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CONVEXITY OF SETS AND FUNCTIONS VIA SECOND-ORDER SUBDIFFERENTIALS

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Dedicated to Professor Gue Myung Lee on the occasion of his 65th birthday

Abstract. It is proved that, for a set belonging to certain classes of closed sets in Asplund spaces, the positive semidefiniteness of the limiting second-order subdifferential of its indicator function at each boundary point is necessary and sufficient for the local convexity of the set. It is also shown that the positive semidefiniteness of the limiting second-order subdifferentials of some kinds of continuous functions can characterize their convexity on closed convex sets.

1. Introduction

The limiting second-order subdifferential ([11], [12, Def. 1.118]) of an extended-real-valued function at a point in the graph of its limiting subdifferential mapping is the normal coderivative in the sense of Mordukhovich of that mapping at the point. This concept extends [12, Prop. 1.119] the notion of the adjoint of the second-order derivative of a C^2-smooth function. Thus, it can be considered as a generalized adjoint second-order derivative which, unlike the classical notion, can be defined for any extended-real-valued function. The concept of Fréchet second-order subdifferential can be defined similarly, replacing the limiting first-order subdifferential mapping and the normal coderivative, respectively, by the Fréchet subdifferential mapping and the Fréchet coderivative. Calculus rules for second-order subdifferentials can be found in [11, 12, 14, 24].

The limiting second-order subdifferential has been applied effectively in the studies of stability of optimization and equilibrium problems; see e.g. [6, 7, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 24, 28, 29, 30], and the references therein. Its uses for deriving second-order necessary and sufficient optimality conditions for $C^{1,1}$-smooth optimization problems, as well as for C^1-smooth optimization problems, can be seen in [3, 8, 17].

To see why the limiting second-order subdifferential plays an important role in the investigation on stability of optimization and equilibrium problems, one can note that many questions concerning the latter can be formulated as questions about the behavior of the solution map of parametric generalized equations (which are parametric constraint systems and variational systems in the terminology of [12]). Namely, given a Banach space X with the dual X^*, a Banach space W, a

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multifunction $F : X \times W \rightrightarrows X^*$, and a closed convex set $K \subset X$, one denotes by $S(w)$ the set of all $x \in X$ satisfying the inclusion

$$0 \in F(x, w) + N(x; K)$$

with $N(x; K)$ being the normal cone to the convex set K at x, i.e., $N(x; K) = \emptyset$ if $x \notin K$, and

$$N(x; K) = \{x^* \in X^* \, : \, \langle x^*, u - x\rangle \le 0 \text{ for every } u \in K\}.$$

To study the behavior of the solution map $S : W \rightrightarrows X$, $w \mapsto S(w)$, by using the Mordukhovich criterion for the Lipschitz-like property of S around a point (w, x) in its graph, one has to compute the normal coderivative of the sum $F(x, w)+N(x; K)$, which is considered as a multifunction on the variable (x, w). Usually, the normal coderivative of $F(x, w)$ can be computed directly by the formula of F. The normal coderivative computation for $N(x; K)$ is, as a rule, more difficult. Since one has $N(x; K) = \partial\delta_K(x)$, where $\delta_K(x) := 0$ for $x \in K$ and $\delta_K(x) := +\infty$ for $x \notin K$ is the indicator function of K, and

$$\partial\delta_K(x) := \{x^* \in X^* \, : \, \langle x^*, u - x\rangle \le \delta_K(u) - \delta_K(x) \;\; \forall u \in K\}$$

is the subdifferential of the convex function δ_K at x, the normal coderivative of $N(\cdot; K)$ at (x, x^*) in the graph of $N(\cdot; K)$ coincides with the limiting second-order subdifferential of the indicator function δ_K at (x, x^*). Knowing the normal coderivative of $F(x, w)$ and the normal coderivative of $N(\cdot; K)$, one can compute the normal coderivative of the sum $F(x, w) + N(x; K)$ by a sum rule. This method of stability research has been applied in many of the above-cited papers and in [12, Chap. 4].

It is well known (see e.g. [23, Theorem 4.5, p. 27]) that convexity of a C^2-smooth function on an open convex set in $\mathbb{R}^n$ can be characterized by the positive semidefiniteness of the Hessian of that function at each point from the set. It is also known [27] that a similar characterization can be obtained for the convexity of a function on an arbitrary convex subset of $\mathbb{R}^n$, provided that the function is defined on a neighborhood of the set and it is twice continuously differentiable at each point of the set. A natural question is that to which extent the positive semidefiniteness of the limiting second-order subdifferential (resp., the Fréchet second-order subdifferential) can characterize convexity of functions. In [1] and [2], the authors have shown that the class of lower semicontinuous functions is too large to have such a characterization, but for piecewise linear functions, as well as for piecewise C^2 functions, convexity can be characterized by the positive semidefiniteness of the limiting second-order subdifferential. Moreover, as proved in [2], convexity of C^1-smooth functions can be characterized by the positive semidefiniteness of the Fréchet second-order subdifferential.

A set C in a Banach space X is completely known by its indicator function. If the set is nonempty and closed, then the indicator function is proper, lower semicontinuous. Hence, by [12, Theorem 3.56], each of the subdifferential mappings $\widehat{\partial}\delta_C : X \rightrightarrows X^*$ and $\partial\delta_C : X \rightrightarrows X^*$, where $\widehat{\partial}\delta(x; C)$ and $\partial\delta(x; C)$ respectively denote the Fréchet and the limiting subdifferential (called the Mordukhovich subdifferential) of $\delta_C(\cdot)$ at x, is monotone if and only if $\delta_C(\cdot)$ is convex, i.e., C is convex. Furthermore, if C is convex and closed, then these subdifferential mappings are maximal monotone [22, Theorem A]. Hence, by [4, Lemma 3.3], if X is a Hilbert space and C is convex then, for any $(\bar{x}, \bar{v})$ in the graph of the multifunction $\partial\delta_C$, the limiting second-order subdifferential $\partial^2\delta_C(\bar{x}, \bar{v})$ of δ_C at $\bar{x}$ relative to $\bar{v}$ is positive semidefinite; see the next section for the definitions of $\partial^2\delta_C(\bar{x}, \bar{v})$ and its positive semidefiniteness.

The aim of this paper is twofold. Firstly, we want to see to which extent the positive semidefiniteness of the limiting second-order subdifferential of the indicator function of a closed set can characterize its convexity. Secondly, we want to know how the convexity of a function defined on a closed convex set can be characterized by means of its second-order subdifferentials.

Since the first-order subdifferential and the second-order subdifferential are local structures, the positive semidefiniteness of the limiting second-order subdifferential of the indicator function of a closed set can serve at most as a certificate of its local convexity, not of its convexity. We will clarify this observation by constructing a suitable example. It is clear that if a set is convex then it is both connected and locally convex. Conversely, by the results of Tietze and Schoenberg (see [25]) we know that convexity of a closed set in a normed space is equivalent to the local convexity and connectedness of that set. Therefore, checking the convexity of a closed set consists of the checks of its connectedness and its local convexity.

