-T^nx\right\|^2,$$

and thus,

$$\mu_n\left(\left\|Ty-y\right\|^2+2\left\langle Ty-y,\ y-T^nx\right\rangle+\left\|y-T^nx\right\|^2\right)\le\mu_n\left\|y-T^nx\right\|^2.$$

This means that

$$\mu_n \left(\|Ty - y\|^2 + 2 \langle Ty - y,\ y - T^n x \rangle \right) \le 0.$$

From (4.1), it holds that

$$\|Ty - y\|^2 + 2 \langle Ty - y,\ y - u \rangle \le 0 \tag{4.4}$$

for all $y \in C$. This implies from (2.2) that $u \in A(T)$.

Case (2). Suppose that $\alpha_{\bullet 0} + \alpha_{\bullet 1} \ge 0$, $\alpha_{\bullet 2} \ge 0$, $\alpha_{\bullet 1} > 0$, $\beta_0 + \beta_1 \ge 0$, $\beta_2 \ge 0$ and $\gamma_0, \gamma_1, \gamma_2 \ge 0$. We can obtain the desired result by replacing the variables y and $T^n x$ in (4.2). $\square$

The following corollary can be easily derived from the previous theorem.

Corollary 4.2. *Let C be a nonempty subset of a real Hilbert space H and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping. Suppose that T satisfies one of the following conditions:*

(1) $\alpha_{0\bullet} + \alpha_{1\bullet} \ge 0,\ \alpha_{2\bullet} \ge 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \ge 0,\ \gamma_0 + \gamma_1 \ge 0,\ \gamma_2 \ge 0;$

(2) $\alpha_{\bullet 0} + \alpha_{\bullet 1} \ge 0,\ \alpha_{\bullet 2} \ge 0,\ \alpha_{\bullet 1} > 0,\ \beta_0 + \beta_1 \ge 0,\ \beta_2 \ge 0,\ \gamma_0, \gamma_1, \gamma_2 \ge 0,$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (3.3)*. Then, the following three statements are equivalent:*

(a) *for any $x \in C$, $\{T^n x\}$ is a bounded sequence in C;*
(b) *there exists an element $z \in C$ such that the sequence $\{T^n z\}$ in C is bounded;*
(c) *$A(T)$ is nonempty.*

Proof. (a) $\Longrightarrow$ (b) obviously holds. (b) $\Longrightarrow$ (c) has already been demonstrated as Theorem 4.1. We will prove that (c) $\Longrightarrow$ (a). Let $x \in C$ and $u \in A(T)$. Then, it holds that

$$\|T^n x - u\| \le \left\|T^{n-1} x - u\right\| \le \cdots \le \|x - u\|$$

for all $n \in \mathbb{N}$, which implies that $\{T^n x\}$ is bounded. $\square$

We can also obtain a fixed point theorem for generic 2-generalized hybrid mappings if C is closed and convex. For that aim, the conditions on the parameters $\alpha_{ij}, \beta_i, \gamma_i \in \mathbb{R}$ $(i, j = 0, 1, 2)$ can be slightly relaxed.

Theorem 4.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $T : C \to C$ be $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid. Suppose that T satisfies one of the following conditions:*

(1) $\alpha_{0\bullet} + \alpha_{1\bullet} \ge 0,\ \alpha_{2\bullet} \ge 0,\ \alpha_{1\bullet} + \beta_0 > 0,\ \beta_1, \beta_2 \ge 0,\ \gamma_0 + \gamma_1 \ge 0,\ \gamma_2 \ge 0;$

(2) $\alpha_{\bullet 0} + \alpha_{\bullet 1} \ge 0,\ \alpha_{\bullet 2} \ge 0,\ \alpha_{\bullet 1} + \gamma_0 > 0,\ \beta_0 + \beta_1 \ge 0,\ \beta_2 \ge 0,\ \gamma_1, \gamma_2 \ge 0,$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (3.3)*. If there exists an element $x \in C$ such that the sequence $\{T^n x\}$ in C is bounded, then $F(T)$ is nonempty. Furthermore, a generic 2-generalized hybrid mapping T has at most one fixed point if $\alpha_{\bullet\bullet} > 0$, where $\alpha_{\bullet\bullet} \equiv \sum_{i,j=0,1,2} \alpha_{ij}$.*

Proof. First, we will prove the existence of a fixed point of T. Let $\mu \in (l^{\infty})^{*}$ be a Banach limit. From Lemma 2.1, it holds that for the bounded sequence $\{T^n x\}$, there exists a unique element $u \in \overline{co}\,\{T^n x\}$ such that

$$\mu_n \langle T^n x,\ v\rangle = \langle u,\ v\rangle \tag{4.5}$$

for all $v \in H$. Note that since C is closed and convex, we have that $\overline{co}\,\{T^n x\} \subset C$. Thus, $u \in C$. It suffices to show that $u \in F(T)$.

