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SURROGATE DUALITY FOR OPTIMIZATION PROBLEMS INVOLVING SET FUNCTIONS

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Abstract. In this paper, we study surrogate duality for optimization problems involving set functions. We investigate the level sets of quasiconvex set functions, and then show surrogate duality for optimization problems involving quasiconvex and convex set functions with the Slater type constraint qualification.

1. Introduction

In convex analysis, various types of functions have been introduced. Convexity and generalized convexity of a real-valued function have been investigated and generalized by various researchers. Vector-valued functions are ones of the generalization for multi-objective optimization problems. Additionally, set-valued functions have been investigated in set optimization problems.

On the other hand, in [20], Morris introduced set functions, which is defined on the class of measurable subsets of an atomless finite measure space satisfying a certain convexity condition. Although a set-valued function is defined on a vector space and the value is a set, a set function is defined on a class of subsets and the value is a real number. For this type of set functions, various results have been introduced, see [3, 4, 10, 11, 14–16, 18, 20, 31]. There are some types of duality results for convex set functions, for example, Lagrange duality in [20], the subdifferential sum formula in [14], and Fenchel-Moreau theorem in [15].

In optimization problems, duality theorems play an important role. It is well known that Lagrange duality for convex optimization have been studied extensively. Additionally, there are so many useful duality results in optimization theory, for example, surrogate duality, Fenchel duality, Mond-Weir duality, Wolfe duality, and so on. However, for set functions, surrogate duality for quasiconvex set functions have not been investigated yet.

In this paper, we study surrogate duality for optimization problems involving set functions. We investigate the level sets of quasiconvex set functions. We show surrogate duality for optimization problems involving quasiconvex and convex set functions with the Slater type constraint qualification.

The paper is organized as follows. In Section 2, we introduce some preliminaries and investigate the level sets of quasiconvex set functions. In Section 3, we show surrogate duality for optimization problems involving quasiconvex and convex

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set functions. By using the separation theorem in the Euclidean space, we show surrogate duality for the problem.

2. Preliminaries

Let $(X, \mathcal{A}, m)$ be an atomless finite measure space with $L_1 := L_1(X, \mathcal{A}, m)$ separable. For $\Omega \in \mathcal{A}$, χ_Ω denotes the characteristic function of Ω. For $\mathcal{S} \subset \mathcal{A}$, we denote $\chi_\mathcal{S} = \{\chi_\Omega \mid \Omega \in \mathcal{S}\}$ and define cl$\mathcal{S}$ is the w^*-closure of $\chi_\mathcal{S}$ in L_∞. In [20], Morris proved that for each Ω, $\Lambda \in \mathcal{A}$ and $\alpha \in [0,1]$, there exist L_∞-sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} (1-\alpha)\chi_{\Omega \setminus \Lambda}, \quad \chi_{\Lambda_n} \xrightarrow{w^*} \alpha \chi_{\Lambda \setminus \Omega},$$

and

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} (1-\alpha)\chi_\Omega + \alpha \chi_\Lambda,$$

consequently, cl$\chi_\mathcal{A}$ contains the convex hull of $\chi_\mathcal{A}$. We call the sequence $\{\Gamma_n = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)\}$ a Morris sequence associated with $(\alpha, \Omega, \Lambda)$.

Definition 2.1. [20] A subfamily $\mathcal{S} \subset \mathcal{A}$ is said to be convex if for every $(\alpha, \Omega, \Lambda) \in [0,1] \times \mathcal{S} \times \mathcal{S}$ and every Morris sequence $\{\Gamma_n\}$ associated with $(\alpha, \Omega, \Lambda)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $\{\Gamma_{n_k}\} \subset \mathcal{S}$.

We show an important property of convex subfamilies.