The paper organization and our results can be outlined as follows. After recalling some basic concepts in Sect. 2, we show in Sect. 3 that the positive semidefiniteness of the limiting second-order subdifferential of the indicator function of a closed set with a C^2-smooth and regular boundary can characterize its local convexity. In addition, we show that such second-order characterization of local convexity is valid for any finite-dimensional closed set which can be locally represented as the epigraphs of piecewise C^2 functions or the epigraphs of C^1 functions. Sect. 4 is devoted to an analysis of the relationships between the convexity of a continuous function defined on a closed convex set and the positive semidefiniteness of its limiting second-order subdifferential with respect to the linear space generated by the set. If the interior of the set is nonempty, then we are able to deal with functions which are C^1-smooth on the interior. If the interior of the given set is empty, then we can carry the analysis just for functions which are $C^{1,1}$-smooth on an open neighborhood of the set.

2. Preliminaries

Let X be a Banach space with the dual and the second dual being denoted respectively by X^* and X^{**}. The value of $x^* \in X^*$ at $x \in X$ is denoted by $\langle x^*, x\rangle$. Every $x \in X$ generates a unique element of $j(x) \in X^{**}$ which is defined by setting $\langle j(x), x^*\rangle := \langle x^*, x\rangle$ for all $x^* \in X^*$. Since $j : X \to X^{**}$ is a linear operator and since $\|j(x)\| = \|x\|$, one usually identifies $j(x)$ with x. So, X can be identified with the closed linear subspace $j(X) \subset X^{**}$. By abuse of notation, we will write $X \subset X^{**}$.

For a subset $\Omega \subset X$, the symbols $\overline{\Omega}$, $\operatorname{int}\Omega$, and $\partial\Omega$ respectively denote the closure of Ω, the interior of Ω, and the boundary of Ω. By $\overline{B}(x,\rho)$ and $B(x,\rho)$, respectively, we abbreviate the closed ball and the open ball centered at x with radius ρ. If $A : X \to Y$ is a bounded linear operator between Banach spaces, then A^* stands for the adjoint of A.

For a multifunction $\Phi : X \rightrightarrows X^*$, the expression $\operatorname*{Lim\,sup}_{x\to\bar{x}} \Phi(x)$ means the sequential Kuratowski-Painlevé upper limit of $\Phi(x)$ as $x \to \bar{x}$ with respect to the norm topology of X and the weak* topology of X^*, that is,

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} \Phi(x) = \{x^* \in X^* : \ \exists \text{ sequences } x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^*, \text{ with } x_k^* \in \Phi(x_k) \text{ for all } k = 1,2,\dots\}.$$

We now recall some fundamental concepts from [12] which will be used in the sequel. The set $\widehat{N}_\varepsilon(x;\Omega)$ of the Fréchet ε-normals to Ω at $x \in \overline{\Omega}$ is given by

$$\widehat{N}_\varepsilon(x;\Omega) = \Big\{x^* \in X^* \, : \, \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le \varepsilon\Big\},$$

where $u \xrightarrow{\Omega} x$ means that $u \to x$ and u remains in Ω. One puts $\widehat{N}_\varepsilon(x;\Omega) = \emptyset$ for all $\varepsilon \ge 0$ whenever $x \notin \overline{\Omega}$. One calls $\widehat{N}_0(x;\Omega)$ the *Fréchet normal cone* to Ω at x, and uses the simpler notation $\widehat{N}(x;\Omega)$ for it. The set

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{x\to\bar{x},\ \varepsilon\downarrow 0} \widehat{N}_\varepsilon(x;\Omega)$$

is the *Mordukhovich normal cone* to Ω at $\bar{x}$. If $\bar{x} \notin \overline{\Omega}$, then one puts $N(\bar{x};\Omega) = \emptyset$.

Let $\Phi : X \rightrightarrows Y$ be a set-valued map between Banach spaces. The set

$$\operatorname{gph}\Phi := \{(x,y) \in X \times Y \, : \, y \in \Phi(x)\}$$

denotes the graph of Φ. For every element $(\bar{x},\bar{y}) \in X \times Y$, the multifunction $D^*\Phi(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$D^*\Phi(\bar{x},\bar{y})(y^*) := \{x^* \in X^* \, : \, (x^*, -y^*) \in N((\bar{x},\bar{y}); \operatorname{gph}\Phi)\},\ y^* \in Y^*,$$

is said to be the *normal coderivative* (called also the *limiting coderivative* and the *coderivative in the sense of Mordukhovich*) of Φ at $(\bar{x},\bar{y})$. In parallel to the normal coderivative $D^*\Phi(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$, one defines the *mixed coderivative*

$$D^*_M\Phi(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$$

by letting $D^*_M\Phi(\bar{x},\bar{y})(y^*)$ be the set of all $x^* \in X^*$ for which there are sequences $\varepsilon_k \downarrow 0$, $(x_k, y_k, y_k^*) \to (\bar{x},\bar{y},y^*)$ and $x_k^* \xrightarrow{w^*} x^*$ with $(x_k,y_k) \in \operatorname{gph}\Phi$ and

$$(x_k^*, -y_k^*) \in \widehat{N}_{\varepsilon_k}((\bar{x},\bar{y}); \operatorname{gph}\Phi)$$

for every k. Clearly, $D^*_M\Phi(\bar{x},\bar{y})(y^*) \subset D^*\Phi(\bar{x},\bar{y})(y^*)$ for all $y^* \in Y^*$, i.e., the graph of the mixed coderivative is contained in the graph of the normal coderivative.

Let $\overline{\mathbb{R}} = [-\infty, +\infty]$ denote the extended real line and let $\mathbb{R}_+ = [0, +\infty)$. Consider a function $\varphi : X \to \overline{\mathbb{R}}$ and suppose that it has a finite value at $\bar{x} \in X$. The set

$$\partial\varphi(\bar{x}) := \{x^* \in X^* \,:\, (x^*, -1) \in N((\bar{x}, \varphi(\bar{x})); \operatorname{epi}\varphi)\}\,,$$

where $\operatorname{epi}\varphi := \{(x, \alpha) \in X \times \mathbb{R} \,:\, \alpha \geq \varphi(x)\}$ denotes the epigraph of φ, is said to be the *limiting subdifferential* (or the *basic subdifferential*) of φ at $\bar{x}$. Let $\bar{v} \in \partial\varphi(\bar{x})$, i.e., $(\bar{x}, \bar{v})$ belongs to the graph of the subdifferential mapping $\partial\varphi : X \rightrightarrows X^*$, $x \mapsto \partial\varphi(x)$. The map $\partial^2\varphi(\bar{x}, \bar{v}) : X^{**} \rightrightarrows X^*$ with the values

$$\partial^2\varphi(\bar{x}, \bar{v})(u) := (D^*\partial\varphi)(\bar{x}, \bar{v})(u), \quad u \in X^{**},$$

is called the *limiting second-order subdifferential* (or the *normal second-order subdifferential*) of φ at $\bar{x}$ relative to $\bar{v}$; see [12, Def. 1.118(i)]. The *mixed second-order subdifferential* $\partial^2_M\varphi(\bar{x}, \bar{v})$ of φ at $\bar{x}$ relative to $\bar{v}$ is defined similarly, provided that the mixed coderivative $(D^*_M\partial\varphi)(\bar{x}, \bar{v})$ is used instead of the normal coderivative $(D^*\partial\varphi)(\bar{x}, \bar{v})$; see [12, Def. 1.118(ii)]. If φ is C^2-smooth on a neighborhood of $\bar{x}$ then, according to [12, Prop. 1.119],

$$\partial^2\varphi(\bar{x}, \bar{v})(u) = \{\nabla^2\varphi(\bar{x})^* u\}, \quad \forall u \in X^{**}.$$

Here $\bar{v} := \nabla\varphi(\bar{x})$ (with $\nabla\varphi(\bar{x})$ denoting the Fréchet derivative of φ at $\bar{x}$) is the unique element of $\partial\varphi(\bar{x})$, and $\nabla^2\varphi(\bar{x})$ means the second-order derivative of φ at $\bar{x}$.

