Linear and Nonlinear Analysis
Volume 5, Number 2, 2019, 201–210
Yokohama Publishers
ISSN 2188-8167

REMARKS ON THE SUCCESSIVE APPROXIMATION OF FIXED POINTS OF QUASI-FIRMLY TYPE NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we present some significant improvements of the recent convergence theorems proved by Song and Li [Math. Commun. 16 (2011), no. 1, 251–264]. Our results are established under weaker assumptions. Moreover, some assumptions as were the case in their results are dropped away.

1. Introduction

Let $E := (E, \|\cdot\|)$ be a real Banach space. For a mapping $T : C \to C$ where C is a nonempty subset of E, we say that an element $x \in C$ is a *fixed point* of T if $x = Tx$. Many problems in mathematics can be reformulated as a problem of finding a fixed point of some suitable mapping. Moreover, if a fixed point of the considered mapping exists, then it is natural to ask whether such a point can be approximated by an iterative scheme.

In this paper, we are interested in the following mappings introduced by Song and Li [7].

Definition 1.1. Let C be a subset of a Banach space E. A mapping $T : C \to C$ is called *quasi-firmly type nonexpansive* if $\mathrm{Fix}(T) := \{p \in C : p = Tp\} \neq \varnothing$ and there exists a real number $k > 0$ such that the following inequality holds for all $(x, p) \in C \times \mathrm{Fix}(T)$:

$$\|Tx - p\|^2 \leq \|x - p\|^2 - k\|x - Tx\|^2.$$

Recall that E satisfies *Opial's condition* [4] if whenever $\{x_n\}$ is a sequence in E which is weakly convergent to $x \in E$ it follows that

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

for all $y \in E$ with $y \neq x$. A mapping $T : C \to C$ with a nonempty fixed point set $\mathrm{Fix}(T)$ is said to satisfies *Senter–Dotson's condition* [6] if there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r > 0$, such that

$$f(d(x, \mathrm{Fix}(T))) \leq \|x - Tx\|$$

for all $x \in C$, where $d(x, \mathrm{Fix}(T)) := \inf\{\|x - p\| : p \in \mathrm{Fix}(T)\}$.

Song and Li [7] proved the weak and strong convergence theorems in the presence of Opial's condition of the spaces or Senter–Dotson's condition of the mapping. The

2010 *Mathematics Subject Classification.* 47H09, 47H10, 47J25.

Key words and phrases. Quasi-firmly type nonexpansive mapping, Ishikawa's iteration, Mann's iteration.

first convergence result (see Theorem 1.2) deals with the Ishikawa's iteration [2] while the second one (see Theorem 1.3) with the Mann's iteration [3].

Theorem 1.2. *Let C be a closed convex subset of a real Banach space E and let $T : C \to C$ be a quasi-firmly type nonexpansive mapping. Suppose that $\{x_n\}$ is a sequence in C iteratively defined as follows: $x_1 \in C$ is arbitrarily chosen and for each $n \geq 1$*

$$\begin{cases} y_n := \beta_n x_n + (1-\beta_n)Tx_n; \\ x_{n+1} := \alpha_n x_n + (1-\alpha_n)Ty_n; \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that

$$\limsup_{n\to\infty} \alpha_n < 1 \text{ and } \limsup_{n\to\infty} \beta_n < 1.$$

Then the following statements are true.

(a) *If E is a reflexive space satisfying Opial's condition and $I-T$ is demiclosed at zero* [1], *that is, $p \in \mathrm{Fix}(T)$ whenever $\{z_n\}$ is a sequence in C such that $\{z_n\}$ converges weakly to $p \in C$ and $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$, then $\{x_n\}$ converges weakly to a fixed point of T (see* [7, Theorem 1]*).*
(b) *If T is continuous and satisfies Senter–Dotson's condition, then $\{x_n\}$ converges strongly to a fixed point of T (see* [7, Theorem 3]*).*
(c) *If C is compact and T is continuous, then $\{x_n\}$ converges strongly to a fixed point of T (see* [7, Theorem 6]*).*
(d) *Suppose that T is continuous. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, \mathrm{Fix}(T)) = 0$ (see* [7, Theorem 5]*).*

Theorem 1.3. *Let C be a closed convex subset of a real Banach space E and let $T : C \to C$ be a quasi-firmly type nonexpansive mapping. Suppose that $\{x_n\}$ is a sequence in C iteratively defined as follows: $x_1 \in C$ is arbitrarily chosen and for each $n \geq 1$*

$$x_{n+1} := \alpha_n x_n + (1-\alpha_n)Tx_n;$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Then the following statements are true.

