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APPROXIMATION OF A COMMON ATTRACTIVE POINT OF NONCOMMUTATIVE NORMALLY 2-GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. This paper presents approximation methods for finding common attractive points of two nonlinear mappings Hilbert spaces. These results are established without assuming that the two mappings are commutative, or that the domain of the mappings is closed. For the case in which the domain is closed, we obtain approximation methods concerning common fixed points. We consider a broad class of mappings called normally 2-generalized hybrid mappings, which includes nonexpansive mappings, generalized hybrid mappings, and other famous classes of nonlinear mappings.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty subset of H, and let T be a mapping from C into H. The sets of fixed points and *attractive points* [33] of T are denoted by

$$\begin{aligned} F(T) &= \{u \in C : Tu = u\} \text{ and} \\ A(T) &= \{u \in H : \|Ty - u\| \leq \|y - u\| \text{ for all } y \in C\}, \end{aligned}$$

respectively. A mapping $T : C \to H$ is said to be

(i) *contractive* if there exists $r \in [0, 1)$ such that $\|Tx - Ty\| \leq r \|x - y\|$ for all $x, y \in C$;

(ii) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

For nonexpansive mappings, several approximation methods for finding fixed points have been proposed. The following iteration was introduced by Mann [23] in 1953:

$$\begin{aligned} x_1 &\in C : \text{given}, \\ x_{n+1} &= \lambda_n x_n + (1 - \lambda_n) T x_n \text{ for all } n \in \mathbb{N}, \end{aligned}$$

where $\mathbb{N}$ is the set of natural numbers and $\{\lambda_n\} \subset [0, 1]$. In 1967, Halpern proposed a different type of iteration [4]:

$$\begin{aligned} x_1 &\in C : \text{given}, \\ x_{n+1} &= \lambda_n x_1 + (1 - \lambda_n) T x_n \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Mann's and Halpern's iterations yield weak and strong convergence (see Reich [25] and Wittmann [37]), respectively.

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In 2010, Kocourek et al. [15] defined a wide class of mappings. A mapping $T : C \to H$ is called

(iii) *generalized hybrid* [15] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta \|Tx - y\|^2 + (1-\beta)\|x-y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. The class of generalized hybrid mappings simultaneously includes nonexpansive mappings, *nonspreading mappings* [17, 18], *hybrid mappings* [31], and λ-*hybrid mappings* [1] as special cases. Note that nonspreading mappings and hybrid mappings are not necessarily continuous (see [12] or [36]).

The class of generalized hybrid mappings has been further extended. A mapping $T : C \to H$ is called

(iv) *normally generalized hybrid* [35] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 + \delta \|x - y\|^2 \le 0$$

for all $x, y \in C$, where (1) $\alpha+\beta+\gamma+\delta \ge 0$ and (2) $\alpha+\beta > 0$ or $\alpha+\gamma > 0$. It is easy to verify that the class of normally generalized hybrid mappings covers generalized hybrid mappings. According to [35], a normally generalized hybrid mapping T has at most one fixed point such as a contraction under the condition $\alpha+\beta+\gamma+\delta > 0$. A mapping $T : C \to C$ is called

(v) 2-*generalized hybrid* [24] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_1 \left\|T^2x - Ty\right\|^2 + \alpha_2 \|Tx - Ty\|^2 + (1-\alpha_1-\alpha_2)\|x - Ty\|^2 \\ \le \beta_1 \left\|T^2x - y\right\|^2 + \beta_2 \|Tx - y\|^2 + (1-\beta_1-\beta_2)\|x-y\|^2\end{aligned}$$

for all $x, y \in C$. It is obvious that the class of 2-generalized hybrid mappings is generalized hybrid if $\alpha_1 = \beta_1 = 0$. Hojo et al. [10] gave examples of 2-generalized hybrid mappings that are not generalized hybrid. A mapping $T : C \to C$ is said to be