Following [17], we say that the limiting second-order subdifferential $\partial^2\varphi(\bar{x}, \bar{v})$ is *positive semidefinite* if $\langle u, z\rangle \geq 0$ for any $u \in X^{**}$ and for any $z \in \partial^2\varphi(\bar{x}, \bar{v})(u)$.

One defines the *Fréchet subdifferential* $\widehat{\partial}\varphi(x)$, *Fréchet coderivative* $\widehat{D}^*\Phi(x, y)$, *Fréchet second-order subdifferential* $\widehat{\partial}^2\varphi(\bar{x}, \bar{v})$, and positive semidefiniteness of the Fréchet second-order subdifferential, in the same manner (see e.g. [1]), replacing the normal cone in the sense of Mordukhovich by the corresponding Fréchet normal cone.

One says that X is an Asplund space [12, Def. 2.17] if every convex continuous function $\varphi : U \to \mathbb{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U. Any reflexive Banach space is an Asplund space. The calculus of normal cones, coderivatives, and subdifferentials, in Asplund spaces is simpler than that in general Banach spaces; see [12, Chap. 3].

According to a result in [2, Theorem 3.1 and its proof] (see also [4]), a C^1-smooth function $\varphi : X \to \mathbb{R}$, where X is an Asplund space, is convex if for every $x \in X$ the Fréchet second-order subdifferential $\widehat{\partial}^2\varphi(x, \nabla\varphi(x))$ is positive semidefinite in the following weak sense: $\langle u, z\rangle \geq 0$ for any $u \in X \subset X^{**}$ and for any element $z \in \widehat{\partial}^2\varphi(x, \nabla\varphi(x))(u)$.

3. Convexity of Sets

We first derive necessary and sufficient second-order conditions for local convexity of sets with C^2-smooth boundaries. Our aim is to investigate the role of the limiting second-order subdifferential of the indicator function of a given closed set in characterizing its local convexity. The obtained results will pave a way for deriving similar conditions for convexity of sets with C^1-smooth boundaries and sets with nonsmooth boundaries. Relationships between local convexity and convexity of closed sets will be briefly discussed in Subsection 3.4 below.

3.1. **Sets with C^2-smooth boundaries.** Let $\psi : X \to \mathbb{R}$ be a C^2-smooth function. We put

$$C = \{x \in X \, : \, \psi(x) \le 0\} \tag{3.1}$$

and assume that $\nabla\psi(x) \neq 0$ for every $x \in \partial C = C \setminus \operatorname{int} C$. Thus, C is a set with a C^2-smooth, regular boundary.

If $\bar{x}$ is an interior point of C, then there is $\rho > 0$ such that $B(\bar{x}, \rho) \subset C$. Hence $\partial\delta_C(u) = \{0\}$ for all $u \in B(\bar{x}, \rho)$; so $\partial^2\delta_C(\bar{x}, 0)(u) = \{0\}$ for any $u \in X^{**}$. In particular, the limiting second-order subdifferential $\partial^2\delta_C(\bar{x}, 0)$ is positive semidefinite.

The next theorem characterizes the positive semidefinite property of the limiting second-order subdifferential $\partial^2\delta_C(\bar{x}, \bar{v})$ of the indicator function $\delta_C(\cdot)$ at a boundary point $\bar{x}$ of C relative to *every* element $\bar{v} \in \partial\delta_C(\bar{x})$.

Theorem 3.1. *Let $\bar{x} \in \partial C$ and let $\nabla\psi(\bar{x})^\perp := \{u \in X^{**} \, : \, \langle u, \nabla\psi(\bar{x})\rangle = 0\}$. The following properties are equivalent:*

(i) *For every $\bar{v} \in \partial\delta_C(\bar{x})$, the limiting second-order subdifferential $\partial^2\delta_C(\bar{x}, \bar{v})$ is positive semidefinite, i.e., $\langle u, z\rangle \ge 0$ for every $u \in X^{**}$ and for every $z \in \partial^2\delta_C(\bar{x}, \bar{v})(u)$.*
(ii) *For every $u \in \nabla\psi(\bar{x})^\perp$, one has $\langle u, \nabla^2\psi(\bar{x})^* u\rangle \ge 0$.*

Proof. Since $\bar{x} \in \partial C$ and $\nabla\psi(\bar{x}) \neq 0$, one has $\partial\delta_C(\bar{x}) = N(\bar{x}; C)$ and for each $\bar{v} \in \partial\delta_C(\bar{x})$ there exists a unique $y^* \in \mathbb{R}_+$ such that $\bar{v} = \nabla\psi(\bar{x})^* y^*$ (see [30, formula (8)]). In what follows, for every $\bar{v} \in \partial\delta_C(\bar{x})$, the corresponding value $y^* = y^*(\bar{v})$ is automatically associated to $\bar{v}$.

By using the central result of the second-order subdifferential calculus in general Banach spaces [12, Theorem 1.127], one can show that

$$\partial^2\delta_C(\bar{x}, \bar{v})(u) = \begin{cases} y^*\nabla^2\psi(\bar{x})^* u + \mathbb{R}\nabla\psi(\bar{x}) & \text{if } y^* > 0, \langle u, \nabla\psi(\bar{x})\rangle = 0 \\ \emptyset & \text{if } y^* > 0, \langle u, \nabla\psi(\bar{x})\rangle \neq 0 \\ \mathbb{R}_+\nabla\psi(\bar{x}) & \text{if } y^* = 0, \langle u, \nabla\psi(\bar{x})\rangle > 0 \\ \mathbb{R}\nabla\psi(\bar{x}) & \text{if } y^* = 0, \langle u, \nabla\psi(\bar{x})\rangle = 0 \\ \{0\} & \text{if } y^* = 0, \langle u, \nabla\psi(\bar{x})\rangle < 0 \\ \emptyset & \text{if } y^* < 0 \end{cases} \tag{3.2}$$

(see [30, p. 211]).

Let us prove that (i) implies (ii). Suppose that for every $\bar{v} \in \partial\delta_C(\bar{x})$, the limiting second-order subdifferential $\partial^2\delta_C(\bar{x}, \bar{v})$ is positive semidefinite. Fix any $u \in \nabla\psi(\bar{x})^\perp$. We have $\langle u, \nabla\psi(\bar{x})\rangle = 0$. Select $\bar{v} = \nabla\psi(\bar{x})^* y^*$ with $y^* > 0$. By (3.2), we have

$$\partial^2\delta_C(\bar{x}, \bar{v})(u) = y^*\nabla^2\psi(\bar{x})^* u + \mathbb{R}\nabla\psi(\bar{x}).$$

Since $z := y^*\nabla^2\psi(\bar{x})^* u$ belongs to $\partial^2\delta_C(\bar{x}, \bar{v})(u)$, the inequality $\langle u, \nabla^2\psi(\bar{x})^* u\rangle \ge 0$ follows from the condition $\langle u, z\rangle \ge 0$.