Case (1). Suppose that $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0$, $\alpha_{2\bullet} \geq 0$, $\alpha_{1\bullet} + \beta_0 > 0$, $\beta_1, \beta_2 \geq 0$, $\gamma_0 + \gamma_1 \geq 0$ and $\gamma_2 \geq 0$. Let $y \in C$. Since T is $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid, we have that

$$\begin{aligned}
&\alpha_{00}\|y - T^n x\|^2 + \alpha_{01}\left\|y - T^{n+1}x\right\|^2 + \alpha_{02}\left\|y - T^{n+2}x\right\|^2 \\
&+\alpha_{10}\|Ty - T^n x\|^2 + \alpha_{11}\left\|Ty - T^{n+1}x\right\|^2 + \alpha_{12}\left\|Ty - T^{n+2}x\right\|^2 \\
&+\alpha_{20}\left\|T^2y - T^n x\right\|^2 + \alpha_{21}\left\|T^2y - T^{n+1}x\right\|^2 + \alpha_{22}\left\|T^2y - T^{n+2}x\right\|^2 \\
&+\beta_0\|y - Ty\|^2 + \beta_1\left\|Ty - T^2y\right\|^2 + \beta_2\left\|T^2y - y\right\|^2 \\
&+\gamma_0\left\|T^n x - T^{n+1}x\right\|^2 + \gamma_1\left\|T^{n+1}x - T^{n+2}x\right\|^2 + \gamma_2\left\|T^{n+2}x - T^n x\right\|^2 \leq 0
\end{aligned} \tag{4.6}$$

for all $n \in \mathbb{N}$. Since $\beta_1, \beta_2, \gamma_2 \geq 0$, we obtain that

$$\begin{aligned}
&\alpha_{00}\|y - T^n x\|^2 + \alpha_{01}\left\|y - T^{n+1}x\right\|^2 + \alpha_{02}\left\|y - T^{n+2}x\right\|^2 \\
&+ \alpha_{10}\|Ty - T^n x\|^2 + \alpha_{11}\left\|Ty - T^{n+1}x\right\|^2 + \alpha_{12}\left\|Ty - T^{n+2}x\right\|^2 \\
&+ \alpha_{20}\left\|T^2y - T^n x\right\|^2 + \alpha_{21}\left\|T^2y - T^{n+1}x\right\|^2 + \alpha_{22}\left\|T^2y - T^{n+2}x\right\|^2 \\
&+ \beta_0\|y - Ty\|^2 + \gamma_0\left\|T^n x - T^{n+1}x\right\|^2 + \gamma_1\left\|T^{n+1}x - T^{n+2}x\right\|^2 \leq 0.
\end{aligned}$$

It holds that

$$\begin{aligned}
&\alpha_{0\bullet}\|y - T^n x\|^2 + \alpha_{01}\left(\left\|y - T^{n+1}x\right\|^2 - \|y - T^n x\|^2\right) \\
&\quad + \alpha_{02}\left(\left\|y - T^{n+2}x\right\|^2 - \|y - T^n x\|^2\right) \\
&+ \alpha_{1\bullet}\|Ty - T^n x\|^2 + \alpha_{11}\left(\left\|Ty - T^{n+1}x\right\|^2 - \|Ty - T^n x\|^2\right) \\
&\quad + \alpha_{12}\left(\left\|Ty - T^{n+2}x\right\|^2 - \|Ty - T^n x\|^2\right) \\
&+ \alpha_{2\bullet}\left\|T^2y - T^n x\right\|^2 + \alpha_{21}\left(\left\|T^2y - T^{n+1}x\right\|^2 - \left\|T^2y - T^n x\right\|^2\right) \\
&\quad + \alpha_{22}\left(\left\|T^2y - T^{n+2}x\right\|^2 - \left\|T^2y - T^n x\right\|^2\right) \\
&+ \beta_0\|y - Ty\|^2 + \gamma_0\left\|T^n x - T^{n+1}x\right\|^2 + \gamma_1\left\|T^{n+1}x - T^{n+2}x\right\|^2 \leq 0.
\end{aligned}$$

