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SOME INEQUALITIES ON WEIGHTED MEANS AND TRACES DEFINED ON SECOND-ORDER CONE

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Abstract. In this paper, we set up the concepts of some weighted means associated with second-order cone. Then, we achieve a few inequalities on the new-extended weighted means and their corresponding traces associated with second-order cone. As a byproduct, a version of Powers-Størmer's inequality is established.

1. Introduction

During the past few decades, many well-known means defined on positive numbers have been generalized to the setting of positive semidefinite matrices. For instance, the arithmetic mean $A(a,b)$, harmonic mean $H(a,b)$, geometric mean $G(a,b)$, logarithm mean $L(a,b)$, see [5, 7, 8, 9]. In particular, the matrix version of Arithmetic-Geometric-Mean Inequality (AGM) is proved in [1, 5, 6], and has attracted much attention. In addition, Lim [23] generalizes the geometric mean from the cone of positive semidefinite matrices to the symmetric cone setting. Some applications are established in [22, 24] accordingly. Recently, Chang et al. [10] also define some well-known means on the second-order cone (SOC for short) and build up several means inequalities under the partial order induced by SOC, denoted by $\mathcal{K}^n$.

In this paper, we are interested in the weighted means and their induced inequalities associated with SOC. More specifically, we set up the concepts of some weighted means in the SOC setting. Then, we achieve a few inequalities on the new-extended weighted means and their corresponding traces associated with second-order cone. As a byproduct, a version of Powers-Størmer's inequality is established. Indeed, for real numbers, there exits a diagraph regarding the weighted means and the weighted Arithmetic-Geometric-Mean inequality, see Figure 1. The direction of arrow in Figure 1 represents the ordered relationship. In Section 3, we shall define these weighted means in the setting of second-order cone and build up the relationship among these SOC weighted means. Moreover, we establish the Heinz mean

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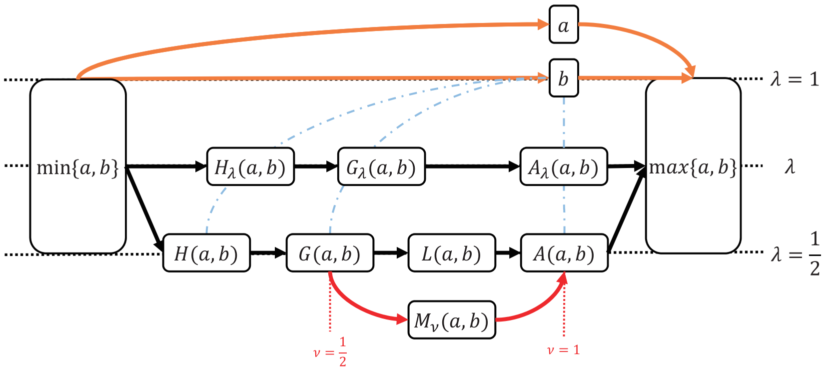

FIGURE 1. Relationship between means defined on real number.

$M_\nu(x,y)$ and the ordered relationship as well. In Section 4, we achieve a version of Powers-Størmer's inequality [17, 18] in the SOC setting.

2. Preliminary

In this section, we briefly review some background materials regarding the second-order cone, also known as Lorentz cone. The second-order cone (SOC for short) in $\mathbb{R}^n$, is defined by

$$\mathcal{K}^n = \left\{ x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid \|x_2\| \le x_1 \right\}.$$

For $n = 1$, $\mathcal{K}^n$ denotes the set of nonnegative real number $\mathbb{R}_+$. For any x, y in $\mathbb{R}^n$, we write $x \succeq_{\mathcal{K}^n} y$ if $x - y \in \mathcal{K}^n$ and write $x \succ_{\mathcal{K}^n} y$ if $x - y \in \text{int}(\mathcal{K}^n)$. In other words, we have $x \succeq_{\mathcal{K}^n} 0$ if and only if $x \in \mathcal{K}^n$ and $x \succ_{\mathcal{K}^n} 0$ if and only if $x \in \text{int}(\mathcal{K}^n)$. The relation $\succeq_{\mathcal{K}^n}$ is a partial ordering but not a linear ordering in $\mathcal{K}^n$, i.e., there exist $x, y \in \mathcal{K}^n$ such that neither $x \succeq_{\mathcal{K}^n} y$ nor $y \succeq_{\mathcal{K}^n} x$. To see this, for $n = 2$, let $x = (1, 1)$ and $y = (1, 0)$, we have $x - y = (0, 1) \notin \mathcal{K}^n$, $y - x = (0, -1) \notin \mathcal{K}^n$.

For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, we define their *Jordan product* as

$$x \circ y = (x^T y \ , \ y_1 x_2 + x_1 y_2).$$

We write x^2 to mean $x \circ x$ and write $x + y$ to mean the usual componentwise addition of vectors. Then, $\circ, +$, together with $e' = (1, 0, \ldots, 0)^T \in \mathbb{R}^n$ and for any $x, y, z \in \mathbb{R}^n$, the following basic properties [14, 15] hold: (1) $e' \circ x = x$, (2) $x \circ y = y \circ x$, (3) $x \circ (x^2 \circ y) = x^2 \circ (x \circ y)$, (4) $(x + y) \circ z = x \circ z + y \circ z$. Notice that the Jordan product is *not associative* in general. However, it is power associative, i.e., $x \circ (x \circ x) = (x \circ x) \circ x$ for all $x \in \mathbb{R}^n$. Thus, we may, without loss of ambiguity, write x^m for the product of m copies of x and $x^{m+n} = x^m \circ x^n$ for all positive integers m and n. Here, we set $x^0 = e'$. Besides, $\mathcal{K}^n$ is *not closed* under Jordan product.