(vi) *normally* 2-*generalized hybrid* [19] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_2 \left\|T^2x - Ty\right\|^2 + \alpha_1 \|Tx - Ty\|^2 + \alpha_0 \|x - Ty\|^2 \\ + \beta_2 \left\|T^2x - y\right\|^2 + \beta_1 \|Tx - y\|^2 + \beta_0 \|x-y\|^2 \le 0\end{aligned}$$

for all $x, y \in C$, where (1) $\sum_{n=0}^{2} (\alpha_n + \beta_n) \ge 0$ and (2) $\alpha_2 + \alpha_1 + \alpha_0 > 0$. The class of normally 2-generalized hybrid mappings contains mappings of types (i)–(v) (see [19]). For a normally 2-generalized hybrid mapping T, Kondo and Takahashi [19] considered the following iteration process:

$$\begin{aligned} x_1 &\in C : \text{given}, \\ x_{n+1} &= a_n x_n + b_n T x_n + c_n T^2 x_n \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

and proved a weak convergence theorem. In another paper [20], they considered the following Halpern type iteration:

$$\begin{aligned} x_1, z &\in C : \text{given}, \\ x_{n+1} &= \lambda_n z + (1-\lambda_n)\left(a_n x_n + b_n T x_n + c_n T^2 x_n\right) \quad \text{for all } n \in \mathbb{N}, \end{aligned}$$

and proved a strong convergence theorem.

Approximation methods for finding common fixed points of nonexpansive mappings have been intensively studied; see, for example, Lions [21], Shimizu and Takahashi [26], Atsushiba and Takahashi [3], Shimoji and Takahashi [27], and Kimura et al. [14]. Iemoto and Takahashi [11] dealt with a nonexpansive mapping and a nonspreading mapping in the framework of a Hilbert space. Very recently, Hojo et al. [6] proved weak and strong convergence theorems for the common attractive and fixed points of two commutative normally 2-generalized hybrid mappings (see also [5, 7, 9, 16]), while Takahashi [32] dealt with two noncommutative generalized hybrid mappings. However, no approximation method for finding the common attractive and fixed points of two noncommutative normally 2-generalized hybrid mappings has yet been developed.

This paper presents approximation methods for finding common attractive and fixed points of two noncommutative normally 2-generalized hybrid mappings. First, Mann type weak convergence theorems are obtained (Theorems 3.1, 3.2, and 3.4). Strong convergence theorems of Halpern type iterations are also established (Theorems 4.1, 4.2, and 4.4). Takahashi's results [32] are derived from our theorems.

2. Preliminaries

This section briefly presents preliminary information and results. In a real Hilbert space H, it is known that

$$2\langle x-y,\ y\rangle \leq \|x\|^2 - \|y\|^2 \leq 2\langle x-y,\ x\rangle \tag{2.1}$$

for all $x, y \in H$. The strong and weak convergence of a sequence $\{x_n\}$ in H to $x\,(\in H)$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. It is also known that a closed and convex subset of H is weakly closed. For a bounded sequence $\{x_n\}$ in H, $\{x_n\}$ is weakly convergent if and only if every weakly convergent subsequence of $\{x_n\}$ has the same weak limit, that is,

$$x_n \rightharpoonup v \Longleftrightarrow [x_{n_i} \rightharpoonup u \text{ implies that } u = v],$$

where $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ and $u, v \in H$. Takahashi and Takeuchi [33] showed that the set of attractive points $A(T)$ is closed and convex in a Hilbert space. For a normally 2-generalized hybrid mapping $T : C \to C$, Kondo and Takahashi [20] demonstrated that $F(T) \subset A(T)$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|Tx - u\| \leq \|x - u\|$ for all $x \in C$ and $u \in F(T)$. We know from [19] that a normally 2-generalized hybrid mapping with $F(T) \neq \emptyset$ is quasi-nonexpansive. We also know that the set of fixed points $F(T)$ of a quasi-nonexpansive mapping is closed and convex (see [13]).