Applying the Banach limit μ, we obtain that

$$\begin{aligned}
&\alpha_{0\bullet}\mu_n\|y - T^n x\|^2 + \alpha_{1\bullet}\mu_n\|Ty - T^n x\|^2 + \alpha_{2\bullet}\mu_n\left\|T^2y - T^n x\right\|^2 \\
&\qquad + \beta_0\|y - Ty\|^2 + (\gamma_0 + \gamma_1)\,\mu_n\left\|T^n x - T^{n+1}x\right\|^2 \leq 0.
\end{aligned}$$

Since $\alpha_{2\bullet} \geq 0$ and $\gamma_0 + \gamma_1 \geq 0$, we have that

$$\alpha_{0\bullet}\mu_n \|y - T^n x\|^2 + \alpha_{1\bullet}\mu_n \|Ty - T^n x\|^2 + \beta_0 \|y - Ty\|^2 \leq 0.$$

Hence,

$$\alpha_{0\bullet}\mu_n \|y - T^n x\|^2 + \alpha_{1\bullet}\mu_n \left(\|Ty - y\|^2 + 2\langle Ty - y,\ y - T^n x\rangle + \|y - T^n x\|^2\right) + \beta_0 \|y - Ty\|^2 \leq 0.$$

Since $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0$, it holds that

$$\alpha_{1\bullet}\mu_n \left(\|Ty - y\|^2 + 2\langle Ty - y,\ y - T^n x\rangle\right) + \beta_0 \|y - Ty\|^2 \leq 0.$$

Using (4.5), we have that

$$\alpha_{1\bullet} \left(\|Ty - y\|^2 + 2\langle Ty - y,\ y - u\rangle\right) + \beta_0 \|y - Ty\|^2 \leq 0$$

for all $y \in C$. Substituting $y = u \in C$,

$$(\alpha_{1\bullet} + \beta_0) \|Tu - u\|^2 \leq 0$$

Since $\alpha_{1\bullet} + \beta_0 > 0$, we obtain that $u \in F(T)$.

Case (2). Suppose that $\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0$, $\alpha_{\bullet 2} \geq 0$, $\alpha_{\bullet 1} + \gamma_0 > 0$, $\beta_0 + \beta_1 \geq 0$, $\beta_2 \geq 0$ and $\gamma_1, \gamma_2 \geq 0$. We can obtain the desired result by replacing the variables y and $T^n x$ in (4.6).

Next, we will prove the uniqueness of a fixed point of T. Let $u, v \in F(T)$. Since T is an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping, we have that

$$\begin{aligned}\alpha_{00} \|u - v\|^2 &+ \alpha_{01} \|u - Tv\|^2 + \alpha_{02} \left\|u - T^2 v\right\|^2 \\ &+ \alpha_{10} \|Tu - v\|^2 + \alpha_{11} \|Tu - Tv\|^2 + \alpha_{12} \left\|Tu - T^2 v\right\|^2 \\ &+ \alpha_{20} \left\|T^2 u - v\right\|^2 + \alpha_{21} \left\|T^2 u - Tv\right\|^2 + \alpha_{22} \left\|T^2 u - T^2 v\right\|^2 \\ &+ \beta_0 \|u - Tu\|^2 + \beta_1 \left\|Tu - T^2 u\right\|^2 + \beta_2 \left\|T^2 u - u\right\|^2 \\ &+ \gamma_0 \|v - Tv\|^2 + \gamma_1 \left\|Tv - T^2 v\right\|^2 + \gamma_2 \left\|T^2 v - v\right\|^2 \leq 0.\end{aligned}$$

Since $u, v \in F(T) \subset F\left(T^2\right)$, we have that $\alpha_{\bullet\bullet} \|u - v\|^2 \leq 0$. Since $\alpha_{\bullet\bullet} > 0$, we obtain that $u = v$. This completes the proof. □

Theorem 4.3 generalizes Theorem 3.3 of Takahashi [18], which asserts that there exists a fixed point of demigeneric generalized hybrid mappings in a Hilbert space. Indeed, let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping. Put $\alpha_{11} = \alpha$, $\alpha_{01} = \beta$, $\alpha_{10} = \gamma$, $\alpha_{00} = \delta$, $\beta_0 = \varepsilon$, $\gamma_0 = \zeta$ and the other coefficients are all 0. Then, T with the condition (2) of Theorem 4.3 is $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-demigeneric generalized hybrid that satisfies $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta > 0$ and $\varepsilon \geq 0$. Thus, Theorem 4.3 is a generalization of Theorem 3.3 of Takahashi [18].