Theorem 2.2. *Let $\mathcal{S}_1$, $\mathcal{S}_2$ be convex subfamilies of $\mathcal{A}$. Then, $\mathcal{S}_1 \cap \mathcal{S}_2$ is convex.*

Proof. Let $(\alpha, \Omega, \Lambda) \in [0,1] \times \mathcal{S} \times \mathcal{S}$ and $\{\Gamma_n\}$ a Morris sequence associated with $(\alpha, \Omega, \Lambda)$. Since $\mathcal{S}_1$ is convex, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $\{\Gamma_{n_k}\} \subset \mathcal{S}_1$. It is clear that the subsequence $\{\Gamma_{n_k}\}$ is also a Morris sequence $\{\Gamma_n\}$ associated with $(\alpha, \Omega, \Lambda)$. Hence there exists a subsequence $\{\Gamma_{n_{k_i}}\}$ of $\{\Gamma_{n_k}\}$ such that $\{\Gamma_{n_{k_i}}\} \subset \mathcal{S}_2$. Then, $\{\Gamma_{n_{k_i}}\}$ is a subsequence of $\{\Gamma_n\}$ and $\{\Gamma_{n_{k_i}}\} \subset \mathcal{S}_1 \cap \mathcal{S}_2$. This completes the proof. □

We introduce definitions of convex and quasiconvex set functions.

Definition 2.3. [20] Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}$. A set function $F : \mathcal{S} \to \mathbb{R}$ is said to be convex if for every $(\alpha, \Omega, \Lambda) \in [0,1] \times \mathcal{S} \times \mathcal{S}$ and every Morris sequence $\{\Gamma_n\}$ associated with $(\alpha, \Omega, \Lambda)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $\{\Gamma_{n_k}\} \subset \mathcal{S}$ and

$$\limsup_{k \to \infty} F(\Gamma_{n_k}) \le (1-\alpha)F(\Omega) + \alpha F(\Lambda).$$

Definition 2.4. [18] Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}$. A set function $F : \mathcal{S} \to \mathbb{R}$ is said to be quasiconvex if for every $(\alpha, \Omega, \Lambda) \in [0,1] \times \mathcal{S} \times \mathcal{S}$ and every Morris sequence $\{\Gamma_n\}$ associated with $(\alpha, \Omega, \Lambda)$, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $\{\Gamma_{n_k}\} \subset \mathcal{S}$ and

$$\limsup_{k \to \infty} F(\Gamma_{n_k}) \le \max\{F(\Omega), F(\Lambda)\}.$$

Each convex set function is quasiconvex, but the converse is not generally true. It is clear that the sum of convex set functions, and a scalar multiple of a convex set function are convex. Furthermore, a scalar multiple of a quasiconvex set function are also quasiconvex. However, the sum of quasiconvex set functions is not always quasiconvex.

The following example is important.

Example 2.5. [18, 20] Let u be a real-valued function on $\mathbb{R}^n$, and $v_1, \dots, v_n \in L_1(X, \mathcal{A}, m)$. We define a function F on $\mathcal{A}$ as follows:

$$F(\Omega) = u\left(\int_\Omega v_1 dm, \dots, \int_\Omega v_n dm\right)$$

Then the following statements hold:

(i) If u is convex, then F is a convex set function.
(ii) If u is upper semicontinuous quasiconvex, then F is a quasiconvex set function.

Morris introduced the above example in Example 3.1 of [20], and proved that F is convex if u is convex. Additionally, in Proposition 3.1 of [18], Lin proved that F is quasiconvex if u is quasiconvex.

In convex analysis, convex functions are characterized by their epigraphs. Similarly, convex set functions are characterized by their epigraphs, see [3]. On the other hand, quasiconvex functions are usually characterized by their level sets. We show characterizations of quasiconvex set functions by their level sets. We define level sets of a real-valued set function F with respect to a binary relation $\diamond$ on $\mathbb{R}$ as

$$L(F, \diamond, \beta) := \{\Omega \in \mathcal{A} \mid F(\Omega) \diamond \beta\}$$

for any $\beta \in \mathbb{R}$.