(a) *If E is a reflexive space satisfying Opial's condition and $I-T$ is demiclosed at zero, then $\{x_n\}$ converges weakly to a fixed point of T (see* [7, Theorem 2]*).*
(b) *If T is continuous and satisfies Senter–Dotson's condition, then $\{x_n\}$ converges strongly to a fixed point of T (see* [7, Theorem 4]*).*
(c) *If C is compact and T is continuous, then $\{x_n\}$ converges strongly to a fixed point of T (see* [7, Theorem 7]*).*
(d) *Suppose that T is continuous. Then $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, \mathrm{Fix}(T)) = 0$ (see* [7, Theorem 5]*).*

We observe that the Ishikawa's iteration includes Mann's iteration as a special case if $\beta_n = 1$ for all $n \geq 1$. However, because of the condition $\limsup_{n\to\infty} \beta_n < 1$, we cannot obtain Theorem 1.3 as a special case of Theorem 1.2. It is the purpose of this paper to present an improvement of Theorem 1.2 so that the results for the Mann's iteration in Theorem 1.3 can be deduced. Moreover, we show that the results are established with a new condition which significantly improves the one proved by Song and Li [7]. In fact, some assumptions assumed in Song and Li's results are superfluous.

2. Main results

We first present the following observation which is a key result of this paper.

Lemma 2.1. *Let E be a real Banach space and let C be a closed convex subset of E. Let $T : C \to C$ be a quasi-firmly type nonexpansive mapping. Suppose that $\{x_n\}$ is a sequence in C iteratively defined as follows: $x_1 \in C$ is arbitrarily chosen and for each $n \geq 1$*

$$\begin{cases} y_n := \beta_n x_n + (1-\beta_n)Tx_n; \\ x_{n+1} := \alpha_n x_n + (1-\alpha_n)Ty_n; \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. Then the following statements are true.

(a) $\|y_n - p\| \leq \|x_n - p\|$ *and* $\|x_{n+1} - p\| \leq \alpha_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\|$ *for all* $n \geq 1$ *and for all* $p \in \mathrm{Fix}(T)$.

(b) $\sum_{n=1}^{\infty}(1-\alpha_n)\|y_n - Ty_n\|^2 < \infty$ *and* $\sum_{n=1}^{\infty}(1-\alpha_n)\|x_n - y_n\|^2 < \infty$.

Proof. Since T is a quasi-firmly type nonexpansive mapping, there exists a constant $k > 0$ such that the following inequality holds for all $(x,p) \in C \times \mathrm{Fix}(T)$:

$$\|Tx - p\|^2 \leq \|x - p\|^2 - k\|x - Tx\|^2.$$

Let $p \in \mathrm{Fix}(T)$. Then

$$\begin{aligned} \|y_n - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)\|Tx_n - p\|^2 \\ &\leq \|x_n - p\|^2 - k(1-\beta_n)\|x_n - Tx_n\|^2 \end{aligned}$$

and hence

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|Ty_n - p\|^2 \\ &\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2 - k(1-\alpha_n)\|y_n - Ty_n\|^2 \\ &\leq \|x_n - p\|^2 - k(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2 - k(1-\alpha_n)\|y_n - Ty_n\|^2. \end{aligned}$$

It follows that (a) holds. In particular, $\{\|x_n - p\|^2\}$ is a nonincreasing sequence and hence it is convergent. Moreover, we have

$$k(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2 + k(1-\alpha_n)\|y_n - Ty_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2.$$

For each $m \geq 1$, we have

$$\begin{aligned} &k\sum_{n=1}^{m}(1-\alpha_n)(1-\beta_n)\|x_n - Tx_n\|^2 + k\sum_{n=1}^{m}(1-\alpha_n)\|y_n - Ty_n\|^2 \\ &\leq \|x_1 - p\|^2 - \|x_{m+1} - p\|^2 \leq \|x_1 - p\|^2. \end{aligned}$$

In particular, $\sum_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n)\|x_n-Tx_n\|^2<\infty$ and $\sum_{n=1}^{\infty}(1-\alpha_n)\|y_n-Ty_n\|^2<\infty$. Finally, we observe that

$$\begin{aligned}\|x_n-y_n\|^2&=\Big((1-\beta_n)\|x_n-Tx_n\|\Big)^2\\&=\Big(\beta_n\cdot 0+(1-\beta_n)\|x_n-Tx_n\|\Big)^2\\&\le(1-\beta_n)\|x_n-Tx_n\|^2.\end{aligned}$$

This implies that

$$\sum_{n=1}^{\infty}(1-\alpha_n)\|x_n-y_n\|^2=\sum_{n=1}^{\infty}(1-\alpha_n)(1-\beta_n)\|x_n-Tx_n\|^2<\infty.$$

Hence (b) holds and the proof is completed. □

2.1. **Weak convergence theorems.** We observe the following lemma which plays an important role in this subsection.