Corollary 4.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping. Suppose that T satisfies one of the following conditions:*

(1) $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0,\ \gamma_0 + \gamma_1 \geq 0,\ \gamma_2 \geq 0;$

(2) $\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0,\ \alpha_{\bullet 2} \geq 0,\ \alpha_{\bullet 1} > 0,\ \beta_0 + \beta_1 \geq 0,\ \beta_2 \geq 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0,$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (3.3). *Then, the following four statements are equivalent:*

(a) *for any $x \in C$, $\{T^n x\}$ is a bounded sequence in C;*
(b) *there exists an element $z \in C$ such that the sequence $\{T^n z\}$ in C is bounded;*
(c) *$A(T)$ is nonempty;*
(d) *$F(T)$ is nonempty.*

Proof. (a) $\Longleftrightarrow$ (b) $\Longleftrightarrow$ (c) can be proved in much the same way as the proof of Corollary 4.2. Also, (b) $\Longrightarrow$ (d) has already been established as Theorem 4.3. We will demonstrate that (d) $\Longrightarrow$ (b). Let $z \in F(T)$. Then, we have that

$$T^n z = T^{n-1} z = \cdots = z$$

for all $n \in \mathbb{N}$, which implies that $\{T^n z\}$ is bounded. □

5. Nonlinear ergodic theorems

In this section, we prove a nonlinear ergodic theorem, which is the main result of this paper. The baseline of the proof was established by Takahashi [14]. See also [4, 8, 12, 13, 20, 22, 23] and [11]. We start with demonstrating the following two lemmas.

Lemma 5.1. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta \geq 0$, and let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $a_n - b_n \to 0$. Then, $\liminf_{n\to\infty} (\alpha a_n + \beta b_n) \geq 0$.*

Proof. If $\alpha = \beta = 0$, the desired result follows. Assume, without loss of generality, that $\alpha > 0$. We will prove that

$$\forall \varepsilon > 0,\ \exists n_0 \in \mathbb{N} \text{ such that } n \geq n_0 \Longrightarrow \alpha a_n + \beta b_n > -\varepsilon.$$

Let $\varepsilon > 0$. Since $a_n - b_n \to 0$, we have that for a positive real number $\varepsilon/\alpha > 0$,

$$\exists n_0 \in \mathbb{N} \text{ such that } n \geq n_0 \Longrightarrow b_n - \frac{\varepsilon}{\alpha} < a_n \left(< b_n + \frac{\varepsilon}{\alpha}\right).$$

Let $n \geq n_0$. Using $\alpha > 0$, $\alpha + \beta \geq 0$ and $b_n \geq 0$, we obtain that

$$\begin{aligned} \alpha a_n + \beta b_n &> \alpha \left(b_n - \frac{\varepsilon}{\alpha}\right) + \beta b_n \\ &= (\alpha + \beta)\, b_n - \varepsilon \geq -\varepsilon. \end{aligned}$$

This completes the proof. □

Lemma 5.2. *Let C be a nonempty subset of a real Hilbert space H and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping. Suppose that T satisfies one of the following conditions:*

$$\begin{array}{ll}(1) & \alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \alpha_{20}, \alpha_{21}, \alpha_{22} \geq 0,\ \alpha_{1\bullet} > 0, \\ & \beta_0, \beta_1, \beta_2 \geq 0,\ \gamma_0 + \gamma_1 \geq 0,\ \gamma_2 \geq 0; \\ (2) & \alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0,\ \alpha_{02}, \alpha_{12}, \alpha_{22} \geq 0,\ \alpha_{\bullet 1} > 0, \\ & \beta_0 + \beta_1 \geq 0,\ \beta_2 \geq 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0,\end{array}$$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (3.3). *Let $x \in C$ such that $\{T^n x\}$ is a bounded sequence in C. Define $S_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x\,(\in H)$ and assume that $S_{n_i}x \rightharpoonup u$, where $\{S_{n_i}x\}$ is a subsequence of $\{S_n x\}$. Then, $u \in A(T)$. Additionally, if C is closed and convex, then $u \in F(T)$.*