For any $x \in \mathcal{K}^n$, it is known that there exists a unique vector in $\mathcal{K}^n$ denoted by $x^{1/2}$ such that $(x^{1/2})^2 = x^{1/2} \circ x^{1/2} = x$. Indeed,

$$x^{1/2} = \left(s, \frac{x_2}{2s}\right), \quad \text{where } s = \sqrt{\frac{1}{2}\left(x_1 + \sqrt{x_1^2 - \|x_2\|^2}\right)}.$$

In the above formula, the term x_2/s is defined to be the zero vector if $x_2 = 0$ and $s = 0$, i.e., $x = 0$. For any $x \in \mathbb{R}^n$, we always have $x^2 \in \mathcal{K}^n$, i.e., $x^2 \succeq_{\mathcal{K}^n} 0$. Hence, there exists a unique vector $(x^2)^{1/2} \in \mathcal{K}^n$ denoted by $|x|$. It is easy to verify that $|x| \succeq_{\mathcal{K}^n} 0$ and $x^2 = |x|^2$ for any $x \in \mathbb{R}^n$. It is also known that $|x| \succeq_{\mathcal{K}^n} x$. For any $x \in \mathbb{R}^n$, we define $[x]_+$ to be the nearest point (in Euclidean norm, since Jordan product does not induce a norm) projection of x onto $\mathcal{K}^n$, which is the same definition as in $\mathbb{R}^n_+$. In other words, $[x]_+$ is the optimal solution of the parametric SOCP: $[x]_+ = \arg\min\{\|x - y\| \mid y \in \mathcal{K}^n\}$. In addition, it can be verified that $[x]_+ = (x + |x|)/2$; see [14, 15].

Recently, there has found many optimization problems involved second-order cones in real world applications. For dealing with second-order cone programs (SOCP) and second-order cone complementarity problems (SOCCP), there needs *spectral decomposition* associated with SOC [13]. More specifically, for any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the vector x can be decomposed as

$$x = \lambda_1 u_x^{(1)} + \lambda_2 u_x^{(2)}, \tag{2.1}$$

where λ_1, λ_2 and $u_x^{(1)}, u_x^{(2)}$ are the spectral values and the associated spectral vectors of x, respectively, given by

$$\lambda_i = x_1 + (-1)^i \|x_2\|, \tag{2.2}$$

$$u_x^{(i)} = \begin{cases} \frac{1}{2}(1, (-1)^i \frac{x_2}{\|x_2\|}) & \text{if} \quad x_2 \neq 0, \\ \frac{1}{2}(1, (-1)^i w) & \text{if} \quad x_2 = 0. \end{cases} \tag{2.3}$$

for $i = 1, 2$ with w being any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\| = 1$. If $x_2 \neq 0$, the decomposition is unique. Accordingly, the determinant, the trace, and the Euclidean norm of x can all be represented in terms of λ_1 and λ_2:

$$\det(x) = \lambda_1\lambda_2, \quad \mathbf{tr}(x) = \lambda_1 + \lambda_2, \quad \|x\|^2 = \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2\right).$$

From the simple calculation, we especially point out that $\mathbf{tr}(x) = 2x_1$, which we frequently use in the following paragraphs.

For any function $f : \mathbb{R} \to \mathbb{R}$, the following vector-valued function associated with $\mathcal{K}^n$ $(n \geq 1)$ was considered in [11, 12]:

$$f^{\text{soc}}(x) = f(\lambda_1)u_x^{(1)} + f(\lambda_2)u_x^{(2)}, \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}. \tag{2.4}$$

If f is defined only on a subset of $\mathbb{R}$, then f^{soc} is defined on the corresponding subset of $\mathbb{R}^n$. The definition (2.4) is unambiguous whether $x_2 \neq 0$ or $x_2 = 0$. The cases of $f^{\rm soc}(x) = x^{1/2}$, x^2, $\exp(x)$ are discussed in [14]. For subsequent analysis, we will frequently use the vector-valued functions corresponding to t^p $(t > 0, p > 0)$. In particular, they can be expressed as

$$x^p = \lambda_1^p u_x^{(1)} + \lambda_2^p u_x^{(2)}, \quad \forall x \in \mathcal{K}^n.$$

The spectral decomposition along with the Jordan algebra associated with SOC entails some basic properties as listed in the following text. We omit the proofs since they can be found in [11, 14, 15].

Lemma 2.1. *For any $x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$ with spectral decomposition (2.1)-(2.3), there have*

(a) $|x| = (x^2)^{1/2} = |\lambda_1| u_x^{(1)} + |\lambda_2| u_x^{(2)}$;

(b) $[x]_+ = [\lambda_1]_+ u_x^{(1)} + [\lambda_2]_+ u_x^{(2)} = \frac{1}{2}(x + |x|)$.

(c) $x \succeq_{\mathcal{K}^n} 0 \iff \langle x, y\rangle \geq 0,\ \forall y \succeq_{\mathcal{K}^n} 0$.

(d) *If $x \succeq_{\mathcal{K}^n} y$, then $\lambda_i(x) \geq \lambda_i(y)$, $\forall i = 1, 2$. Hence, $\mathbf{tr}(x) \geq \mathbf{tr}(y)$. Moreover, if $x \succeq_{\mathcal{K}^n} y \succeq_{\mathcal{K}^n} 0$, then $\det(x) \geq \det(y)$.*

We point out that the relation $\succeq_{\mathcal{K}^n}$ is not a linear ordering. Hence, it is not possible to compare any two vectors (elements) via $\succeq_{\mathcal{K}^n}$. Nonetheless, we note that

$$\begin{aligned} \max\{a, b\} &= b + [a - b]_+ = \frac{1}{2}(a + b + |a - b|), \\ \min\{a, b\} &= a - [a - b]_- = \frac{1}{2}(a + b - |a - b|). \end{aligned}$$

This motivates us to define supremum and infimum of $\{x, y\}$, denoted by $x \vee y$ and $x \wedge y$ respectively, in the setting of second-order cone as follows. For any $x, y \in \mathbb{R}^n$,

$$\begin{aligned} x \vee y &:= y + [x - y]_+ = \frac{1}{2}(x + y + |x - y|), \\ x \wedge y &:= \begin{cases} x - [x - y]_+ = \frac{1}{2}(x + y - |x - y|), & \text{if } x + y \succeq_{\mathcal{K}^n} |x - y|; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Next, we recall the concepts of SOC-monotone and SOC-convex functions which are introduced in [11] and will be needed for subsequent analysis. For a real valued function $f : \mathbb{R} \to \mathbb{R}$, f is said to be *SOC-monotone* of order n if its corresponding vector-valued function $f^{\rm soc}$ defined as in (2.4) satisfies

$$x \succeq_{\mathcal{K}^n} y \implies f^{\rm soc}(x) \succeq_{\mathcal{K}^n} f^{\rm soc}(y).$$

The function f is said to be SOC-monotone if f is SOC-monotone of all order n. The function f is said to be *SOC-convex* of order n if its corresponding vector-valued function $f^{\rm soc}$ defined as in (2.4) satisfies

$$f^{\rm soc}((1 - \lambda)x + \lambda y) \preceq_{\mathcal{K}^n} (1 - \lambda) f^{\rm soc}(x) + \lambda f^{\rm soc}(y)$$

for all $x, y \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$. The function f is said to be SOC-convex if f is SOC-convex of all order n.