Lemma 2.2. *Suppose that $\{s_n\}$ and $\{t_n\}$ are two sequences of nonnegative real numbers and $\{\alpha_n\}$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty}s_n=s$ and $\limsup_{n\to\infty}\alpha_n<1$. If $s_{n+1}\le\alpha_n s_n+(1-\alpha_n)t_n$ for all $n\ge 1$, then $\liminf_{n\to\infty}t_n\ge s$.*

Proof. Let $\alpha:=\frac{1}{2}(1+\limsup_{n\to\infty}\alpha_n)$ and let $\varepsilon>0$ be given. Since $\lim_{n\to\infty}s_n=s\ge 0$ and $\limsup_{n\to\infty}\alpha_n<1$, there exists a natural number N such that

- $s-\varepsilon<s_n<s+\varepsilon$;
- $\alpha_n<\alpha<1$;

for all $n\ge N$. In particular, if $n\ge N$, then $1-\alpha_n>1-\alpha>0$ and

$$s-\varepsilon<s_{n+1}\le\alpha_n s_n+(1-\alpha_n)t_n<\alpha_n(s+\varepsilon)+(1-\alpha_n)t_n.$$

This implies that

$$t_n>s-\frac{(1+\alpha_n)\varepsilon}{1-\alpha_n}>s-\frac{2\varepsilon}{1-\alpha}.$$

Hence $\liminf_{n\to\infty}t_n\ge s$. □

We now present the following improvement of Theorem 1.2. It is worth mentioning that no assumption on $\{\beta_n\}$ is required. In particular, we immediately obtain the result for the Mann's iteration.

Theorem 2.3. *Let E be a reflexive real Banach space satisfying Opial's condition. Let C be a closed convex subset of E and $T:C\to C$ be a quasi-firmly type nonexpansive mapping such that $I-T$ is demiclosed at zero. Suppose that $\{x_n\}$ is a sequence in C iteratively defined as follows: $x_1\in C$ is arbitrarily chosen and for each $n\ge 1$*

$$\begin{cases} y_n:=\beta_n x_n+(1-\beta_n)Tx_n;\\ x_{n+1}:=\alpha_n x_n+(1-\alpha_n)Ty_n;\end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that $\limsup_{n\to\infty}\alpha_n<1$. Then $\{x_n\}$ converges weakly to a fixed point of T.

Proof. The proof is divided into three steps.

Step 1: $\lim_{n\to\infty} \|y_n - p\|$ exists for all $p \in \mathrm{Fix}(T)$. To see this, let $p \in \mathrm{Fix}(T)$ and let $s_n := \|x_n - p\|$ and $t_n := \|y_n - p\|$ for all $n \geq 1$. It follows from Lemma 2.1 that $s := \lim_{n\to\infty} s_n$ exists, $\limsup_{n\to\infty} t_n \leq s$, and $s_{n+1} \leq \alpha_n s_n + (1-\alpha_n)t_n$ for all $n \geq 1$. It follows from Lemma 2.2 that $\liminf_{n\to\infty} t_n \geq s$ and hence $\lim_{n\to\infty} t_n = s$.