Theorem 2.6. *Let F be a real-valued set function from $\mathcal{A}$. Then (i) and (ii) are equivalent.*

(i) *F is quasiconvex,*
(ii) *for each $\beta \in \mathbb{R}$, $L(F, <, \beta)$ is convex.*

Proof. We show that (i) implies (ii). Let $\beta \in \mathbb{R}$, $(\alpha, \Omega, \Lambda) \in [0,1] \times L(F, <, \beta) \times L(F, <, \beta)$, and $\{\Gamma_n\}$ a Morris sequence associated with $(\alpha, \Omega, \Lambda)$. Since F is quasiconvex, there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that $\{\Gamma_{n_k}\} \subset \mathcal{A}$ and

$$\limsup_{k\to\infty} F(\Gamma_{n_k}) \le \max\{F(\Omega), F(\Lambda)\} < \beta.$$

Hence, there exists a subsequence $\{\Gamma_{n_{k_j}}\}$ of $\{\Gamma_{n_k}\}$ such that $\{\Gamma_{n_{k_j}}\} \subset L(F, <, \beta)$. This shows that $L(F, <, \beta)$ is convex.

Next, we show that (ii) implies (i). Let $(\alpha, \Omega, \Lambda) \in [0,1] \times \mathcal{A} \times \mathcal{A}$, $\{\Gamma_n\}$ a Morris sequence associated with $(\alpha, \Omega, \Lambda)$ and $\beta = \max\{F(\Omega), F(\Lambda)\}$. Then, for each $k \in \mathbb{N}$, $\Omega, \Lambda \in L(F, <, \beta + \frac{1}{k})$. Since $L(F, <, \beta + \frac{1}{k})$ is convex, there exists a subsequence $\{\Gamma^k_{n_m}\}$ of $\{\Gamma_n\}$ such that $\{\Gamma^k_{n_m}\} \subset L(F, <, \beta + \frac{1}{k})$.

Let $\Gamma_{n_1} = \Gamma^1_{n_1}$. For each $k \geq 2$, there exists $\bar{k} \in \mathbb{N}$ such that $\Gamma_{n_{k-1}} = \Gamma_{\bar{k}}$. Put $\Gamma_{n_k} = \Gamma^k_{n_{k_0}}$ satisfying $\hat{k} > \bar{k}$ where $\Gamma^k_{n_{k_0}} = \Gamma_{\hat{k}}$. Then, Γ_{n_k} is a subsequence of Γ_n and for each $k \in \mathbb{N}$, $F(\Gamma_{n_k}) < \beta + \frac{1}{k}$. Hence,

$$\limsup_{k\to\infty} F(\Gamma_{n_k}) \leq \beta = \max\{F(\Omega), F(\Lambda)\}.$$

This shows that F is quasiconvex. □

Remark 2.7. We consider the following statement (iii):

(iii) for each $\beta \in \mathbb{R}$, $L(F, \leq, \beta)$ is convex.

In [18], Lin proved that (iii) implies (i) in Theorem 2.6. In [16], Lee proved that (i) implies (ii), and (iii) implies (i). However, the other implications have not been investigated yet. The following conjecture is important.

(iv) If $\mathcal{S}_k$ be a convex subfamily of $\mathcal{A}$ for each $k \in \mathbb{N}$, then $\bigcap_{k\in\mathbb{N}} \mathcal{S}_k$ is convex.

If the conjecture (iv) is true, then (i), (ii), and (iii) are equivalent since

$$L(F, \leq, \beta) = \bigcap_{k\in\mathbb{N}} L\left(F, <, \beta + \frac{1}{k}\right).$$

Definition 2.8. [11] Let $\mathcal{S}$ be a convex subfamily of $\mathcal{A}$. A set function $F : \mathcal{S} \to \mathbb{R}$ is said to be w^*-upper semicontinuous (w^*-usc) if for every $\Omega \in \mathcal{S}$, and a net $\{\Omega_\alpha\} \subset \mathcal{S}$ satisfying $\chi_{\Omega_\alpha} \xrightarrow{w^*} \chi_\Omega$,

$$F(\Omega) \geq \limsup_{\alpha} F(\Omega_\alpha).$$

Remark 2.9. In [11], Hsia and Lee defined a w^*-usc function as follows: F is said to be w^*-usc if for each $\Omega \in \mathcal{S}$,

$$F(\Omega) = \inf_{U\in N(\Omega)} \sup_{\Omega_0\in U\cap\mathcal{S}} F(\Omega_0),$$

where $N(\Omega_0)$ is the family of all w^*-neighborhoods of Ω. This condition is equivalent to w^*-upper semicontinuity in this paper. On the other hand, in [14, 15], Lai and Lin defined a w^*-usc function by a sequence $\{\Omega_n\} \subset \mathcal{S}$, they did not define w^*-usc by a net. This type of semicontinuity is not equivalent to w^*-upper semicontinuity in this paper.