The concepts of SOC-monotone and SOC-convex functions are analogous to matrix-monotone and matrix-convex functions [4, 16], and can be viewed special cases of operator monotone and operator convex functions [3, 9, 20]. Examples and characterizations of SOC-monotone and SOC-convex functions are given in [11, 12].

Lemma 2.2. ([11, Proposition 3.3]) *Let* $f : (0, \infty) \to (0, \infty)$ *be* $f(t) = 1/t$. *Then,*

(a) $-f$ *is SOC-monotone on* $(0, \infty)$*;*
(b) f *is SOC-convex on* $(0, \infty)$.

Lemma 2.3. ([11, Proposition 3.7]) *Let* $f : [0, \infty) \to [0, \infty)$ *be* $f(t) = t^r$, $0 \le r \le 1$. *Then,*

(a) f *is SOC-monotone on* $[0, \infty)$*;*
(b) $-f$ *is SOC-convex on* $[0, \infty)$.

Let V be a Euclidean Jordan algebra, $\mathcal{K}$ be the set of all square elements of V (the associated symmetric cone in V), and $\Omega := \mathrm{int}\mathcal{K}$ (the interior symmetric cone). For $x \in V$, let $\mathcal{L}(x)$ denote the linear operator given by $\mathcal{L}(x)y := x \circ y$, and let $P(x) := 2\mathcal{L}(x)^2 - \mathcal{L}(x^2)$. The mapping P is called the *quadratic representation* of V. If x is invertible, then we have

$$P(x)\mathcal{K} = \mathcal{K} \quad \text{and} \quad P(x)\Omega = \Omega.$$

Here, we list some properties of the mapping P whose proofs can be found in Proposition II.3.1, Proposition II.3.2, and Proposition II.3.4 of [14] and Lemma 2.3 of [22].

Proposition 2.4. *Let* V *be a Jordan algebra with an identity element* e.

(a) *An element* x *is invertible if and only if* $P(x)$ *is invertible. In this case,* $P(x)x^{-1} = x$ *and* $P(x)^{-1} = P(x^{-1})$.
(b) *If* x *and* y *are invertible, then* $P(x)y$ *is invertible and* $(P(x)y)^{-1} = P(x^{-1})y^{-1}$.
(c) *For any elements* x *and* y, $P(P(x)y) = P(x)P(y)P(x)$. *In particular,* $P(x^2) = P(x)^2$.
(d) *For any elements* $x, y \in \overline{\Omega}$, $x \preceq y$ *if and only if* $P(x) \preceq P(y)$, *which means* $P(y) - P(x)$ *is a positive semidefinite matrix.*

3. Inequality for weighted mean

As mentioned in Section 1, some inequalities for SOC means have been established in [10]. In this section, we set up more general SOC means, i.e., the weighted means associated with SOC including SOC weighted arithmetic mean, SOC weighted geometric mean, SOC weighted harmonic mean, and SOC weighted Heinz mean. A few inequalities based on these means are established accordingly.

In the setting of second-order cone, we call a binary operation $(x, y) \mapsto M(x, y)$ defined on $\mathrm{int}(\mathcal{K}^n) \times \mathrm{int}(\mathcal{K}^n)$ an SOC mean if the following are satisfied:

(i) $M(x,y) \succ_{\mathcal{K}^n} 0$;
(ii) $x \wedge y \preceq_{\mathcal{K}^n} M(x,y) \preceq_{\mathcal{K}^n} x \vee y$;
(iii) $M(x,y)$ is monotone in x, y;
(iv) $M(\alpha x, \alpha y) = \alpha M(x,y)$, $\alpha > 0$;
(v) $M(x,y)$ is continuous in x, y.

As introduced in [10], the SOC arithmetic mean $A(x,y) : \mathrm{int}(\mathcal{K}^n) \times \mathrm{int}(\mathcal{K}^n) \to \mathrm{int}(\mathcal{K}^n)$ is defined by

$$A(x,y) = \frac{x+y}{2};$$

the SOC harmonic mean $H(x,y) : \mathrm{int}(\mathcal{K}^n) \times \mathrm{int}(\mathcal{K}^n) \to \mathrm{int}(\mathcal{K}^n)$ is defined as

$$H(x,y) = \left(\frac{x^{-1} + y^{-1}}{2} \right)^{-1};$$

whereas the geometric mean $G(x,y)$ is defined as

$$G(x,y) := P(x^{\frac{1}{2}}) \left(P(x^{-\frac{1}{2}}) y \right)^{\frac{1}{2}}.$$

Compared with the definition in [10], we omit the symmetric property, i.e., the condition $M(x,y) = M(y,x)$ is dropped out, since the SOC weighted mean is not symmetric in general. Now, we consider the SOC weighted means as below. For $0 \leq \lambda \leq 1$, we let

$$A_\lambda(x,y) := (1-\lambda)x + \lambda y \tag{3.1}$$

$$H_\lambda(x,y) := \left((1-\lambda)x^{-1} + \lambda y^{-1} \right)^{-1} \tag{3.2}$$

$$G_\lambda(x,y) := P\left(x^{\frac{1}{2}}\right) \left(P(x^{-\frac{1}{2}}) y \right)^{\lambda}, \tag{3.3}$$

denote the SOC weighted arithmetic mean, the SOC weighted harmonic mean, and the SOC weighted geometric mean, respectively. According to the definition, it is clear that

$$\begin{aligned} A_{1-\lambda}(x,y) &= A_\lambda(y,x) \\ H_{1-\lambda}(x,y) &= H_\lambda(y,x) \\ G_{1-\lambda}(x,y) &= G_\lambda(y,x) \end{aligned}$$

We note that when $\lambda = 1/2$, these SOC weighted means coincide with the SOC arithmetic mean $A(x,y)$, the SOC harmonic mean $H(x,y)$, and the SOC geometric mean $G(x,y)$, respectively. For more details, please refer to [10].

To achieve inequalities for the aforementioned SOC weighted means, we need the following technical lemma [10, Lemma 4], which is indeed a symmetric cone version of Bernoulli inequality.