We now show that (ii) implies (i). Suppose that $\langle u, \nabla^2\psi(\bar{x})^*u\rangle \geq 0$ every vector $u \in \nabla\psi(\bar{x})^{\perp}$. Take any $u \in X^{**}$ and any $z \in \partial^2\delta_C(\bar{x}, \bar{v})(u)$. In accordance with (3.2), such vector z does not exist if $y^* < 0$, or $y^* > 0$ and $\langle u, \nabla\psi(\bar{x})\rangle \neq 0$.

In the case where $y^* = 0$ and $\langle u, \nabla\psi(\bar{x})\rangle > 0$, by (3.2) we have $z \in \mathbb{R}_+\nabla\psi(\bar{x})$. Then the inequality $\langle u, z\rangle \geq 0$ is valid.

If $y^* = 0$ and $\langle u, \nabla\psi(\bar{x})\rangle = 0$, then by (3.2) we have $z \in \mathbb{R}\nabla\psi(\bar{x})$. Thus the inequality $\langle u, z\rangle \geq 0$ is also valid in this case.

Finally, consider the case where $y^* = 0$ and $\langle u, \nabla\psi(\bar{x})\rangle < 0$, by (3.2) we have $z = 0$. The desired inequality $\langle u, z\rangle \geq 0$ is now obvious.

The equivalence between (i) and (ii) has been established. □

Necessary conditions for convexity of C can be stated as follows.

Proposition 3.2. *If X is a Hilbert space and if the set C given by* (3.1) *is convex and nonempty then, for any $\bar{x} \in \partial C$, the properties* (i) *and* (ii) *in the formulation of Theorem 3.1 are valid.*

Proof. By the nonemptyness, closedness, and convexity of C, the indicator function δ_C is a proper, lower semicontinous, and convex. Hence, according to [12, Theorem 3.56], the subdifferential mapping $\partial\delta_C : X \rightrightarrows X^* \equiv X$ is monotone. Moreover, it is maximal monotone. Consequently, by [4, Lemma 3.3] (which is an infinite dimensional extension of [17, Theorem 2.1]), for any $(\bar{x}, \bar{v}) \in \mathrm{gph}\,\partial\delta_C$, the second-order subdifferential $\partial^2\delta_C(\bar{x}, \bar{v})$ of δ_C at $\bar{x}$ relative to $\bar{v}$ is positive semidefinite. This means that the property (i) in Theorem 3.1 is valid. Then, by that theorem, the property (ii) is valid too. □

According to [25, p. 432], the notion of local convexity is due to H. Tietze (1928).

Definition 3.3. One says that a set $\Omega \subset X$ is *locally convex* around $\bar{x} \in \Omega$ if the exists $\rho > 0$ such that $\Omega \cap B(\bar{x}, \rho)$ is a convex set. If Ω is locally convex around any point $\bar{x} \in \Omega$, then Ω is said to be a *locally convex set.*

Since the sets $\partial^2\delta_C(\bar{x}, \bar{v})(u)$, with $\bar{v} \in \partial\delta_C(\bar{x})$ and $u \in X^{**}$ being chosen arbitrarily, depend only on the local structure of C around $\bar{x}$, the replacement of C by $C \cap \overline{B}(\bar{x}, \rho)$ for any $\rho > 0$ does not change the sets $\partial^2\delta_C(\bar{x}, \bar{v})(u)$, where $\bar{v} \in \partial\delta_C(\bar{x})$ and $u \in X^{**}$. This observation allows us to derive from Proposition 3.2 the following.

Proposition 3.4. *If X is a Hilbert space and if the set C given by* (3.1) *is locally convex around a point $\bar{x} \in \partial C$, then the properties* (i) *and* (ii) *in the formulation of Theorem 3.1 are valid.*

Proposition 3.2 gives us second-order necessary conditions for the convexity of C. It is of interest to have some second-order sufficient conditions for the latter property.

Before going further, let us show that the necessary conditions for the convexity of sets with C^2-smooth boundaries provided by Proposition 3.2 are not sufficient ones.

Example 3.5. Let $X = \mathbb{R}^2$ and $\psi(x) = \psi_1(x)\psi_2(x)$, where $\psi_1(x) = x_1^2 + x_2^2 - \frac{1}{4}$ and $\psi_2(x) = (x_1 - 2)^2 + x_2^2 - 1$ for every $x = (x_1, x_2) \in X$. It is easy to see that the set C defined by (3.1) is the disjoint union of the closed balls centered at $(0, 0)$ and at $(2, 0)$ with the radiuses $\frac{1}{2}$ and 1, which are denoted respectively by C_1 and C_2. Hence C is compact, nonconvex. For every $x \in \partial C_1$, we have

$$\nabla\psi(x) = \psi_2(x)\nabla\psi_1(x) + \psi_1(x)\nabla\psi_2(x) = \psi_2(x)\nabla\psi_1(x)$$

and $\psi_2(x) > 0$. Thus $\nabla\psi(x) = 2\psi_2(x)(x_1, x_2) \neq 0$. Similarly, for every $x \in \partial C_2$, it holds that

$$\nabla\psi(x) = \psi_1(x)\nabla\psi_2(x) = 2\psi_1(x)(x_1 - 2, x_2) \neq 0.$$

Furthermore, since

$$\nabla^2\psi(x) = 2\nabla\psi_2(x)^T\nabla\psi_1(x) + \psi_1(x)\nabla^2\psi_2(x) + \psi_2(x)\nabla^2\psi_1(x),$$

where the superscript ^{T} stands for matrix transposition, for any $\bar{x} \in \partial C_1$ and $u = (u_1, u_2) \in \nabla\psi(\bar{x})^\perp$ we have

$$\langle\nabla^2\psi(\bar{x})^*u, u\rangle = \psi_2(\bar{x})\langle\nabla^2\psi_1(\bar{x})u, u\rangle = 2\psi_2(\bar{x})(u_1^2 + u_2^2) \geq 0.$$

This means that the property (ii) in Theorem 3.1 is valid; hence for every element $\bar{v} \in \partial\delta_C(\bar{x})$, the limiting second-order subdifferential $\partial^2\delta_C(\bar{x}, \bar{v})$ is positive semidefinite. We have seen that the necessary conditions provided by Proposition 3.2 do not guarantee that C is convex.

It turns out that the converse statement of that one in Proposition 3.4 holds true in a general infinite-dimensional setting. The precise formulation of the result is as follows.

Theorem 3.6. *If the property* (i) *(or the equivalent property* (ii)*) in the formulation of Theorem 3.1 is valid at any $\bar{x} \in \partial C$, where C is given by* (3.1)*, then C is a locally convex set.*

Proof. The proof is divided into two steps.