Proof. **Case (1)**. Suppose that $\alpha_{0\bullet}+\alpha_{1\bullet} \geq 0, \alpha_{20}, \alpha_{21}, \alpha_{22} \geq 0, \alpha_{1\bullet} > 0, \beta_0, \beta_1, \beta_2 \geq 0$, $\gamma_0 + \gamma_1 \geq 0$ and $\gamma_2 \geq 0$. Let $y \in C$. We will prove that $\|Ty - y\|^2 + 2\langle Ty - y,\ y - u\rangle \leq 0$. Since T is $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid, the following holds:

$$\begin{aligned}\alpha_{00}&\left\|y - T^k x\right\|^2 + \alpha_{01}\left\|y - T^{k+1}x\right\|^2 + \alpha_{02}\left\|y - T^{k+2}x\right\|^2 \\ &+\alpha_{10}\left\|Ty - T^k x\right\|^2 + \alpha_{11}\left\|Ty - T^{k+1}x\right\|^2 + \alpha_{12}\left\|Ty - T^{k+2}x\right\|^2 \\ &+\alpha_{20}\left\|T^2y - T^k x\right\|^2 + \alpha_{21}\left\|T^2y - T^{k+1}x\right\|^2 + \alpha_{22}\left\|T^2y - T^{k+2}x\right\|^2 \\ &+\beta_0\|y - Ty\|^2 + \beta_1\left\|Ty - T^2y\right\|^2 + \beta_2\left\|T^2y - y\right\|^2 \\ &+\gamma_0\left\|T^k x - T^{k+1}x\right\|^2 + \gamma_1\left\|T^{k+1}x - T^{k+2}x\right\|^2 + \gamma_2\left\|T^{k+2}x - T^k x\right\|^2 \leq 0\end{aligned} \tag{5.1}$$

for all $k \in \mathbb{N}\cup\{0\}$. Since $\alpha_{20}, \alpha_{21}, \alpha_{22} \geq 0$ and $\beta_0, \beta_1, \beta_2, \gamma_2 \geq 0$, we have that

$$\begin{aligned}\alpha_{00}&\left\|y - T^k x\right\|^2 + \alpha_{01}\left\|y - T^{k+1}x\right\|^2 + \alpha_{02}\left\|y - T^{k+2}x\right\|^2 \\ &+ \alpha_{10}\left\|Ty - T^k x\right\|^2 + \alpha_{11}\left\|Ty - T^{k+1}x\right\|^2 + \alpha_{12}\left\|Ty - T^{k+2}x\right\|^2 \\ &+ \gamma_0\left\|T^k x - T^{k+1}x\right\|^2 + \gamma_1\left\|T^{k+1}x - T^{k+2}x\right\|^2 \leq 0,\end{aligned}$$

and hence,

$$\begin{aligned}\alpha_{00}&\left\|y - T^k x\right\|^2 + \alpha_{01}\left\|y - T^{k+1}x\right\|^2 + \alpha_{02}\left\|y - T^{k+2}x\right\|^2 \\ &+ \alpha_{10}\left(\|Ty - y\|^2 + 2\left\langle Ty - y,\ y - T^k x\right\rangle + \left\|y - T^k x\right\|^2\right) \\ &+ \alpha_{11}\left(\|Ty - y\|^2 + 2\left\langle Ty - y,\ y - T^{k+1}x\right\rangle + \left\|y - T^{k+1}x\right\|^2\right) \\ &+ \alpha_{12}\left(\|Ty - y\|^2 + +2\left\langle Ty - y,\ y - T^{k+2}x\right\rangle + \left\|y - T^{k+2}x\right\|^2\right)\end{aligned}$$

$$+\gamma_0 \left\|T^k x - T^{k+1}x\right\|^2 + \gamma_1 \left\|T^{k+1}x - T^{k+2}x\right\|^2 \le 0.$$

We obtain that

$$\begin{aligned}(\alpha_{00}+\alpha_{10})\left\|y-T^kx\right\|^2 &+ (\alpha_{01}+\alpha_{11})\left\|y-T^{k+1}x\right\|^2 + (\alpha_{02}+\alpha_{12})\left\|y-T^{k+2}x\right\|^2 \\ &+ \alpha_{10}\left(\|Ty-y\|^2 + 2\left\langle Ty-y,\ y-T^kx\right\rangle\right) \\ &+ \alpha_{11}\left(\|Ty-y\|^2 + 2\left\langle Ty-y,\ y-T^{k+1}x\right\rangle\right) \\ &+ \alpha_{12}\left(\|Ty-y\|^2 + 2\left\langle Ty-y,\ y-T^{k+2}x\right\rangle\right) \\ &+ \gamma_0 \left\|T^k x - T^{k+1}x\right\|^2 + \gamma_1 \left\|T^{k+1}x - T^{k+2}x\right\|^2 \le 0.\end{aligned}$$