Lemma 3.1. *Let V be a Euclidean Jordan algebra, $\mathcal{K}$ be the associated symmetric cone, and e be the Jordan identity. Then,*

$$(e+s)^t \preceq_{\mathcal{K}} e + ts,$$

where $0 \le t \le 1$*,* $s \succeq_{\mathcal{K}} -e$*, and the partial order is induced by the closed convex cone* $\mathcal{K}$*.*

Theorem 3.2. *Suppose* $0 \le \lambda \le 1$*. Let* $A_\lambda(x,y)$*,* $H_\lambda(x,y)$*, and* $G_\lambda(x,y)$ *be defined as in (3.1), (3.2), and (3.3), respectively. Then, for any* $x \succ_{\mathcal{K}^n} 0$ *and* $y \succ_{\mathcal{K}^n} 0$*, there holds*

$$x \wedge y \preceq_{\mathcal{K}^n} H_\lambda(x,y) \preceq_{\mathcal{K}^n} G_\lambda(x,y) \preceq_{\mathcal{K}^n} A_\lambda(x,y) \preceq_{\mathcal{K}^n} x \vee y.$$

Proof. (i) To verify the first inequality, we discuss two cases. For $\frac{1}{2}(x+y-|x-y|) \notin \mathcal{K}^n$, the inequality holds automatically. For $\frac{1}{2}(x+y-|x-y|) \in \mathcal{K}^n$, we note that $\frac{1}{2}(x+y-|x-y|) \preceq_{\mathcal{K}^n} x$ and $\frac{1}{2}(x+y-|x-y|) \preceq_{\mathcal{K}^n} y$. Then, using the SOC-monotonicity of $f(t) = -t^{-1}$ (Lemma 2.3), we obtain

$$x^{-1} \preceq_{\mathcal{K}^n} \left(\frac{x+y-|x-y|}{2}\right)^{-1} \quad \text{and} \quad y^{-1} \preceq_{\mathcal{K}^n} \left(\frac{x+y-|x-y|}{2}\right)^{-1},$$

which imply

$$(1-\lambda)x^{-1} + \lambda y^{-1} \preceq_{\mathcal{K}^n} \left(\frac{x+y-|x-y|}{2}\right)^{-1}.$$

Next, applying the SOC-monotonicity again to this inequality, we conclude that

$$\frac{x+y-|x-y|}{2} \preceq_{\mathcal{K}^n} \left((1-\lambda)x^{-1} + \lambda y^{-1}\right)^{-1}.$$

(ii) For the second and third inequalities, it suffices to verify the third inequality (the second one can be deduced thereafter). Let $s = P(x^{-\frac{1}{2}})y - e$, which gives $s \succeq_{\mathcal{K}^n} -e$. Then, applying Lemma 3.1 yields

$$\left(e + P(x^{-\frac{1}{2}})y - e\right)^\lambda \preceq_{\mathcal{K}^n} e + \lambda\left[P(x^{-\frac{1}{2}})y - e\right],$$

which is equivalent to

$$0 \preceq_{\mathcal{K}^n} (1-\lambda)e + \lambda\left[P(x^{-\frac{1}{2}})y\right] - \left(P(x^{-\frac{1}{2}})y\right)^\lambda.$$

Since $P(x^{\frac{1}{2}})$ is invariant on $\mathcal{K}^n$, we have

$$\begin{aligned} 0 \quad &\preceq_{\mathcal{K}^n} \quad P(x^{\frac{1}{2}})\left((1-\lambda)e + \lambda\left[P(x^{-\frac{1}{2}})y\right] - \left(P(x^{-\frac{1}{2}})y\right)^\lambda\right) \\ &= \quad (1-\lambda)x + \lambda y - P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^\lambda, \end{aligned}$$

and hence

$$P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^\lambda \preceq_{\mathcal{K}^n} (1-\lambda)x + \lambda y. \tag{3.4}$$

For the second inequality, replacing x and y in (3.4) by x^{-1} and y^{-1}, respectively, gives

$$P(x^{-\frac{1}{2}})\left(P(x^{\frac{1}{2}})y^{-1}\right)^\lambda \preceq_{\mathcal{K}^n} (1-\lambda)x^{-1} + \lambda y^{-1}.$$

Using the SOC-monotonicity again, we conclude

$$\left((1-\lambda)x^{-1}+\lambda y^{-1}\right)^{-1} \preceq_{\mathcal{K}^n} \left(P(x^{-\frac{1}{2}})\left(P(x^{\frac{1}{2}})y^{-1}\right)^{\lambda}\right)^{-1} = P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^{\lambda}.$$

where the equality holds by Proposition 2.4(b).

(iii) To see the last inequality, we observe that $x \preceq_{\mathcal{K}^n} \frac{1}{2}(x+y+|x-y|)$ and $y \preceq_{\mathcal{K}^n} \frac{1}{2}(x+y+|x-y|)$, which imply

$$(1-\lambda)x+\lambda y \preceq_{\mathcal{K}^n} \frac{x+y+|x-y|}{2}.$$

Then, the desired result follows. □

In [19], Huang et al. established three SOC trace versions of Young inequalities. Based on Theorem 3.2, we have the following SOC determinant version of Young inequality.

Theorem 3.3. (Young inequality) *For any $x \succ_{\mathcal{K}^n} 0$ and $y \succ_{\mathcal{K}^n} 0$, there holds*

$$\det(x\circ y) \le \det\left(\frac{x^p}{p}+\frac{y^q}{q}\right)$$

where $1<p,q<\infty$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$.

Proof. Since $\frac{x^p}{p}+\frac{y^q}{q} \succeq_{\mathcal{K}^n} G_{\frac{1}{q}}(x^p,y^q) = P(x^{\frac{p}{2}})\left(P(x^{-\frac{p}{2}})y^q\right)^{\frac{1}{q}}$, and hence

$$\det\left(\frac{x^p}{p}+\frac{y^q}{q}\right) \ge \det\left(P(x^{\frac{p}{2}})\left(P(x^{-\frac{p}{2}})y^q\right)^{\frac{1}{q}}\right) = \det(x)\det(y) \ge \det(x\circ y)$$

by Proposition III.4.2 of [14] and Proposition 2.2(b) of [11]. □

Now, we consider the family of Heinz means

$$M_\nu(a,b) := \frac{a^\nu b^{1-\nu}+a^{1-\nu}b^\nu}{2}$$

for $a,b>0$ and $0\le\nu\le 1$. Following the idea of Kubo-Ando extension in [21], the SOC Heinz mean can be defined as

$$M_\nu(x,y) := \frac{G_\nu(x,y)+G_\nu(y,x)}{2} \tag{3.5}$$

where $x,y \succ_{\mathcal{K}^n} 0$ and $0\le\nu\le 1$. We point out that an obvious "naive" extension could be

$$B_\nu(x,y) := \frac{x^\nu\circ y^{1-\nu}+x^{1-\nu}\circ y^\nu}{2}. \tag{3.6}$$

Unfortunately, B_ν may not always satisfy the definition of SOC mean. Although it is not an SOC mean, we still are interested in seeking the trace or norm inequality about B_ν and other SOC means, and it will be discussed in the next section.