Let D be a nonempty, closed, and convex subset of H. For any $x \in H$, there exists a unique nearest point $u \in D$, that is, $\|x - u\| = \inf_{z \in D} \|x - z\|$. This correspondence is called the *metric projection* from H onto D, and is denoted by P_D. For the metric projection P_D from H onto D, it holds that $\langle x - P_D x,\ P_D x - z\rangle \geq 0$ for all $x \in H$ and $z \in D$. For more details, see Takahashi [29] and [30].

Regarding the existence of common attractive and fixed points, Hojo [5] found the following result.

Theorem 2.1 ([5])**.** *Let C be a nonempty subset of H, and let S and T be commutative normally 2-generalized hybrid mappings of C into itself. Suppose that there exists an element $z \in C$ such that $\{S^k T^l z : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then, $A(S) \cap A(T)$ is nonempty. Additionally, if C is closed and convex, then $F(S) \cap F(T)$ is nonempty.*

The following lemma will be used in the proofs of the main theorems in this paper.

Lemma 2.2 ([34])**.** *Let D be a nonempty, closed, and convex subset of H, let P_D be the metric projection from H onto D, and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - q\| \leq \|x_n - q\|$ for all $q \in D$ and $n \in \mathbb{N}$, then $\{P_D x_n\}$ is convergent in D.*

In the following lemma, parts (a) and (b) were proved by Takahashi [30] and Maruyama et al. [24], respectively. As was recognized in [24], parts (c), (d), and more generalized equalities hold. For completeness, we provide the proof of (c) in this paper.

Lemma 2.3 ([24,30])**.** *Let $x, y, z, w, v \in H$ and $a, b, c, d, e \in \mathbb{R}$. Then, the following hold:*

(a) *If $a + b = 1$, then $\|ax + by\|^2 = a\|x\|^2 + b\|y\|^2 - ab\|x - y\|^2$.*

(b) *If $a + b + c = 1$, then*

$$\begin{aligned}&\|ax + by + cz\|^2\\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2.\end{aligned}$$

(c) *If $a + b + c + d = 1$, then*

$$\begin{aligned}\|ax + by + cz + dw\|^2 \;=\;& a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2\\ & -ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2\\ & -bc\|y - z\|^2 - bd\|y - w\|^2 - cd\|z - w\|^2.\end{aligned}$$

(d) *If $a + b + c + d + e = 1$, then*

$$\begin{aligned}&\|ax + by + cz + dw + ev\|^2\\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 + d\|w\|^2 + e\|v\|^2\\ &\quad - ab\|x - y\|^2 - ac\|x - z\|^2 - ad\|x - w\|^2 - ae\|x - v\|^2\\ &\quad - bc\|y - z\|^2 - bd\|y - w\|^2 - be\|y - v\|^2\\ &\quad - cd\|z - w\|^2 - ce\|z - v\|^2 - de\|w - v\|^2.\end{aligned}$$

Proof. We prove part (c). From the relationship between the inner product and norm in Hilbert spaces, it follows that

$$\begin{aligned}& \|ax + by + cz + dw\|^2\\ =\;& \langle ax + by + cz + dw,\ ax + by + cz + dw\rangle\\ =\;& a^2\|x\|^2 + b^2\|y\|^2 + c^2\|z\|^2 + d^2\|w\|^2\\ & +ab\langle x,\ y\rangle + ac\langle x,\ z\rangle + ad\langle x,\ w\rangle + ab\langle y,\ x\rangle + bc\langle y,\ z\rangle + bd\langle y,\ w\rangle\\ & +ac\langle z,\ x\rangle + bc\langle z,\ y\rangle + cd\langle z,\ w\rangle + ad\langle w,\ x\rangle + bd\langle w,\ y\rangle + cd\langle w,\ z\rangle\end{aligned}$$

$$\begin{aligned} &= a^2\|x\|^2+b^2\|y\|^2+c^2\|z\|^2+d^2\|w\|^2 \\ &\quad +2ab\langle x,\ y\rangle+2ac\langle x,\ z\rangle+2ad\langle x,\ w\rangle \\ &\quad +2bc\langle y,\ z\rangle+2bd\langle y,\ w\rangle+2cd\langle z,\ w\rangle . \end{aligned}$$