Step 2: The weak cluster point set of the sequence $\{y_n\}$ is a singleton. To see this, it follows from Step 1 that $\{y_n\}$ is bounded. Moreover, it follows from $\limsup_{n\to\infty} \alpha_n < 1$ and Lemma 2.1(b) that $\lim_{n\to\infty} \|y_n - Ty_n\| = 0$. Since E is reflexive, $\{y_n\}$ has a weakly convergent subsequence. We now suppose that $\{y_{n_k}\}$ and $\{y_{m_j}\}$ are two subsequences of $\{y_n\}$ such that they converge weakly to p and to q, respectively. Since $I - T$ is demiclosed at zero, we have $p, q \in \mathrm{Fix}(T)$. It follows from Step 1 that $\lim_{n\to\infty} \|y_n - p\|$ and $\lim_{n\to\infty} \|y_n - q\|$ both exist. We show that $p = q$. Otherwise, it follows from Opial's condition that

$$\begin{aligned}\lim_{n\to\infty} \|y_n - p\| &= \lim_{k\to\infty} \|y_{n_k} - p\| < \lim_{k\to\infty} \|y_{n_k} - q\| = \lim_{n\to\infty} \|y_n - q\| \\ &= \lim_{j\to\infty} \|y_{m_j} - q\| < \lim_{j\to\infty} \|y_{m_j} - p\| = \lim_{n\to\infty} \|y_n - p\|.\end{aligned}$$

This is a contradiction.

Step 3: The sequence $\{x_n\}$ converges weakly to a fixed point of T. To see this, we note from Step 2 that $\{y_n\}$ converges weakly to a fixed point of T. It follows from Lemma 2.1(b) and $\limsup_{n\to\infty} \alpha_n < 1$ that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Hence the statement follows. □

Remark 2.4. Our Theorem 2.3 improves Theorem 1 of [7] (see Theorem 1.2(a) in this paper) because the assumption $\limsup_{n\to\infty} \beta_n < 1$ is dropped. If we let $\beta_n = 1$ for all $n \geq 1$, then our Theorem 2.3 is the same as Theorem 2 of [7] (see Theorem 1.3(a) of this paper).

2.2. **Strong convergence theorems.** First, we note that the continuity of the mapping T is superfluous for the closedness of $\mathrm{Fix}(T)$ when T is a quasi-firmly type nonexpansive mapping. In fact, we have the following result.

Proposition 2.5. *Let C be a closed subset of a real Banach space E. If $T : C \to C$ is a quasi-firmly type nonexpansive mapping, then* $\mathrm{Fix}(T)$ *is closed.*

Proof. Suppose that $\{p_n\}$ is a sequence in $\mathrm{Fix}(T)$ such that $\lim_{n\to\infty} p_n = p$ for some $p \in E$. Since C is closed, we have $p \in C$. Moreover, since T is quasi-firmly type nonexpansive, there exists a constant $k > 0$ such that

$$\|Tp - p_n\|^2 \leq \|p - p_n\|^2 - k\|p - Tp\|^2$$

for all $n \geq 1$. In particular, since $\lim_{n\to\infty} \|Tp - p_n\|^2 = \|Tp - p\|^2$ and $\lim_{n\to\infty} \|p - p_n\|^2 = 0$, we have $\|p - Tp\|^2 = 0$, that is, $p \in \mathrm{Fix}(T)$. Hence $\mathrm{Fix}(T)$ is closed. □

The following lemma is known in the literature (for example, see [9]).

Lemma 2.6. *Suppose that $\{s_n\}$ and $\{t_n\}$ are two sequences of nonnegative real numbers. If $\sum_{n=1}^{\infty} s_n t_n < \infty$ and $\sum_{n=1}^{\infty} s_n = \infty$, then* $\liminf_{n\to\infty} t_n = 0$.

Proof. We give the proof for the sake of completeness. Let $t := \liminf_{n\to\infty} t_n \geq 0$ and let $\varepsilon > 0$ be given. Then there exists a natural number N such that $t_n \geq t - \varepsilon$ for all $n \geq N$. This implies that $\sum_{n=N}^{m} s_n t_n \geq (t-\varepsilon)\sum_{n=N}^{m} s_n$ for all $m \geq N$. Note that $\lim_{m\to\infty} \sum_{n=N}^{m} s_n t_n < \infty$ and $\lim_{m\to\infty} \sum_{n=N}^{m} s_n = \infty$. This implies that $t - \varepsilon \leq 0$, that is, $t \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $t = 0$. This completes the proof. $\square$

We now present the following improvement of strong convergence theorem of Song and Li. It is worth mentioning that no assumption on $\{\beta_n\}$ is required. In particular, we immediately obtain the result for the Mann's iteration.

Theorem 2.7. *Let E be a real Banach space. Let C be a closed convex subset of E and $T : C \to C$ be a quasi-firmly type nonexpansive mapping. Suppose that $\{x_n\}$ is a sequence in C iteratively defined as follows: $x_1 \in C$ is arbitrarily chosen and for each $n \geq 1$*

$$\begin{cases} y_n := \beta_n x_n + (1-\beta_n)Tx_n; \\ x_{n+1} := \alpha_n x_n + (1-\alpha_n)Ty_n; \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. The following statements are true.