3. Surrogate duality

In this section, we consider the following optimization problem involving set functions:

$$\begin{cases} \text{minimize } F(x), \\ \text{subject to } G_i(x) \leq 0, \forall i \in I, \end{cases}$$

where $I = \{1, \ldots, m\}$, F is a real-valued w^*-usc quasiconvex set function from $\mathcal{A}$, and G_i is a real-valued convex set function from $\mathcal{A}$ for each $i \in I$. Let $\mathcal{S} = \{\Omega \in \mathcal{A} \mid G_i(\Omega) \leq 0, \forall i \in I\}$, and assume that there exists $\Omega_1 \in \mathcal{A}$ such that $G_i(\Omega_1) < 0$ for each $i \in I$.

Surrogate duality has been studied for various types of optimization problems, for example, zero-one integer programming problem, quasiconvex optimization, robust optimization, and so on. For more details, see [5, 7–9, 19, 25, 30] and references therein.

We show the following surrogate duality theorem for optimization problems involving quasiconvex and convex set functions.

Theorem 3.1. *Let $I = \{1, \dots, m\}$, F a real-valued w^*-usc quasiconvex set function on $\mathcal{A}$, G_i a real-valued convex set function on $\mathcal{A}$ for each $i \in I$, and $\mathcal{S} = \{\Omega \in \mathcal{A} \mid G_i(\Omega) \leq 0, \forall i \in I\}$. Assume that there exists $\Omega_1 \in \mathcal{A}$ such that $G_i(\Omega_1) < 0$ for each $i \in I$.*

Then,

$$\inf_{\Omega \in \mathcal{S}} F(\Omega) = \max_{\lambda \in \mathbb{R}^m_+} \inf \left\{ F(\Omega) \,\middle|\, \sum_{i=1}^m \lambda_i G_i(\Omega) \leq 0 \right\}.$$

Proof. Let $\mu = \inf_{\Omega \in \mathcal{S}} F(\Omega)$. At first, we show surrogate weak duality. Let $\lambda \in \mathbb{R}^m_+$. We can easily see that $\mathcal{S} \subset \{\Omega \in \mathcal{A} \mid \sum_{i=1}^m \lambda_i G_i(\Omega) \leq 0\}$. This shows that

$$\mu \geq \sup_{\lambda \in \mathbb{R}^m_+} \inf \left\{ F(\Omega) \,\middle|\, \sum_{i=1}^m \lambda_i G_i(\Omega) \leq 0 \right\},$$

that is, surrogate weak duality holds.

If $\mu = -\infty$, then putting $\lambda = 0$, the equality holds.

Assume that $\mu > -\infty$. Let

$$\begin{aligned} A &= \left\{ z \in \mathbb{R}^m \,\middle|\, \exists \Omega \in \mathcal{A} \text{ s.t. } \begin{array}{l} G_i(\Omega) \leq z_i, \forall i \in \{1, \dots, m\}, \\ F(\Omega) < \mu \end{array} \right\}, \\ N &= \{z \in \mathbb{R}^m \mid z_i \leq 0, \forall i \in \{1, \dots, m\}\}, \end{aligned}$$

where $\mathbb{R}^m$ is the m-dimensional Euclidean space. It is clear that N is a closed convex cone in $\mathbb{R}^m$.