For any positive numbers a, b, it is well-known that

$$\sqrt{ab} \le M_\nu(a,b) \le \frac{a+b}{2}. \tag{3.7}$$

Together with the proof of Theorem 3.2, we can obtain the following inequality accordingly.

Theorem 3.4. *Suppose* $0 \le \nu \le 1$ *and* $\lambda = \frac{1}{2}$. *Let* $A_{\frac{1}{2}}(x,y)$, $G_{\frac{1}{2}}(x,y)$, *and* $M_\nu(x,y)$ *be defined as in (3.1), (3.3), and (3.5), respectively. Then, for any* $x \succ_{\mathcal{K}^n} 0$ *and* $y \succ_{\mathcal{K}^n} 0$, *there holds*

$$G_{\frac{1}{2}}(x,y) \preceq_{\mathcal{K}^n} M_\nu(x,y) \preceq_{\mathcal{K}^n} A_{\frac{1}{2}}(x,y).$$

Proof. Consider $x \succ_{\mathcal{K}^n} 0$, $y \succ_{\mathcal{K}^n} 0$ and $0 \le \nu \le 1$, from Theorem 3.2, we have

$$\begin{aligned} M_\nu(x,y) &= \frac{G_\nu(x,y) + G_\nu(y,x)}{2} \\ &\preceq_{\mathcal{K}^n} \frac{A_\nu(x,y) + A_\nu(y,x)}{2} \\ &= A_{\frac{1}{2}}(x,y). \end{aligned}$$

On the other hand, we note that

$$\begin{aligned} M_\nu(x,y) &= \frac{G_\nu(x,y) + G_{1-\nu}(x,y)}{2} \\ &= \frac{P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^{\nu} + P(x^{\frac{1}{2}})\left(P(x^{-\frac{1}{2}})y\right)^{1-\nu}}{2} \\ &= P(x^{\frac{1}{2}})\left(\frac{\left(P(x^{-\frac{1}{2}})y\right)^{\nu} + \left(P(x^{-\frac{1}{2}})y\right)^{1-\nu}}{2}\right) \\ &\succeq_{\mathcal{K}^n} P(x^{\frac{1}{2}})\left(\left(P(x^{-\frac{1}{2}})y\right)^{\frac{\nu}{2}} \circ \left(P(x^{-\frac{1}{2}})y\right)^{\frac{1-\nu}{2}}\right) \\ &= G_{\frac{1}{2}}(x,y) \end{aligned}$$

where the inequality holds due to the fact $\frac{u+v}{2} \succeq_{\mathcal{K}^n} u^{\frac{1}{2}} \circ v^{\frac{1}{2}}$ for any $u, v \in \mathcal{K}^n$ and the invariant property of $P(x^{\frac{1}{2}})$ on $\mathcal{K}^n$. ☐

Over all, we could have a picture regarding the ordered relationship of these SOC weighted means as depicted in Figure 2.

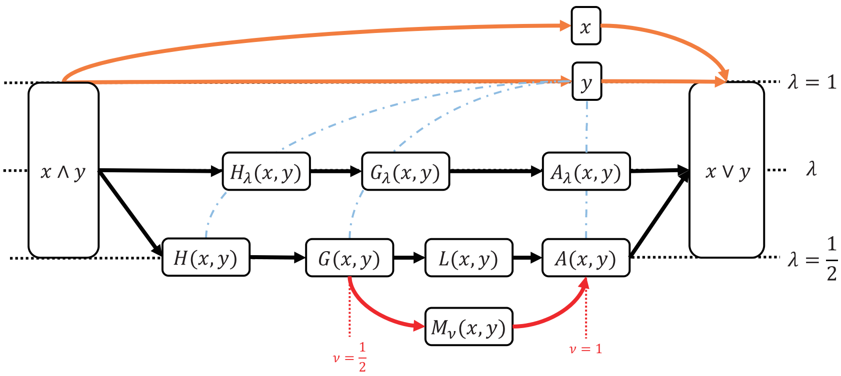

FIGURE 2. Relationship between means defined on second-order cone.

4. Inequalities for trace of SOC weighted means

Up to now, we have extended the weighted harmonic mean, weighted geometric mean, weighted Heinz mean, and weighted arithmetic mean to second-order cone setting. In this section, we explore some other inequalities associated with traces of these SOC weighted means. First, by applying Lemma 2.1(d), we immediately obtain the following trace inequalities for SOC weighted means.

Theorem 4.1. *Suppose* $0 \leq \lambda \leq 1$. *Let* $A_\lambda(x, y)$, $H_\lambda(x, y)$, *and* $G_\lambda(x, y)$ *be defined as in (3.1), (3.2), and (3.3), respectively. For, any* $x \succ_{\mathcal{K}^n} 0$ *and* $y \succ_{\mathcal{K}^n} 0$, *there holds*

$$\mathbf{tr}(x \wedge y) \leq \mathbf{tr}(H_\lambda(x, y)) \leq \mathbf{tr}(G_\lambda(x, y)) \leq \mathbf{tr}(A_\lambda(x, y)) \leq \mathbf{tr}(x \vee y).$$

Theorem 4.2. *Suppose* $0 \leq \nu \leq 1$ *and* $\lambda = \frac{1}{2}$. *Let* $A_{\frac{1}{2}}(x, y)$, $H_{\frac{1}{2}}(x, y)$, $G_{\frac{1}{2}}(x, y)$, *and* $M_\nu(x, y)$ *be defined as in (3.1), (3.2), (3.3), and (3.5), respectively. Then, for any* $x \succ_{\mathcal{K}^n} 0$ *and* $y \succ_{\mathcal{K}^n} 0$, *there holds*

$$\mathbf{tr}(x \wedge y) \leq \mathbf{tr}(H_{\frac{1}{2}}(x, y)) \leq \mathbf{tr}(G_{\frac{1}{2}}(x, y)) \leq \mathbf{tr}(M_\nu(x, y)) \leq \mathbf{tr}(A_{\frac{1}{2}}(x, y)) \leq \mathbf{tr}(x \vee y).$$

As mentioned earlier, there are some well-known means, like Heinz mean

$$M_\nu(a, b) = \frac{a^\nu b^{1-\nu} + a^{1-\nu} b^\nu}{2} \quad \text{for } 0 < \nu < 1$$

which cannot serve as SOC means albeit it is a natural extension. Even though they are not SOC means, it is still possible to deriving some trace or norm inequality about these means.