(a) *The sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, \mathrm{Fix}(T)) = 0.$$

(b) *If $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and T satisfies Senter–Dotson's condition, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. (a) ($\Rightarrow$) is trivial. We prove ($\Leftarrow$). The proof method given below is modified from [5]. We assume that $\liminf_{n\to\infty} d(x_n, \mathrm{Fix}(T)) = 0$. It follows from Lemma 2.1(a) that $d(x_{n+1}, \mathrm{Fix}(T)) \leq d(x_n, \mathrm{Fix}(T))$ for all $n \geq 1$ and hence $\lim_{n\to\infty} d(x_n, \mathrm{Fix}(T))$ exists. This implies that $\lim_{n\to\infty} d(x_n \, \mathrm{Fix}(T)) = 0$. We now prove that $\{x_n\}$ is a Cauchy sequence. To see this, let $p \in \mathrm{Fix}(T)$. It follows that

$$\|x_n - x_{n+k}\| \leq \|x_n - p\| + \|x_{n+k} - p\| \leq 2\|x_n - p\|.$$

Since $p \in \mathrm{Fix}(T)$ is arbitrary, we have

$$\|x_n - x_{n+k}\| \leq 2d(x_n, \mathrm{Fix}(T)).$$

It follows from $\lim_{n\to\infty} d(x_n \, \mathrm{Fix}(T)) = 0$ that $\{x_n\}$ is a Cauchy sequence. It follows from the closedness of C that $\lim_{n\to\infty} x_n = q$ for some $q \in C$. In particular, $d(q, \mathrm{Fix}(T)) = 0$. Since $\mathrm{Fix}(T)$ is closed by Proposition 2.5, we have $q \in \mathrm{Fix}(T)$. Hence the statement (a) is proved.

(b) We assume that $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and T satisfies Senter–Dotson's condition. To prove the statement, it suffices to prove that $\liminf_{n\to\infty} d(x_n, \mathrm{Fix}(T)) = 0$. Since T satisfies Senter–Dotson's condition, there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r > 0$, such that

$$f(d(y_n, \mathrm{Fix}(T))) \leq \|y_n - Ty_n\|$$

for all $n \geq 1$. It follows from Lemma 2.1(b) that

$$\sum_{n=1}^{\infty}(1-\alpha_n)(\|y_n - Ty_n\|^2 + \|x_n - y_n\|^2) < \infty.$$

It follows from Lemma 2.6 that

$$\liminf_{n\to\infty}(\|y_n - Ty_n\|^2 + \|x_n - y_n\|^2) = 0.$$

In particular, there exist a strictly increasing sequence $\{n_k\}$ of natural numbers such that

$$\lim_{k\to\infty}(\|y_{n_k} - Ty_{n_k}\| + \|x_{n_k} - y_{n_k}\|) = 0.$$

This implies that

$$\lim_{k\to\infty} f(d(y_{n_k}, \mathrm{Fix}(T))) \leq \lim_{k\to\infty}\|y_{n_k} - Ty_{n_k}\| = 0.$$

Since f is nondecreasing, we have

$$\lim_{k\to\infty} d(y_{n_k}, \mathrm{Fix}(T)) = 0.$$

Since $\lim_{k\to\infty}\|x_{n_k} - y_{n_k}\| = 0$, we have

$$\liminf_{n\to\infty} d(x_n, \mathrm{Fix}(T)) \leq \lim_{k\to\infty} d(x_{n_k}, \mathrm{Fix}(T)) = 0.$$

This completes the proof. □

Remark 2.8. Our Theorem 2.7 improves the strong convergence results of Song and Li in the following ways.

(a) The assumptions $\limsup_{n\to\infty}\alpha_n < 1$ and $\limsup_{n\to\infty}\beta_n < 1$ in Theorem 3 and Theorem 5 of [7] (see Theorem 1.2(b) and (c) of this paper) are replaced by the more general assumption $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$.

(b) The continuity of the mapping is not assumed as were the cases in Theorem 3 and Theorem 5 of [7] (see Theorem 1.2(b) and (c) of this paper).

(c) If we let $\beta_n = 1$ for all $n \geq 1$, then our Theorem 2.7 improves Theorem 4 and Theorem 5 of [7] (see Theorem 1.3(b) and (c) of this paper).