This yields that

$$\begin{aligned}(\alpha_{00}+\alpha_{10})\left\|y-T^kx\right\|^2 &+ (\alpha_{01}+\alpha_{11})\left\|y-T^{k+1}x\right\|^2 + (\alpha_{02}+\alpha_{12})\left\|y-T^{k+2}x\right\|^2 \\ &+ \alpha_{1\bullet}\|Ty-y\|^2 + 2\alpha_{10}\left\langle Ty-y,\ y-T^kx\right\rangle + 2\alpha_{11}\left\langle Ty-y,\ y-T^{k+1}x\right\rangle \\ &+ 2\alpha_{12}\left\langle Ty-y,\ y-T^{k+2}x\right\rangle + \gamma_0 \left\|T^k x - T^{k+1}x\right\|^2 + \gamma_1 \left\|T^{k+1}x - T^{k+2}x\right\|^2 \le 0.\end{aligned}$$

It holds that

$$\begin{aligned}(\alpha_{0\bullet}+\alpha_{1\bullet})\left\|y-T^kx\right\|^2 &+ (\alpha_{01}+\alpha_{11})\left(\left\|y-T^{k+1}x\right\|^2 - \left\|y-T^kx\right\|^2\right) \\ &+ (\alpha_{02}+\alpha_{12})\left(\left\|y-T^{k+2}x\right\|^2 - \left\|y-T^kx\right\|^2\right) \\ &+ \alpha_{1\bullet}\|Ty-y\|^2 + 2\left\langle Ty-y,\ \alpha_{1\bullet}y - \left(\alpha_{10}T^kx + \alpha_{11}T^{k+1}x + \alpha_{12}T^{k+2}x\right)\right\rangle \\ &+ \gamma_0 \left\|T^k x - T^{k+1}x\right\|^2 + \gamma_1 \left\|T^{k+1}x - T^{k+2}x\right\|^2 \le 0.\end{aligned}$$

Since $\alpha_{0\bullet}+\alpha_{1\bullet}\ge 0$, we have that

$$\begin{aligned}(\alpha_{01}+\alpha_{11})&\left(\left\|y-T^{k+1}x\right\|^2 - \left\|y-T^kx\right\|^2\right) \\ &+ (\alpha_{02}+\alpha_{12})\left(\left\|y-T^{k+2}x\right\|^2 - \left\|y-T^kx\right\|^2\right) + \alpha_{1\bullet}\|Ty-y\|^2 \\ &+ 2\left\langle Ty-y,\ \alpha_{1\bullet}y - \left\{\alpha_{1\bullet}T^kx + \alpha_{11}\left(T^{k+1}x - T^kx\right) + \alpha_{12}\left(T^{k+2}x - T^kx\right)\right\}\right\rangle \\ &+ \gamma_0 \left\|T^k x - T^{k+1}x\right\|^2 + \gamma_1 \left\|T^{k+1}x - T^{k+2}x\right\|^2 \le 0.\end{aligned}$$

Summing these inequalities with respect to k from 0 to $n-1$, we obtain that

$$\begin{aligned}n\alpha_{1\bullet}\|Ty-y\|^2 &+ (\alpha_{01}+\alpha_{11})\left(\|y-T^nx\|^2 - \|y-x\|^2\right) \\ &+ (\alpha_{02}+\alpha_{12})\left(\left\|y-T^{n+1}x\right\|^2 + \|y-T^nx\|^2 - \|y-Tx\|^2 - \|y-x\|^2\right)\end{aligned}$$

$$+ 2\langle Ty - y,\ n\alpha_{1\bullet} y - \{\alpha_{1\bullet} \sum_{k=0}^{n-1} T^k x + \alpha_{11}\left(T^n x - x\right)$$

$$+ \alpha_{12}\left(T^{n+1}x + T^n x - Tx - x\right)\}\rangle$$

$$+ \gamma_0 \sum_{k=0}^{n-1} \left\|T^k x - T^{k+1}x\right\|^2 + \gamma_1 \sum_{k=0}^{n-1} \left\|T^{k+1}x - T^{k+2}x\right\|^2 \leq 0.$$

Dividing it by n, we have that

$$\alpha_{1\bullet} \|Ty - y\|^2 + \frac{1}{n}\left(\alpha_{01} + \alpha_{11}\right)\left(\|y - T^n x\|^2 - \|y - x\|^2\right) \tag{5.2}$$

$$+\frac{1}{n}\left(\alpha_{02} + \alpha_{12}\right)\left(\left\|y - T^{n+1}x\right\|^2 + \|y - T^n x\|^2 - \|y - Tx\|^2 - \|y - x\|^2\right)$$

$$+2\langle Ty - y,\ \alpha_{1\bullet} y - \{\alpha_{1\bullet} S_n x + \frac{1}{n}\alpha_{11}\left(T^n x - x\right)$$

$$+\frac{1}{n}\alpha_{12}\left(T^{n+1}x + T^n x - Tx - x\right)\}\rangle$$

$$+\gamma_0 \frac{1}{n} \sum_{k=0}^{n-1} \left\|T^k x - T^{k+1}x\right\|^2 + \gamma_1 \frac{1}{n} \sum_{k=0}^{n-1} \left\|T^{k+1}x - T^{k+2}x\right\|^2 \leq 0.$$