Next, we pay attention to another special inequality. The Powers-Størmer's inequality asserts that for $s \in [0, 1]$ the following inequality

$$2\mathbf{Tr}\left(A^s B^{1-s}\right) \geq \mathbf{Tr}\left(A + B - |A - B|\right).$$

holds for any pair of positive definite matrices A, B. This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory [2]. In [17, 18], Hou, Osaka and Tomiyama investigate the generalized Powers-Størmer inequality. More specifically, for any positive matrices A, B and matrix-concave function f, they prove that

$$\mathbf{Tr}(A) + \mathbf{Tr}(B) - \mathbf{Tr}(|A - B|) \leq 2\mathbf{Tr}\left(f(A)^{\frac{1}{2}} g(B) f(A)^{\frac{1}{2}}\right).$$

where $g(t) = \begin{cases} \frac{t}{f(t)}, & t \in (0, +\infty) \\ 0, & t = 0 \end{cases}$. Moreover, Hou et al. also show that the Powers-Størmer's inequality characterizes the trace property for a normal linear positive functional on a von Neumann algebras and for a linear positive functional on a C^*-algebra. Motivated by the above facts, we establish a version of the Powers-Størmer's inequality for SOC-monotone function on $[0, +\infty)$ in the SOC setting. To proceed, we need a technical lemma.

Lemma 4.3. *For* $0 \preceq_{\mathcal{K}^n} u \preceq_{\mathcal{K}^n} x$ *and* $0 \preceq_{\mathcal{K}^n} v \preceq_{\mathcal{K}^n} y$, *there holds*

$$0 \leq \mathbf{tr}(u \circ v) \leq \mathbf{tr}(x \circ y).$$

Proof. Suppose $0 \preceq_{\mathcal{K}^n} u \preceq_{\mathcal{K}^n} x$ and $0 \preceq_{\mathcal{K}^n} v \preceq_{\mathcal{K}^n} y$, we have

$$\begin{aligned} & \mathbf{tr}(x \circ y) - \mathbf{tr}(u \circ v) \\ = \ & \mathbf{tr}(x \circ y - u \circ v) \\ = \ & \mathbf{tr}(x \circ y - x \circ v + x \circ v - u \circ v) \\ = \ & \mathbf{tr}(x \circ (y - v) + (x - u) \circ v) \\ = \ & \mathbf{tr}(x \circ (y - v)) + \mathbf{tr}((x - u) \circ v) \\ \geq \ & 0, \end{aligned}$$

where the inequality holds by Lemma 2.1(c). □

Recall that the Jordan product is not associative in general. However, it would be associative after taking trace.

Proposition 4.4. *For any* $x, y, z \in \mathbb{R}^n$, *there holds* $\mathbf{tr}((x \circ y) \circ z) = \mathbf{tr}(x \circ (y \circ z))$.

Proof. From direct computation, we have $x \circ y = \left(x_1 y_1 + x_2^T y_2, x_1 y_2 + y_1 x_2\right)$ and

$$\mathbf{tr}((x \circ y) \circ z) = 2\left(x_1 y_1 z_1 + z_1 x_2^T y_2 + x_1 y_2^T z_2 + y_1 x_2^T z_2\right).$$

Similarly, we also have $y \circ z = \left(y_1 z_1 + y_2^T z_2, y_1 z_2 + z_1 y_2\right)$ and

$$\mathbf{tr}(x \circ (y \circ z)) = 2\left(x_1 y_1 z_1 + x_1 y_2^T z_2 + y_1 x_2^T z_2 + z_1 x_2^T y_2\right).$$

Therefore, we conclude the desired result. □

According to the proof in [17, 18], the crucial point is under what conditions of $f(t)$, there holds the SOC-monotonicity of $\frac{t}{f(t)}$. For establishing the SOC version of Powers-Størmer's inequality, it is also a key, which is answered in next proposition.

Proposition 4.5. *Let f be a strictly positive, continuous function on $[0, +\infty)$. The function $g(t) := \dfrac{t}{f(t)}$ is SOC-monotone if one of the following conditions holds.*

(a) *f is matrix-monotone of order 4.*
(b) *f is matrix-concave of order 3.*
(c) *For any contraction $T : \mathcal{K}^n \mapsto \mathcal{K}^n$ and $z \in \mathcal{K}^n$, there holds*

$$f^{\rm soc}(Tz) \succeq_{\mathcal{K}^n} Tf^{\rm soc}(z). \tag{4.1}$$

Proof. (a) According to [17, Proposition 2.1], the 4-matrix-monotonicity of f would imply the 2-matrix-monotonicity of g, which coincides with the SOC-monotonicity by [12, Corollary 3.1].

(b) From [18, Theorem 2.1], the 3-matrix-concavity of f implies the 2-matrix-monotonicity of g, which coincides with the SOC-monotonicity as well.

(c) Suppose $0 \preceq_{\mathcal{K}^n} x \preceq_{\mathcal{K}^n} y$, we have $P(x) \preceq P(y)$ by Proposition 2.4(d). This together with [4, Lemma V.1.7] implies $\|P(x^{\frac{1}{2}})P(y^{-\frac{1}{2}})\| \leq 1$. Hence, $P(x^{\frac{1}{2}})P(y^{-\frac{1}{2}})$ is an contraction. Then

$$\begin{aligned}
& x = P(x^{\frac{1}{2}})(P(y^{-\frac{1}{2}})y) \\
\Longrightarrow \quad & f^{\rm soc}(x) = f^{\rm soc}(P(x^{\frac{1}{2}})(P(y^{-\frac{1}{2}})y)) \\
\Longrightarrow \quad & f^{\rm soc}(x) \succeq_{\mathcal{K}^n} P(x^{\frac{1}{2}})(P(y^{-\frac{1}{2}})f^{\rm soc}(y)) \\
\Longleftrightarrow \quad & P(x^{-\frac{1}{2}})f^{\rm soc}(x) \succeq_{\mathcal{K}^n} P(y^{-\frac{1}{2}})f^{\rm soc}(y) \\
\Longleftrightarrow \quad & x^{-1} \circ f^{\rm soc}(x) \succeq_{\mathcal{K}^n} y^{-1} \circ f^{\rm soc}(y) \\
\Longleftrightarrow \quad & x \circ (f^{\rm soc}(x))^{-1} \preceq_{\mathcal{K}^n} y \circ (f^{\rm soc}(y))^{-1} \\
\Longleftrightarrow \quad & g^{\rm soc}(x) \preceq_{\mathcal{K}^n} g^{\rm soc}(y).
\end{aligned}$$

where the second implication holds by setting $T = P(x^{\frac{1}{2}})P(y^{-\frac{1}{2}})$ and the first equivalence holds by the invariant property of $P(x^{-\frac{1}{2}})$ on $\mathcal{K}^n$. □