Step 1. We will prove that C can be locally presented as epigraphs of C^2-smooth functions. Fix any point $\bar{x} \in \partial C$. Since $\nabla\psi(\bar{x}) \neq 0$, there exists a unit vector $z \in X$ satisfying $\langle\nabla\psi(\bar{x}), z\rangle \neq 0$. Setting

$$X_0 = \ker\nabla\psi(\bar{x}) = \{u \in X \,:\, \langle\nabla\psi(\bar{x}), u\rangle = 0\}$$

and $Y = \operatorname{span}\{z\} = \mathbb{R}z$, we observe that X_0 and Y are closed subspaces of X. Moreover, $X = X_0 \oplus Y$ and the formula $\|u + v\|_1 := \|u\| + \|v\|$, for $u \in X_0$ and $v \in Y$, defines a new norm on X which is equivalent to the given norm $\|\cdot\|$. Thus X can be interpreted as the product $X_0 \times Y$ of the Banach spaces X_0 and Y with the norm $\|(u, v)\| = \|u\| + \|v\|$. Identifying every vector $x = u + v$ in X, where $u \in X_0$ and $v \in Y$, with the vector $(u, v) \in X_0 \times Y$, we can rewrite the equation $\psi(x) = 0$ equivalently as $\psi(u, v) = 0$. Let $\bar{x} = \bar{u} + \bar{v}$ with $\bar{u} \in X_0$ and $\bar{v} \in Y$. Since

$$\begin{aligned}\frac{\partial\psi}{\partial v}(\bar{u}, \bar{v}) = \lim_{t\to 0}\frac{\psi(\bar{u}, \bar{v} + tz) - \psi(\bar{u}, \bar{v})}{t} &= \lim_{t\to 0}\frac{\psi(\bar{x} + tz) - \psi(\bar{x})}{t}\\ &= \langle\nabla\psi(\bar{x}), z\rangle \neq 0,\end{aligned}$$

by the implicit function theorem (see, e.g., [9, p. 29]) there exist open neighborhoods U and V respectively of $\bar{u} \in X_0$ and $\bar{v} \in Y$ such that for every $u \in U$ there is a unique $v = v(u) \in V$ such that $\psi(u, v(u)) = 0$, and the function $u \mapsto v(u)$, is C^2-smooth on U. Since $Y = \operatorname{span}\{z\}$, for each $u \in U$ there is a unique $\varphi(u) \in \mathbb{R}$ such that $v(u) = \varphi(u)z$. It is clear that the function $\varphi : U \to \mathbb{R}$ is C^2-smooth.

As $\langle \nabla\psi(\bar{x}), z\rangle \neq 0$, there are two possibilities: (a) $\frac{\partial\psi}{\partial v}(\bar{u}, \bar{v}) < 0$; (b) $\frac{\partial\psi}{\partial v}(\bar{u}, \bar{v}) > 0$.

First, suppose that (a) occurs. Replacing U and V by smaller neighborhoods, if necessary, we can assume that $\frac{\partial\psi}{\partial v}(u, v) < 0$ for all $(u, v) \in U \times V$. We are going to show that

$$C \cap (U \times V) = (\operatorname{epi}\widetilde{\varphi}) \cap (U \times V), \tag{3.3}$$

where $\widetilde{\varphi}(u) := \varphi(u)z$ and

$$\operatorname{epi}\widetilde{\varphi} := \{(u, v) \in X_0 \times Y \,:\, v = \alpha z,\ \alpha \geq \varphi(u)\}. \tag{3.4}$$

Let $(u, v) \in U \times V$ be such that $\psi(u, v) \leq 0$, that is, (u, v) belongs to the left-hand-side of (3.3). Let $v = \alpha z$, $\alpha \in \mathbb{R}$. By the mean-value theorem we can find $\xi \in (\varphi(u)z, v) = \{(1-t)\varphi(u)z + tv \,:\, 0 < t < 1\}$ such that

$$\psi(u, v) - \psi(u, \varphi(u)z) = \frac{\partial\psi}{\partial v}(u, \xi)(\alpha - \varphi(u)). \tag{3.5}$$

Since $\psi(u, \varphi(u)z) = 0$, $\psi(u, v) \leq 0$, and $\frac{\partial\psi}{\partial v}(u, \xi) < 0$, this yields $\alpha - \varphi(u) \geq 0$. Hence (u, v) belongs to the right-hand-side of (3.3). Conversely, if this property is valid, then $v = \alpha z$ and $\alpha \geq \varphi(u)$. Choose $\xi \in (\varphi(u)z, v)$ such that (3.5) is fulfilled. Using the properties $\psi(u, \varphi(u)z) = 0$, $\alpha \geq \varphi(u)$, and $\frac{\partial\psi}{\partial v}(u, \xi) < 0$, by (3.5) we can infer that $\psi(u, v) \leq 0$. Hence (u, v) belongs to the left-hand-side of (3.3).

If (b) occurs, then by the above arguments we show that

$$C \cap (U \times V) = (\operatorname{hypo}\widetilde{\varphi}) \cap (U \times V), \tag{3.6}$$

where

$$\operatorname{hypo}\widetilde{\varphi} := \{(u, v) \in X_0 \times Y \,:\, v = \alpha z,\ \alpha \leq \varphi(u)\}.$$

Step 2. We now prove that C is locally convex around any given point $\bar{x} \in \partial C$. By the construction given in Step 1, we select a unit vector z with $\langle \nabla\psi(\bar{x}), z\rangle \neq 0$, decompose $X = X_0 \oplus Y$, write $\bar{x} = \bar{u} + \bar{v}$ with $\bar{u} \in X_0$ and $\bar{v} \in Y$, find open convex neighborhoods of U and V respectively of $\bar{u}$ and $\bar{v}$, and find a C^2-smooth function $\widetilde{\varphi} : U \to V$, $\widetilde{\varphi}(u) = \varphi(u)z$, such that either the representation (3.3) or the representation (3.6) takes place. Consider the first case. We want to prove that the function $\varphi : U \to \mathbb{R}$ is convex on U. For doing so, it suffices to show that $\nabla^2\varphi(u)$ is positive semidefinite at every $u \in U$. Indeed, if the latter is valid then, for any segment $[u^1, u^2] \subset U$, $u^2 \neq u^1$, the function $g(t) := \varphi((1-t)u^1 + tu^2),\ t \in [0, 1]$, is continuous on $[0, 1]$, twice differentiable at any $t \in (0, 1)$, and

$$\begin{aligned} g'(t) &= \langle \nabla\varphi((1-t)u^1 + tu^2), u^2 - u^1\rangle, \\ g''(t) &= \langle \nabla^2\varphi((1-t)u^1 + tu^2)(u^2 - u^1), u^2 - u^1\rangle. \end{aligned}$$

Hence, by the PSD property of $\nabla^2\varphi(u)$ on U, one has $g''(t) \geq 0$ for every $t \in (0,1)$. It follows that g is convex on $(0,1)$. By the continuity of g, one can assert that g is convex on $[0,1]$. It follows that φ is convex on U. It remains to prove that, for every $u \in U$, $\nabla^2\varphi(u)$ is positive semidefinite. In accordance with (3.3), by identifying $\mathbb{R}z$ with $\mathbb{R}$ and αz with $\alpha \in \mathbb{R}$, we can represent $\psi(u,v)$ for $(u,v) \in U \times V$ in the form

$$\psi(u,v) = \widetilde{\psi}(u,\alpha) := \varphi(u) - \alpha,$$

where $\alpha \in \mathbb{R}$ is defined uniquely by the condition $v = \alpha z$. Let $u \in U$ and $\alpha \in \mathbb{R}$ be such that $\varphi(u) - \alpha = 0$. Since

$$\nabla\widetilde{\psi}(u,\alpha) = (\nabla\varphi(u), -1),$$

we have

$$\{(\widetilde{u}, \nabla\varphi(u)\widetilde{u}) \,:\, \widetilde{u} \in X_0\} \subset \nabla\widetilde{\psi}(u,\alpha)^{\perp}.$$