Since $\gamma_0 + \gamma_1 \geq 0$, it holds that

$$\liminf_{n\to\infty} \left(\gamma_0 \frac{1}{n} \sum_{k=0}^{n-1} \left\|T^k x - T^{k+1}x\right\|^2 + \gamma_1 \frac{1}{n} \sum_{k=0}^{n-1} \left\|T^{k+1}x - T^{k+2}x\right\|^2\right) \geq 0. \tag{5.3}$$

Indeed, since $\{T^n x\}$ is bounded, we obtain that

$$\begin{aligned} &\frac{1}{n} \sum_{k=0}^{n-1} \left\|T^k x - T^{k+1}x\right\|^2 - \frac{1}{n} \sum_{k=0}^{n-1} \left\|T^{k+1}x - T^{k+2}x\right\|^2 \\ = &\frac{1}{n}\left(\sum_{k=0}^{n-1} \left\|T^k x - T^{k+1}x\right\|^2 - \sum_{k=0}^{n-1} \left\|T^{k+1}x - T^{k+2}x\right\|^2\right) \\ = &\frac{1}{n}\left(\|x - Tx\|^2 - \left\|T^n x - T^{n+1}x\right\|^2\right) \to 0 \text{ as } n \to \infty. \end{aligned}$$

From Lemma 5.1, the inequality (5.3) holds under the condition $\gamma_0 + \gamma_1 \geq 0$. Replacing n by n_i and taking the lim inf as $i \to \infty$ in (5.2), we obtain that

$$\alpha_{1\bullet} \|Ty - y\|^2 + 2\alpha_{1\bullet}\langle Ty - y,\ y - u\rangle \leq 0.$$

Since $\alpha_{1\bullet} > 0$, it holds that

$$\|Ty - y\|^2 + 2\langle Ty - y,\ y - u\rangle \leq 0$$

for all $y \in C$. From (2.2), we obtain that $u \in A(T)$.

Additionally, suppose that C is closed and convex. Then, $\{S_n x\}$ is a sequence in C. Since C is weakly closed, we have that $u \in A(T) \cap C$. From Lemma 2.2, we obtain that $u \in F(T)$.

Case (2). Suppose that $\alpha_{\bullet 0}+\alpha_{\bullet 1}\geq 0$, $\alpha_{02},\alpha_{12},\alpha_{22}\geq 0$, $\alpha_{\bullet 1}>0$, $\beta_0+\beta_1\geq 0$, $\beta_2\geq 0$ and $\gamma_0,\gamma_1,\gamma_2\geq 0$. We can derive the desired result by replacing y and $T^k x$ in (5.1). □

Using Lemma 5.2, we can demonstrate our main theorem of this paper.

Theorem 5.3. *Let C be an nonempty subset of a real Hilbert space H and let $T: C\to C$ be a $(\alpha_{ij},\beta_i,\gamma_i;\, i,j=0,1,2)$-generic 2-generalized hybrid mapping with $A(T)\neq\emptyset$. Let $P_{A(T)}$ be the metric projection from H onto $A(T)$. Suppose that T satisfies one of the following conditions:*

$$\begin{array}{ll}(1) & \alpha_{0\bullet}+\alpha_{1\bullet}\geq 0,\ \alpha_{20},\alpha_{21},\alpha_{22}\geq 0,\ \alpha_{1\bullet}>0,\\ & \beta_0,\beta_1,\beta_2\geq 0,\ \gamma_0+\gamma_1\geq 0,\ \gamma_2\geq 0;\\ (2) & \alpha_{\bullet 0}+\alpha_{\bullet 1}\geq 0,\ \alpha_{02},\alpha_{12},\alpha_{22}\geq 0,\ \alpha_{\bullet 1}>0,\\ & \beta_0+\beta_1\geq 0,\ \beta_2\geq 0,\ \gamma_0,\gamma_1,\gamma_2\geq 0;\end{array}$$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (3.3). Then, for any $x\in C$, the sequence $\left\{S_n x\equiv\frac{1}{n}\sum_{k=0}^{n-1}T^k x\right\}$ converges weakly to $v\in A(T)$, where $v\equiv\lim_{n\to\infty}P_{A(T)}T^n x$. Additionally, suppose that C is closed and convex. Then, for any $x\in C$, the sequence $\{S_n x\}$ converges weakly to a fixed point v of T.