We show that $\mathrm{cl}A$ is a convex subset of $\mathbb{R}^m$. Let x, $y \in \mathrm{cl}A$, and $\alpha \in [0, 1]$. For each $\varepsilon > 0$, there exist x_ε, $y_\varepsilon \in A$ such that $\|x - x_\varepsilon\| < \varepsilon$ and $\|y - y_\varepsilon\| < \varepsilon$. Also, there exist Ω_{x_ε}, $\Omega_{y_\varepsilon} \in \mathcal{A}$ such that $F(\Omega_{x_\varepsilon}) < \mu$, $F(\Omega_{y_\varepsilon}) < \mu$, $G_i(\Omega_{x_\varepsilon}) \leq x_{\varepsilon,i}$ for each $i \in I$, and $G_i(\Omega_{y_\varepsilon}) \leq y_{\varepsilon,i}$ for each $i \in I$. Hence

$$G_i(\Omega_{x_\varepsilon}) - \varepsilon < x_i, \text{ and } G_i(\Omega_{y_\varepsilon}) - \varepsilon < y_i.$$

Then, there exists a Morris sequence $\{\Gamma_n\}$ associated with $(\alpha, \Omega_{x_\varepsilon}, \Omega_{y_\varepsilon})$ such that for each $i \in I$,

$$\limsup_{n \to \infty} G_i(\Gamma_n) \leq (1 - \alpha) G_i(\Omega_{x_\varepsilon}) + \alpha G_i(\Omega_{y_\varepsilon}),$$

and

$$\limsup_{n \to \infty} F(\Gamma_n) \leq \max\{F(\Omega_{x_\varepsilon}), F(\Omega_{y_\varepsilon})\} < \mu$$

since F is quasiconvex, and G_i is convex. Also,

$$\limsup_{n \to \infty} G_i(\Gamma_n) \quad \leq \quad (1 - \alpha) G_i(\Omega_{x_\varepsilon}) + \alpha G_i(\Omega_{y_\varepsilon})$$

$$\begin{aligned} &< (1-\alpha)(x_i+\varepsilon)+\alpha(y_i+\varepsilon) \\ &= (1-\alpha)x_i+\alpha y_i+\varepsilon. \end{aligned}$$

Hence, there exists $n_0 \in \mathbb{N}$ such that for each $i \in I$,

$$G_i(\Gamma_{n_0}) < (1-\alpha)x_i+\alpha y_i+\varepsilon,$$

and

$$F(\Gamma_{n_0}) \le \max\{F(\Omega_{x_\varepsilon}), F(\Omega_{y_\varepsilon})\} < \mu.$$

This shows that

$$(1-\alpha)x+\alpha y+(\varepsilon,\varepsilon,\dots,\varepsilon) \in A,$$

that is, $(1-\alpha)x+\alpha y \in \mathrm{cl}A$. Hence, $\mathrm{cl}A$ is convex.

Also, we can prove that $(\mathrm{cl}A)\cap(\mathrm{int}N)=\emptyset$ since $\mu=\inf_{\Omega\in\mathcal{S}}F(\Omega)$. Hence, by the separation theorem between $\mathrm{cl}A$ and N, there exist $w \in \mathbb{R}^m \setminus \{0\}$ and $\beta \in \mathbb{R}$ such that for each $z \in A$ and $y \in N$,

$$\langle w,z\rangle \ge \beta \ge \langle w,y\rangle .$$

By the definition of the cone N, we can show that $\beta=0$ and $w \in \mathbb{R}^m_+ \setminus \{0\}$.

Next, we show that for each $z \in A$, $\langle w,z\rangle > 0$.

Assume that there exists $z_0 \in A$ such that $\langle w,z_0\rangle = 0$. Then, there exists $\Omega_0 \in \mathcal{A}$ such that $G_i(\Omega_0) \le z_{0,i}$ for each $i \in \{1,\dots,m\}$, and $F(\Omega_0)<\mu$. Let

$$z_1 = (G_1(\Omega_1), G_2(\Omega_1), \dots, G_m(\Omega_1)) \in \mathbb{R}^m,$$

$\varepsilon = \dfrac{\min\{|z_{1,i}| \mid i \in I\}}{2}$, and $e=(\varepsilon,\varepsilon,\dots,\varepsilon) \in \mathbb{R}^m_+$. Then, we can check that $z_{1,i}+\varepsilon < 0$ for each $i \in I$, and $z_1+e \notin A$. For each $\alpha \in (0,1]$, let

$$z_\alpha := (1-\alpha)z_0+\alpha(z_1+e).$$

Since $w \in \mathbb{R}^m_+ \setminus \{0\}$, $z_{1,i}+\varepsilon<0$, and $\langle w,z_0\rangle=0$,

$$\langle w,z_\alpha\rangle = (1-\alpha)\langle w,z_0\rangle+\alpha\langle w,z_1+e\rangle = \alpha\langle w,z_1+e\rangle < 0.$$