In addition, since

$$\nabla^2\widetilde{\psi}(u,\alpha) = \begin{bmatrix} \nabla^2\varphi(u) & 0 \\ 0 & 0 \end{bmatrix},$$

the assumption

$$\langle \nabla^2\widetilde{\psi}(u,\alpha)^*\widetilde{w}, \widetilde{w}\rangle \geq 0, \quad \forall \widetilde{w} = (\widetilde{u}, \widetilde{\alpha}) \in \nabla\widetilde{\psi}(u,\alpha)^{\perp},$$

which follows from the assumption made on ψ in the formulation of the theorem, implies that

$$\langle \nabla^2\varphi(u)^*\widetilde{u}, \widetilde{u}\rangle \geq 0, \quad \forall \widetilde{u} \in X_0.$$

As $U \times V$ is convex, the convexity of $C \cap (U \times V)$ is a direct consequence of (3.3) and the convexity of φ, because $\widetilde{\varphi}(u) = \varphi(u)z$.

If the representation (3.6) takes place, arguing similarly we can show that φ is concave on U. Then the convexity of $C \cap (U \times V)$ follows from (3.6) and the concavity of φ.

Since the local convexity of C around any point $\bar{x} \in \operatorname{int} C$ is obvious, the proof is complete. □

3.2. **Sets with C^1-smooth boundaries.** In this subsection, we consider the set C given by (3.1), where X is an Asplund space and $\varphi : X \to \mathbb{R}$ is a continuous function.

The proof of Theorem 3.6 gives hint to the following definition.

Definition 3.7. One says that a closed set $\Omega \subset X$ can be *locally represented as the epigraph of a C^1-smooth function* around $\bar{x} \in \partial\Omega$ if one can decompose X into a direct sum $X = X_0 \oplus Y$ with X_0 being a closed linear subspace and $Y = \mathbb{R}z$ an 1-dimensional space, $\bar{x} = \bar{u} + \bar{v}$ with $\bar{u} \in X_0$ and $\bar{v} \in Y$, and there exist open neighborhoods U of $\bar{x}$, V of $\bar{v}$, together with a C^1-smooth function $\varphi : X_0 \to \mathbb{R}$, such that

$$\Omega \cap (U \times V) = (\operatorname{epi}\widetilde{\varphi}) \cap (U \times V), \tag{3.7}$$

where $\widetilde{\varphi}(u) := \varphi(u)z$ and $\operatorname{epi}\widetilde{\varphi}$ is defined by (3.4).

Theorem 3.8. *Suppose that for any $\bar{x} \in \partial C$ the set C can be locally represented in the form (3.7) where $\Omega = C$ and the Fréchet second-order subdifferential $\widehat{\partial}^2\varphi(u, \nabla\varphi(u))$ is positive semidefinite at any point on $(u, \nabla\varphi(u)) \in (U \times \mathbb{R}) \cap (\mathrm{gph}\,\varphi)$ in the following weaker sense:*

$$\langle u^*, \widetilde{u}\rangle \geq 0, \quad \forall \widetilde{u} \in X_0 \subset X_0^{**}, \ \forall u^* \in \widehat{\partial}^2\varphi(u, \nabla\varphi(u))(\widetilde{u}). \tag{3.8}$$

Then C is a locally convex set.

Proof. It suffices to prove that C is locally convex around any point $\bar{x} \in \partial C$. Given such a point, we represent C in the form (3.7) where $\Omega = C$. Since (3.7) remains valid if one replaces U and V by smaller neighborhoods of $\bar{u}$ and $\bar{v}$, there is no loss of generality in assuming that U and V are open convex sets of X_0 and Y, respectively. Since the property (3.8) is fulfilled at any $u \in U$, the arguments used in proving Theorem 3.1 in [2] show that the function $\varphi : U \to \mathbb{R}$ is convex. Then equality (3.7), where $\Omega = C$, implies that $C \cap (U \times V)$ is convex. □

3.3. **Sets with nonsmooth boundaries.** As in [1], a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be *piecewise linear* (or *piecewise affine*) if there exist families $\{P_1, ..., P_k\}$, $\{a_1, ..., a_k\}$, and $\{b_1, ..., b_k\}$ of polyhedral convex sets in $\mathbb{R}^n$, points in $\mathbb{R}^n$, and points in $\mathbb{R}$, respectively, such that $\mathbb{R}^n = \bigcup\limits_{i=1}^{k} P_i$, $\mathrm{int}P_i \cap \mathrm{int}P_j = \emptyset$ for all $i \neq j$, and

$$\varphi(x) = \varphi_i(x) := \langle a_i, x\rangle + b_i \quad \forall x \in P_i, \ \forall i \in \{1, ..., k\}. \tag{3.9}$$

From (3.9) one has $\varphi_i(x) = \varphi_j(x)$ for any $x \in P_i \cap P_j$ and $i, j \in \{1, ..., k\}$.

Similarly, one says [5] that a function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is *piecewise* C^2 if there exist families $\{P_1, ..., P_k\}$, $\{a_1, ..., a_k\}$, and $\{b_1, ..., b_k\}$ of polyhedral convex sets in $\mathbb{R}^n$ and twice continuously differentiable functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$, such that $\mathbb{R}^n = \bigcup\limits_{i=1}^{k} P_i$, $\mathrm{int}P_i \cap \mathrm{int}P_j = \emptyset$ for all $i \neq j$, and

$$\varphi(x) = \varphi_i(x) \quad \forall x \in P_i, \ \forall i \in \{1, ..., k\}. \tag{3.10}$$

Condition (3.10) forces $\varphi_i(x) = \varphi_j(x)$ for any $x \in P_i \cap P_j$ and $i, j \in \{1, ..., k\}$.

From definitions it is clear that a piecewise linear function is a piecewise C^2 function. Therefore, next theorem is an extension of [1, Theorem 4.9].

Theorem 3.9. (See [5, Theorem 3.3]) *Suppose that $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a piecewise C^2 function. Then, φ is convex if and only if*

$$\langle u^*, \widetilde{u}\rangle \geq 0 \quad \forall \widetilde{u} \in \mathbb{R}^n, \ \forall u^* \in \partial^2\varphi(u, \xi)(\widetilde{u}) \ \textit{with} \ (u, \xi) \in \mathrm{gph}\,\partial\varphi.$$

Consider a set C defined by (3.1), where $X = \mathbb{R}^n$ and $\varphi : X \to \mathbb{R}$ is a continuous function. In analogy with Definition 3.7, we have the following.