Proof. We know that $A(T)$ is closed and convex in H. Since $A(T)\neq\emptyset$ is assumed, there exists the metric projection $P_{A(T)}$ from H onto $A(T)$. Let $x\in C$ and define $S_n x\equiv\frac{1}{n}\sum_{k=0}^{n-1}T^k x\in H$ for all $n\in\mathbb{N}$. From Corollary 4.2, $\{T^n x\}$ is a bounded sequence in C. Thus, the sequence $\{S_n x\}$ is also bounded in H. It is obvious that

$$\left\|T^{n+1}x-q\right\|\leq\left\|T^n x-q\right\| \tag{5.4}$$

for any $q\in A(T)$ and $n\in\mathbb{N}\cup\{0\}$. From Lemma 2.3, the sequence $\left\{P_{A(T)}T^n x\right\}$ is convergent in $A(T)$. Define $v\equiv\lim_{n\to\infty}P_{A(T)}T^n x\in A(T)$. Our aim is to prove that $S_n x\rightharpoonup v$. Let $\{S_{n_i}x\}$ be a subsequence of $\{S_n x\}$ such that $S_{n_i}x\rightharpoonup u$. Since $\{S_n x\}$ is bounded, from (2.1), it is enough to show that $u=v$. We have from Lemma 5.2 that $u\in A(T)$. It is easy to verify that the sequence of real numbers $\left\{\left\|T^n x-P_{A(T)}T^n x\right\|\right\}$ is monotone decreasing. Indeed, it holds from $P_{A(T)}T^n x\in A(T)$ and (5.4) that

$$\begin{aligned}\left\|T^{n+1}x-P_{A(T)}T^{n+1}x\right\| &\leq \left\|T^{n+1}x-P_{A(T)}T^n x\right\|\\ &\leq \left\|T^n x-P_{A(T)}T^n x\right\|\end{aligned} \tag{5.5}$$

for any $n\in\mathbb{N}\cup\{0\}$. This means that the sequence $\left\{\left\|T^n x-P_{A(T)}T^n x\right\|\right\}$ is monotone decreasing. Since $u\in A(T)$, we have

$$\left\langle T^k x-P_{A(T)}T^k x,\ P_{A(T)}T^k x-u\right\rangle\geq 0$$

for all $k\in\mathbb{N}\cup\{0\}$. Thus,

$$\left\langle T^k x-P_{A(T)}T^k x,\ P_{A(T)}T^k x-v+v-u\right\rangle\geq 0.$$

Using Schwarz's inequality and (5.5), we have that

$$\left\langle T^k x-P_{A(T)}T^k x,\ -(v-u)\right\rangle$$

$$
\begin{aligned}
&\leq \left\langle T^k x - P_{A(T)} T^k x,\ P_{A(T)} T^k x - v \right\rangle \\
&\leq \left\| T^k x - P_{A(T)} T^k x \right\| \left\| P_{A(T)} T^k x - v \right\| \\
&\leq \left\| x - P_{A(T)} x \right\| \left\| P_{A(T)} T^k x - v \right\| .
\end{aligned}
$$

Summing these inequalities with respect to k from 0 to $n-1$, we obtain that

$$
\left\langle \sum_{k=0}^{n-1} T^k x - \sum_{k=0}^{n-1} P_{A(T)} T^k x,\ \ -(v-u) \right\rangle \leq \left\| x - P_{A(T)} x \right\| \sum_{k=0}^{n-1} \left\| P_{A(T)} T^k x - v \right\| .
$$

Dividing it by n, we have that

$$
\left\langle S_n x - \frac{1}{n} \sum_{k=0}^{n-1} P_{A(T)} T^k x,\ \ -(v-u) \right\rangle \leq \left\| x - P_{A(T)} x \right\| \frac{1}{n} \sum_{k=0}^{n-1} \left\| P_{A(T)} T^k x - v \right\| .
$$

Replacing n by n_i and taking the limit as $i \to \infty$, we obtain that

$$
\langle u - v,\ \ -(v-u) \rangle \leq 0
$$

since $P_{A(T)} T^n x \to v$ and $S_{n_i} x \rightharpoonup u$. Hence, we have that $u = v$.

Additionally, suppose that C is closed and convex. Then, $\{S_n x\}$ is a sequence in C. Since C is weakly closed and $S_n x \rightharpoonup v$, we have that $v \in C \cap A(T)$. From Lemma 2.2, we obtain that $v \in F(T)$. This completes the proof. $\square$

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