Remark 4.6. We elaborate more about Proposition 4.5. We notice that the SOC-monotonicity and SOC-concavity of f are not strong enough to guarantee the SOC-monotonicity of g. Indeed, the SOC-monotonicity and SOC-concavity only coincides with the 2-matrix-monotonicity and 2-matrix-concavity, respectively. Hence, we need stronger condition on f to assure the SOC-monotonicity of g. Another point to mention is that the condition (4.1) in Proposition 4.5(c) is a similar idea for SOC setting parallel to the following condition:

$$C^* f(A) C \preceq f(C^* A C) \tag{4.2}$$

for any positive semidefinite A and a contraction C in the space of matrices. This inequality (4.2) plays a key role in proving matrix-monotonicity and matrix-convexity. For more details about this condition, please refer to [17, 18]. To the contrast, it is not clear about how to define $(\cdot)^*$ associated with SOC. Nonetheless, we figure out that the condition (4.1) may act as a role like (4.2).

Theorem 4.7. *Let $f : [0, +\infty) \longrightarrow (0, +\infty)$ be SOC-monotone and satisfy one of the conditions in Proposition 4.5. Then, for any $x, y \in \mathcal{K}^n$, there holds*

$$\mathbf{tr}(x+y) - \mathbf{tr}(|x-y|) \leq 2\mathbf{tr}\left(f^{\rm soc}(x)^{\frac{1}{2}} \circ g^{\rm soc}(y) \circ f^{\rm soc}(x)^{\frac{1}{2}}\right), \tag{4.3}$$

where $g(t) = \frac{t}{f(t)}$ if $t > 0$, and $g(0) = 0$.

Proof. For any $x, y \in \mathcal{K}^n$, it is known that $x-y$ can be expressed as $[x-y]_+ - [x-y]_-$. Let us denote by $\mathbf{p} := [x-y]_+$ and $\mathbf{q} := [x-y]_-$. Then we have

$$x - y = \mathbf{p} - \mathbf{q} \quad \text{and} \quad |x-y| = \mathbf{p} + \mathbf{q}$$

and the inequality (4.3) is equivalent to the following

$$\mathbf{tr}(x) - \mathbf{tr}\left(f^{\rm soc}(x)^{\frac{1}{2}} \circ g^{\rm soc}(y) \circ f^{\rm soc}(x)^{\frac{1}{2}}\right) \leq \mathbf{tr}(\mathbf{p}).$$

Since $y + \mathbf{p} \succeq_{\mathcal{K}^n} y \succeq_{\mathcal{K}^n} 0$ and $y + \mathbf{p} = x + \mathbf{q} \succeq_{\mathcal{K}^n} x \succeq_{\mathcal{K}^n} 0$, we have $g^{\rm soc}(x) \preceq_{\mathcal{K}^n} g^{\rm soc}(y + \mathbf{p})$ and by Proposition 4.4

$$\begin{aligned}
& \mathbf{tr}(x) - \mathbf{tr}\left(f^{\rm soc}(x)^{\frac{1}{2}} \circ g^{\rm soc}(y) \circ f^{\rm soc}(x)^{\frac{1}{2}}\right) \\
= \;& \mathbf{tr}\left(f^{\rm soc}(x)^{\frac{1}{2}} \circ g^{\rm soc}(x) \circ f^{\rm soc}(x)^{\frac{1}{2}}\right) - \mathbf{tr}\left(f^{\rm soc}(x)^{\frac{1}{2}} \circ g^{\rm soc}(y) \circ f^{\rm soc}(x)^{\frac{1}{2}}\right) \\
\leq \;& \mathbf{tr}\left(f^{\rm soc}(x)^{\frac{1}{2}} \circ g^{\rm soc}(y+\mathbf{p}) \circ f^{\rm soc}(x)^{\frac{1}{2}}\right) - \mathbf{tr}\left(f^{\rm soc}(x)^{\frac{1}{2}} \circ g^{\rm soc}(y) \circ f^{\rm soc}(x)^{\frac{1}{2}}\right) \\
= \;& \mathbf{tr}\left(f^{\rm soc}(x)^{\frac{1}{2}} \circ (g^{\rm soc}(y+\mathbf{p}) - g^{\rm soc}(y)) \circ f^{\rm soc}(x)^{\frac{1}{2}}\right) \\
\leq \;& \mathbf{tr}\left(f^{\rm soc}(y+\mathbf{p})^{\frac{1}{2}} \circ (g^{\rm soc}(y+\mathbf{p}) - g^{\rm soc}(y)) \circ f^{\rm soc}(y+\mathbf{p})^{\frac{1}{2}}\right) \\
= \;& \mathbf{tr}\left(f^{\rm soc}(y+\mathbf{p})^{\frac{1}{2}} \circ g^{\rm soc}(y+\mathbf{p}) \circ f^{\rm soc}(y+\mathbf{p})^{\frac{1}{2}}\right) \\
& -\mathbf{tr}\left(f^{\rm soc}(y+\mathbf{p})^{\frac{1}{2}} \circ g^{\rm soc}(y) \circ f^{\rm soc}(y+\mathbf{p})^{\frac{1}{2}}\right) \\
\leq \;& \mathbf{tr}(y+\mathbf{p}) - \mathbf{tr}\left(f^{\rm soc}(y)^{\frac{1}{2}} \circ g^{\rm soc}(y) \circ f^{\rm soc}(y)^{\frac{1}{2}}\right) \\
= \;& \mathbf{tr}(y+\mathbf{p}) - \mathbf{tr}(y) \\
= \;& \mathbf{tr}(\mathbf{p}).
\end{aligned}$$

Hence, we prove the assertion. □

As an application we get the SOC version of Powers-Størmer's inequality.

Theorem 4.8. *For any $x, y \in \mathcal{K}^n$ and $0 \leq \lambda \leq 1$, there holds*

$$\mathbf{tr}\left(x + y - |x-y|\right) \leq 2\mathbf{tr}\left(x^{\lambda} \circ y^{1-\lambda}\right) \leq \mathbf{tr}\left(x + y + |x-y|\right).$$

Proof. (i) For the first inequality, taking $f(t) = t^{\lambda}$ for $0 \leq \lambda \leq 1$ and applying Theorem 4.7. It is known that f is matrix-monotone with $f((0, +\infty)) \subseteq (0, +\infty)$ and $g(t) = \frac{t}{f(t)} = t^{1-\lambda}$. Then, the inequality follows from (4.3) in Theorem 4.7.