3. A further generalized form of quasi-firmly type nonexpansive mappings

In this section, we use Xu's characterization of uniform convexity in terms of an inequality [10]. Recall that a real Banach space X is *uniformly convex* (see [8, 11]) if

$$\delta_X(\varepsilon) := \inf\left\{1 - \frac{1}{2}\|x+y\| : \|x\| = \|y\| = 1 \text{ and } \|x-y\| = \varepsilon\right\} > 0$$

for all $0 < \varepsilon \leq 2$. The following result is Theorem 2 of [10].

Lemma 3.1. *Let r be a positive real number. A Banach space X is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in X$ with $\|x\|, \|y\| \le r$ and for all $\lambda \in [0, 1]$.

Inspired by the preceding result, we now introduce the following mappings.

Definition 3.2. Let C be a subset of a Banach space E. A mapping $T : C \to C$ is called *generalized quasi-firmly type nonexpansive* if $\mathrm{Fix}(T) \neq \varnothing$ and for each $r > 0$ there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$\|Tx - p\|^2 \le \|x - p\|^2 - g(\|x - Tx\|)$$

for all $(x, p) \in C \times \mathrm{Fix}(T)$ with $\|x\| \le r$ and $\|p\| \le r$.

Remark 3.3. If $T : C \to C$ is quasi-firmly type nonexpansive, then it is generalized quasi-firmly type nonexpansive.

The following result shows that every averaged mapping in a uniformly convex Banach space is generalized quasi-firmly type nonexpansive.

Theorem 3.4. *Let X be a uniformly convex Banach space and C be a convex subset of X. Suppose that $T : C \to C$ is a quasi-nonexpansive mapping and $S := \lambda I + (1-\lambda)T$ where I is an identity mapping and $\lambda \in (0, 1)$. Then S is generalized quasi-firmly type nonexpansive and* $\mathrm{Fix}(S) = \mathrm{Fix}(T)$.

Proof. Obviously, $\mathrm{Fix}(S) = \mathrm{Fix}(T)$ and hence $\mathrm{Fix}(S) \neq \varnothing$. To show that S is generalized quasi-firmly type nonexpansive, let $r > 0$. It follows from Lemma 3.1 that there exists a continuous, strictly increasing, and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and

$$\|\lambda x + (1-\lambda)y\|^2 \le \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $x, y \in X$ with $\|x\|, \|y\| \le r$ and for all $\lambda \in [0, 1]$. Let $(x, p) \in C \times \mathrm{Fix}(T)$ such that $\|x\| \le r$ and $\|p\| \le r$. Then

$$\begin{aligned} \|Sx - p\|^2 &= \|\lambda(x-p) + (1-\lambda)(Tx-p)\|^2 \\ &\le \lambda\|x-p\|^2 + (1-\lambda)\|Tx-p\|^2 - \lambda(1-\lambda)g(\|x-Tx\|) \\ &\le \|x-p\|^2 - h(\|x-Tx\|) \end{aligned}$$

where $h(t) := \lambda(1-\lambda)g(t)$. It is clear that h is a continuous, strictly increasing, and convex function such that $h(0) = 0$. This completes the proof. □

We obtain the following convergence result whose proof follows exactly the same as the proof of the corresponding results in the preceding section.

Theorem 3.5. *Let C be a closed convex subset of a Banach space X and $T : C \to C$ be a generalized quasi-firmly type nonexpansive mapping. Suppose that $\{x_n\}$ is a sequence in C iteratively defined as follows: $x_1 \in C$ is arbitrarily chosen and for each $n \geq 1$*

$$\begin{cases} y_n := \beta_n x_n + (1-\beta_n)Tx_n; \\ x_{n+1} := \alpha_n x_n + (1-\alpha_n)Ty_n; \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$. The following statements are true.

(1) *If E is reflexive and satisfies Opial's condition; $I - T$ is demiclosed at zero; and $\limsup_{n\to\infty} \alpha_n < 1$, then $\{x_n\}$ converges weakly to a fixed point of T*
(2) *If $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ and T satisfies Senter–Dotson's condition, then $\{x_n\}$ converges strongly to a fixed point of T.*

Acknowledgements

The first author's work is supported by the Science Achievement Scholarship of Thailand. The second author's work is supported by the Thailand Research Fund and Khon Kaen University under grant number RSA6280002.

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Manuscript received 10 December 2018
revised 5 May 2019

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