Definition 3.10. One says that a closed set $\Omega \subset \mathbb{R}^n$ can be *locally represented as the epigraph of a piecewise C^2 function* around $\bar{x} \in \partial\Omega$ if one can decompose $X = \mathbb{R}^n$ into a direct sum $X = X_0 \oplus Y$ with X_0 being a linear subspace and $Y = \mathbb{R}z$ an 1-dimensional space, $\bar{x} = \bar{u} + \bar{v}$ with $\bar{u} \in X_0$ and $\bar{v} \in Y$, and there exist open neighborhoods U of $\bar{x}$, V of $\bar{v}$, together with a piecewise C^2 function $\varphi : X_0 \to \mathbb{R}$, such that

$$\Omega \cap (U \times V) = (\operatorname{epi} \widetilde{\varphi}) \cap (U \times V), \tag{3.11}$$

where $\widetilde{\varphi}(u) := \varphi(u)z$ and $\operatorname{epi} \widetilde{\varphi}$ is defined by (3.4).

Theorem 3.11. *Suppose that for any $\bar{x} \in \partial C$ the set C can be locally represented in the form* (3.11) *where $\Omega = C$ and the limiting second-order subdifferential $\partial^2\varphi(u, \nabla\varphi(u))$ is positive semidefinite at any point on $(u, \nabla\varphi(u)) \in (U\times\mathbb{R})\cap(\operatorname{gph}\varphi)$ in the following sense:*

$$\langle u^*, \widetilde{u}\rangle \geq 0 \quad \forall \widetilde{u} \in \mathbb{R}^n,\ \forall u^* \in \partial^2\varphi(u,\xi)(\widetilde{u}) \ \text{ with } \ (u,\xi) \in (\operatorname{gph} \partial\varphi) \cap (U \times \mathbb{R}^{n-1}).$$

Then C is a locally convex set.

Proof. The proof is similar to the one of Theorem 3.8. The only difference is that we have to use the representation (3.11) instead of the representation (3.7), and the above Theorem 3.9 instead of Theorem 3.1 in [2]. □

Let us consider an illustrative example for Theorem 3.11.

Example 3.12. Take C in the form of a disjoint union of a triangle and a quadrangle in $\mathbb{R}^2$. Then $\psi : \mathbb{R}^2 \to \mathbb{R}$ can be defined by the formula $\psi(x) = \psi_1(x)\psi_2(x)$, where

$$\psi_1(x) = \max\{\langle a_i, x\rangle + \alpha_i \,:\, i = 1, 2, 3\}, \quad \psi_2(x) = \max\{\langle a_i, x\rangle + \alpha_i \,:\, i = 4, \dots, 7\}$$

with $a_i \in \mathbb{R}^2$ and $\alpha_i \in \mathbb{R}$ for $i = 1, \dots, 7$. Note that C can be locally represented as the epigraph of a piecewise linear function around any $\bar{x} \in \partial C$. Since that piecewise linear function is convex, its second-order subdifferentials are positive semidefinite.

3.4. **Local convexity and convexity.** Tietze [26] proved that *a closed and connected set in $\mathbb{R}^n$ which is locally convex is also convex.* By suggesting a new, simpler proof scheme for that theorem, Schoenberger [25, Theorem 2] showed that *a closed and connected set in a Hilbert space which is locally convex is also convex.* The author also stated [25, p. 342] that his extension of Tietze's result is valid for any normed space. The proof given in [25] is based on the notion of δ-convexity, a geometrical lemma valid in Euclidean spaces, and the fact that if a closed locally convex set in a Hilbert space is connected then it is connected by line segments (i.e., any two points of the set can be joined by a polygonal lying wholly in it).

The just cited results of Schoenberger reduce the problem of checking convexity of a closed set in a normed space to checking the set's connectedness and local convexity. The first task is a topological problem (recall that image of a connected set via a continuous mapping is a connected set). The second one can be solved by using second-order subdifferential mappings as we have done in this section.

If one uses first-order subdifferential mappings, then one can verify convexity of a closed set in a Asplund space by checking to the monotonicity of the Fréchet subdifferential or the limiting subdifferential of the indicator function; see [12, Theorem 3.56].

4. Convexity of functions

Let $C \subset X$ be a nonempty closed convex subset in a Banach space, $\varphi : X \to \overline{\mathbb{R}}$ an extended-real-valued function. Sufficient conditions for φ to be convex on C can be given by using second-order subdifferentials of φ. Note that the first assertion of the following theorem does not require that C must have nonempty interior.

Theorem 4.1. *If X is an Asplund space, then each one of the following two conditions is sufficient for the convexity of φ on Ω:*

(i) *φ is a $C^{1,1}$-function on some open set U containing C (i.e., the Fréchet derivative $\nabla\varphi(x)$ exists for every $x \in U$ and the map $\nabla\varphi : U \to X^*$ is locally Lipschitz), and*

$$\langle z, u\rangle \geq 0 \quad \forall u \in C - C \subset X^{**}, \ \ \forall z \in \partial^2_M \varphi(x, \nabla\varphi(x))(u) \ \ \textit{with} \ \ x \in C. \tag{4.1}$$

(ii) *$\operatorname{int} C$ is nonempty, φ is continuous on C, φ is C^1-smooth on $\operatorname{int} C$, and*

$$\langle z, u\rangle \geq 0 \quad \forall u \in \operatorname{int}C - \operatorname{int}C \subset X^{**}, \ \ \forall z \in \widehat{\partial}^2 \varphi(x, \nabla\varphi(x))(u) \ \ \textit{with} \ \ x \in \operatorname{int}C.$$

Proof. We first consider the case where (i) holds. Since we will differentiate φ only on $C \subset U$, there is no loss of generality in assuming that $U = X$. Suppose to the contrary that φ is nonconvex on C. Define the function $\tilde{\varphi} : X \to \mathbb{R} \cup \{+\infty\}$ by putting $\tilde{\varphi}(x) := \varphi(x) + \delta\big(x; C\big)$ for all $x \in X$. As C is a closed convex subset of X and φ is nonconvex on C, $\tilde{\varphi}$ is lower semicontinuous and nonconvex. So, according to [12, Theorem 3.56], $\widehat{\partial}\tilde{\varphi}(\cdot)$ is nonmonotone, that is, there exist $a, b \in X$ and $\tilde{x}^*_a \in \widehat{\partial}\tilde{\varphi}(a)$, $\tilde{x}^*_b \in \widehat{\partial}\tilde{\varphi}(b)$ such that

$$\langle \tilde{x}^*_b - \tilde{x}^*_a, b - a\rangle < 0. \tag{4.2}$$

Since $\operatorname{dom}\tilde{\varphi} = C$, we must have that $a, b \in C$. By the sum rule for the Fréchet subdifferential in [12, Proposition 1.107(i)],

$$\widehat{\partial}\tilde{\varphi}(a) = \nabla\varphi(a) + N(a; C), \quad \widehat{\partial}\tilde{\varphi}(b) = \nabla\varphi(b) + N(b; C).$$

Hence there exist $x^*_a \in N(a; C)$ and $x^*_b \in N(b; C)$ such that $\tilde{x}^*_a = \nabla\varphi(a) + x^*_a$ and $\tilde{x}^*_a = \nabla\varphi(b) + x^*_b$. According to (4.2),

$$\langle \nabla\varphi(b) - \nabla\varphi(a), b - a\rangle + \langle x^*_b - x^*_a, b - a\rangle < 0. \tag{4.3}$$

Due to the convexity of C, the set-valued mapping $N(\cdot; C) : X \rightrightarrows X^*$ is monotone. So $\langle x^*_b - x^*_a, b - a\rangle \geq 0$. Hence, (4.3) yields

$$\langle \nabla\varphi(b) - \nabla\varphi(a), b - a\rangle < 0. \tag{4.4}$$

Consider the function

$$\psi(x) := \langle \nabla\varphi(a) - \nabla\varphi(b), x\rangle + \langle \nabla\varphi(x), b - a\rangle, \quad x \in X.$$

Since $\psi(a) = \psi(b)$ and ψ is continuous on $[a, b]$, there exists $\bar{x} \in (a, b)$ satisfying $\psi(\bar{x}) = \min\limits_{x \in [a,b]} \psi(x)$ or $\psi(\bar{x}) = \max\limits_{x \in [a,b]} \psi(x)$.