This shows that $z_\alpha \notin A$ for each $\alpha \in (0,1]$. Let $U \in D := \{U : w^*\text{-nbd. of } \chi_{\Omega_0}\}$. Then, there exists $\alpha_0 \in (0,1]$ such that

$$(1-\alpha_0)\chi_{\Omega_0}+\alpha_0\chi_{\Omega_1} \in U.$$

Also, there exists a neighborhood U_{α_0} of $(1-\alpha_0)\chi_{\Omega_0}+\alpha_0\chi_{\Omega_1}$ such that $U_{\alpha_0} \subset U$. Since G_i is convex, there exists a Morris sequence $\{\Omega_n\}$ associated with $(\alpha_0,\Omega_0,\Omega_1)$ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} (1-\alpha_0)\chi_{\Omega_0}+\alpha_0\chi_{\Omega_1},$$

and for each $i \in I$,

$$\limsup_{n\to\infty} G_i(\Omega_n) \le (1-\alpha_0)G_i(\Omega_0)+\alpha_0 G_i(\Omega_1).$$

Then,

$$\begin{aligned} \limsup_{n\to\infty} G_i(\Omega_n) &\le (1-\alpha_0)G_i(\Omega_0)+\alpha_0 G_i(\Omega_1) \\ &\le (1-\alpha_0)z_{0,i}+\alpha_0 z_{1,i} \end{aligned}$$

$$< \quad (1-\alpha_0)z_{0,i} + \alpha_0(z_{1,i} + \varepsilon).$$

Hence, there exists $n_0 \in \mathbb{N}$ such that $\chi_{\Omega_{n_0}} \in U_{\alpha_0} \subset U$, and for each $i \in I$,

$$G_i(\Omega_{n_0}) < (1-\alpha_0)z_{0,i} + \alpha_0(z_{1,i} + \varepsilon).$$

Since $z_{\alpha_0} \notin A$, $F(\Omega_{n_0}) \geq \mu$. Put $\Omega_U = \Omega_{n_0}$. Then, we can check that $\{\chi_{\Omega_U}\}_{U \in D}$ is a net and $\chi_{\Omega_U} \xrightarrow{w^*} \chi_{\Omega_0}$. Since F is w^*-usc,

$$F(\Omega_0) \geq \limsup_U F(\Omega_U) \geq \mu.$$

This is a contradiction. Hence, $\langle w, z \rangle > 0$ for each $z \in A$.

By the separation inequality, for each $\Omega \in \mathcal{A}$ with $\sum_{i=1}^m w_i G_i(\Omega) \leq 0$,

$$z = (G_1(\Omega), G_2(\Omega), \ldots, G_m(\Omega)) \notin A.$$

This shows that $F(\Omega) \geq \mu$. Hence,

$$\mu \leq \inf \left\{ F(\Omega) \,\middle|\, \sum_{i=1}^m w_i G_i(\Omega) \leq 0 \right\} \leq \sup_{\lambda \in \mathbb{R}^m_+} \inf \left\{ F(\Omega) \,\middle|\, \sum_{i=1}^m \lambda_i G_i(\Omega) \leq 0 \right\} \leq \mu.$$

This completes the proof. □

Remark 3.2. In Theorem 3.1, we assume the following condition:

$$\text{there exists } \Omega_1 \in \mathcal{A} \text{ such that } G_i(\Omega_1) < 0 \text{ for each } i \in I.$$

The above condition is the Slater type constraint qualification for set functions. In the research of duality theorems, various types of constraint qualifications have been investigated, see [1, 2, 6, 12, 13, 17, 21–30]. Especially, necessary and sufficient constraint qualifications for surrogate duality have been investigated, see [25, 30]. Necessary and sufficient constraint qualifications for surrogate duality via set functions are future research.

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