(ii) For the second inequality, we note that

$$\begin{aligned} 0 &\preceq_{\mathcal{K}^n} \quad x \preceq_{\mathcal{K}^n} \frac{x+y+|x-y|}{2}, \\ 0 &\preceq_{\mathcal{K}^n} \quad y \preceq_{\mathcal{K}^n} \frac{x+y+|x-y|}{2}. \end{aligned}$$

Moreover, for $0 \leq \lambda \leq 1$, $f(t) = t^\lambda$ is SOC-monotone on $[0, +\infty)$. This implies that

$$\begin{aligned} 0 &\preceq_{\mathcal{K}^n} \quad x^\lambda \preceq_{\mathcal{K}^n} \left(\frac{x+y+|x-y|}{2}\right)^\lambda, \\ 0 &\preceq_{\mathcal{K}^n} \quad y^{1-\lambda} \preceq_{\mathcal{K}^n} \left(\frac{x+y+|x-y|}{2}\right)^{1-\lambda}. \end{aligned}$$

Then, applying Lemma 4.3 gives

$$\mathbf{tr}\left(x^\lambda \circ y^{1-\lambda}\right) \leq \mathbf{tr}\left(\frac{x+y+|x-y|}{2}\right),$$

which is the desired result. □

According to the definition of B_λ, we observe that

$$B_0(x,y) = B_1(x,y) = \frac{x+y}{2} = A_{\frac{1}{2}}(x,y).$$

This together with Theorem 4.7 leads to

$$\mathbf{tr}(x \wedge y) \leq \mathbf{tr}\left(B_\lambda(x,y)\right) \leq \mathbf{tr}(x \vee y).$$

In fact, we can sharpen the upper bound of $\mathbf{tr}(B_\lambda(x,y))$ as shown in the following theorem, which also shows when the maximum occurs. Moreover, the inequality (3.7) remains true for second-order cone, in the following trace version.

Theorem 4.9. *For any $x, y \in \mathcal{K}^n$ and $0 \leq \lambda \leq 1$, there holds*

$$2\mathbf{tr}\left(x^{\frac{1}{2}} \circ y^{\frac{1}{2}}\right) \leq \mathbf{tr}\left(x^\lambda \circ y^{1-\lambda} + x^{1-\lambda} \circ y^\lambda\right) \leq \mathbf{tr}(x+y),$$

which is equivalent to $\mathbf{tr}\left(x^{\frac{1}{2}} \circ y^{\frac{1}{2}}\right) \leq \mathbf{tr}\left(B_\lambda(x,y)\right) \leq \mathbf{tr}(A_{\frac{1}{2}}(x,y))$. *In particular,*

$$\mathbf{tr}\left(x^{1-\lambda} \circ y^\lambda\right) \leq \mathbf{tr}\left(A_\lambda(x,y)\right).$$

Proof. It is clear that the inequalities hold when $\lambda = 0, 1$. Suppose that $\lambda \neq 0, 1$, we set $p = \frac{1}{\lambda}$, $q = \frac{1}{1-\lambda}$.

For the first inequality, we write $x = \xi_1 u_x^{(1)} + \xi_2 u_x^{(2)}$, $y = \mu_1 u_y^{(1)} + \mu_2 u_y^{(2)}$ by spectral decomposition (2.1)-(2.3). We note that $\xi_i, \mu_j \geq 0$ and $u_x^{(i)}, u_y^{(j)} \in \mathcal{K}^n$ for all $i, j = 1, 2$. Then

$$x^\lambda \circ y^{1-\lambda} + x^{1-\lambda} \circ y^\lambda - 2x^{\frac{1}{2}} \circ y^{\frac{1}{2}} = \sum_{i,j=1}^{2} \left(\xi_i^\lambda \mu_j^{1-\lambda} + \xi_i^{1-\lambda}\mu_j^\lambda - 2\sqrt{\xi_i \mu_j}\right) u_x^{(i)} \circ u_y^{(j)},$$

which implies

$$\begin{aligned}
& \mathbf{tr}\left(x^{\lambda}\circ y^{1-\lambda}+x^{1-\lambda}\circ y^{\lambda}-2x^{\frac{1}{2}}\circ y^{\frac{1}{2}}\right) \\
= & \ \mathbf{tr}\left(\sum_{i,j=1}^{2}\left(\xi_i^{\lambda}\mu_j^{1-\lambda}+\xi_i^{1-\lambda}\mu_j^{\lambda}-2\sqrt{\xi_i\mu_j}\right)u_x^{(i)}\circ u_y^{(j)}\right) \\
= & \ \sum_{i,j=1}^{2}\mathbf{tr}\left(\left(\xi_i^{\lambda}\mu_j^{1-\lambda}+\xi_i^{1-\lambda}\mu_j^{\lambda}-2\sqrt{\xi_i\mu_j}\right)u_x^{(i)}\circ u_y^{(j)}\right) \\
= & \ \sum_{i,j=1}^{2}\left(\xi_i^{\lambda}\mu_j^{1-\lambda}+\xi_i^{1-\lambda}\mu_j^{\lambda}-2\sqrt{\xi_i\mu_j}\right)\mathbf{tr}\left(u_x^{(i)}\circ u_y^{(j)}\right) \\
\geq & \ 0,
\end{aligned}$$

where the inequality holds by (3.7) and Lemma 2.1(c).

For the second inequality, by the trace version of Young inequality [19], we have

$$\begin{aligned}
\mathbf{tr}\left(x^{\lambda}\circ y^{1-\lambda}\right) &\leq \mathbf{tr}\left(\frac{(x^{\lambda})^p}{p}+\frac{(y^{1-\lambda})^q}{q}\right)=\mathbf{tr}\left(\frac{x}{p}+\frac{y}{q}\right), \\
\mathbf{tr}\left(x^{1-\lambda}\circ y^{\lambda}\right) &\leq \mathbf{tr}\left(\frac{(x^{1-\lambda})^q}{q}+\frac{(y^{\lambda})^p}{p}\right)=\mathbf{tr}\left(\frac{x}{q}+\frac{y}{p}\right).
\end{aligned}$$

Adding up these two inequalities together yields the desired result. □

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