Case 1. $\psi(\bar{x}) = \min_{x\in[a,b]} \psi(x)$. Put $\hat{\psi}(x) = \psi(x) + \delta(x;[a,b])$ for all $x \in X$. Obviously, we have $\hat{\psi}(\bar{x}) = \min_{x\in[a,b]} \hat{\psi}(x)$. According to the Fermat rule [12, Proposition 1.114], $0 \in \partial\hat{\psi}(\bar{x})$. Since the function $\langle\nabla\varphi(\cdot), b-a\rangle$ is locally Lipschitz by the assumptions made, according to [12, Theorem 2.33] we have

$$\partial\hat{\psi}(\bar{x}) \subset \partial\psi(\bar{x}) + N(\bar{x};[a,b]) = \nabla\varphi(a) - \nabla\varphi(b) + \partial\langle\nabla\varphi(\cdot), b-a\rangle(\bar{x}) + N(\bar{x};[a,b]);$$

hence

$$0 \in \nabla\varphi(a) - \nabla\varphi(b) + \partial\langle\nabla\varphi(\cdot), b-a\rangle(\bar{x}) + N(\bar{x};[a,b]). \tag{4.5}$$

Since the function $\langle\nabla\varphi(\cdot), b-a\rangle$ is locally Lipschitz around $\bar{x}$, using the scalarization formula in [12, Theorem 1.90] we have

$$\partial\big\langle\nabla\varphi(\cdot), b-a\big\rangle(\bar{x}) = D^*_M\nabla\varphi(\bar{x},\nabla\varphi(\bar{x}))(b-a).$$

By definition of the mixed second-order subdifferential,

$$\partial^2_M\varphi(\bar{x},\nabla\varphi(\bar{x}))(b-a) = D^*_M\nabla\varphi(\bar{x},\nabla\varphi(\bar{x}))(b-a). \tag{4.6}$$

From (4.5)–(4.6) it follows that

$$\nabla\varphi(b) - \nabla\varphi(a) \in \partial^2_M\varphi(\bar{x},\nabla\varphi(\bar{x}))(b-a) + N(\bar{x};[a,b]).$$

Thus, there exists $\hat{x}^* \in N(\bar{x};[a,b])$ satisfying

$$\nabla\varphi(b) - \nabla\varphi(a) - \hat{x}^* \in \partial^2_M\varphi(\bar{x},\nabla\varphi(\bar{x}))(b-a). \tag{4.7}$$

Since $b-a \in C-C$ and $\bar{x} \in C$, by (4.1) and (4.7) we have

$$\langle\nabla\varphi(b) - \nabla\varphi(a) - \hat{x}^*, b-a\rangle \geq 0. \tag{4.8}$$

Since $\bar{x} \in (a,b)$ and $\hat{x}^* \in N(\bar{x};[a,b])$, $\langle\hat{x}^*, b-a\rangle = 0$. From (4.8) it follows that

$$\langle\nabla\varphi(b) - \nabla\varphi(a), b-a\rangle \geq 0.$$

This contradicts (4.4).

Case 2. $\psi(\bar{x}) = \max_{x\in[a,b]} \psi(x)$. Using the same arguments as in Case 1 for

$$\hat{\psi}(x) := -\psi(x) + \delta(x;[a,b]),$$

we obtain $\langle\nabla\varphi(a) - \nabla\varphi(b), a-b\rangle \geq 0$, which contradicts (4.4). Thus φ is a convex function on C.

Now, assume that (ii) is valid. Arguing as in the proof of [2, Theorem 3.1], we can show that φ is convex on $\operatorname{int} C$. Given any pair $x, u \in C$ and a value $t \in (0,1)$, we can find some sequences $\{x^k\} \subset \operatorname{int} C$ and $\{u^k\} \subset \operatorname{int} C$ converging respectively to x and u. Passing the inequality

$$\varphi((1-t)x^k + tu^k) \leq (1-t)\varphi(x^k) + t\varphi(u^k)$$

to the limit as $k \to \infty$, by the continuity of φ on C we obtain

$$\varphi((1-t)x + tu) \leq (1-t)\varphi(x) + t\varphi(u).$$

This establishes the convexity of φ on C.

The proof is complete. $\square$

Finally, let us establish a necessary condition for the convexity of φ on C in the case C is a smooth-boundary convex set in term of limiting second-order subdifferentials of an extended real-valued function.

Theorem 4.2. *Let C be given by* (3.1)*, where X is a Hilbert space. Suppose that* int$C$ *is nonempty, and $\nabla\psi(x) \neq 0$ for every $x \in \partial C$. Let $\varphi : X \to \mathbb{R}$ be a C^2-smooth function which is convex on C. Then, for each $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial(\varphi + \delta_C)$, one has*

$$\langle z, u\rangle \geq 0 \quad \forall z \in \partial^2(\varphi + \delta_C)(\bar{x}, \bar{v})(u), \ \ \forall u \in X = X^{**}.$$

Proof. Suppose that φ is a convex function. Take any $(\bar{x}, \bar{v}) \in \operatorname{gph} \partial(\varphi+\delta_C)$ and $z \in \partial^2(\varphi+\delta_C)(\bar{x}, \bar{v})(u)$ with $u \in X = X^{**}$. Since φ is twice continuously differentiable, by [12, Proposition 1.121] we have

$$\partial^2(\varphi + \delta_C)(\bar{x}, \bar{v})(u) = \nabla^2\varphi(\bar{x})^*(u) + \partial^2\delta_C\big(\bar{x}, \bar{v} - \nabla\varphi(\bar{x})\big)(u).$$

Thus $z - \nabla^2\varphi(x)^*(u) \in \partial^2\delta_C\big(\bar{x}, \bar{v} - \nabla\varphi(\bar{x})\big)(u)$. Since φ is convex on C, according to Proposition 3.2 we have $\langle z - \nabla^2\varphi(\bar{x})^*(u), u\rangle \geq 0$, or, equivalently,

$$\langle z, u\rangle \geq \langle \nabla^2\varphi(\bar{x})^*(u), u\rangle. \tag{4.9}$$

Since $\bar{x} \in C$ and C is a convex set with int$C \neq \emptyset$, there exists a sequence $\{x_k\} \subset$ int$C$ such that $x_k \to \bar{x}$. By the convexity of φ on int$C$, $\langle \nabla^2\varphi(x_k)^*(u), u\rangle \geq 0$ for all k. Taking the limit as $k \to \infty$, we have $\langle \nabla^2\varphi(\bar{x})^*(u), u\rangle \geq 0$. Thus, by (4.9), $\langle z, u\rangle \geq 0$. This finishes the proof. $\square$

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