Using $\|u-v\|^2=\|u\|^2-2\langle u,\ v\rangle+\|v\|^2$ for all $u,v\in H$, we have that

$$\begin{aligned} &\|ax+by+cz+dw\|^2 \\ &= a^2\|x\|^2+b^2\|y\|^2+c^2\|z\|^2+d^2\|w\|^2 \\ &\quad +ab\left(\|x\|^2+\|y\|^2-\|x-y\|^2\right)+ac\left(\|x\|^2+\|z\|^2-\|x-z\|^2\right) \\ &\quad +ad\left(\|x\|^2+\|w\|^2-\|x-w\|^2\right)+bc\left(\|y\|^2+\|z\|^2-\|y-z\|^2\right) \\ &\quad +bd\left(\|y\|^2+\|w\|^2-\|y-w\|^2\right)+cd\left(\|z\|^2+\|w\|^2-\|z-w\|^2\right) \\ &= a(a+b+c+d)\|x\|^2+b(b+a+c+d)\|y\|^2 \\ &\quad +c(c+a+b+d)\|z\|^2+d(d+a+b+c)\|w\|^2 \\ &\quad -ab\|x-y\|^2-ac\|x-z\|^2-ad\|x-w\|^2 \\ &\quad -bc\|y-z\|^2-bd\|y-w\|^2-cd\|z-w\|^2 . \end{aligned}$$

Because $a+b+c+d=1$, we obtain

$$\begin{aligned} \|ax+by+cz+dw\|^2 &= a\|x\|^2+b\|y\|^2+c\|z\|^2+d\|w\|^2 \\ &\quad -ab\|x-y\|^2-ac\|x-z\|^2-ad\|x-w\|^2 \\ &\quad -bc\|y-z\|^2-bd\|y-w\|^2-cd\|z-w\|^2 . \end{aligned}$$

This completes the proof of (c). Similarly, (d) can be derived from long and somewhat tedious calculations. $\square$

The proof of the following lemma was developed in [15] and [24], among others.

Lemma 2.4 ([19,35])**.** *Let C be a nonempty subset of H, let T be a mapping from C into itself, and let $\{x_n\}$ be a sequence in C.*

(a) *Suppose that T is a normally generalized hybrid mapping. If $\{x_n\}$ satisfies $Tx_n-x_n\to 0$ and $x_n\rightharpoonup u$, then $u\in A(T)$.*

(b) *Suppose that T is a normally 2-generalized hybrid mapping. If $\{x_n\}$ satisfies $Tx_n-x_n\to 0$, $T^2x_n-x_n\to 0$ and $x_n\rightharpoonup u$, then $u\in A(T)$.*

The following lemma was demonstrated by Takahashi and Takeuchi [33].

Lemma 2.5 ([33])**.** *Let C be a nonempty subset of H, and let T be a mapping from C into H. Then, $A(T)\cap C\subset F(T)$.*

The following two lemmas will be exploited to derive the strong convergence in Theorem 4.1.

Lemma 2.6 ([2]; see also [38])**.** *Let $\{X_n\}$ be a sequence of nonnegative real numbers, let $\{Y_n\}$ be a sequence of real numbers such that $\limsup_{n\to\infty}Y_n\le 0$, and let $\{Z_n\}$ be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty}Z_n<\infty$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ such that $\sum_{n=1}^{\infty}\lambda_n=\infty$. If $X_{n+1}\le(1-\lambda_n)X_n+\lambda_nY_n+Z_n$ for all $n\in\mathbb{N}$, then $X_n\to 0$ as $n\to\infty$.*

Lemma 2.7 ([22]). *Let $\{X_n\}$ be a sequence of real numbers. Assume that $\{X_n\}$ is not monotone decreasing for sufficiently large $n \in \mathbb{N}$, in other words, there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let n_0 be a natural number such that $\{k \leq n_0 : X_k < X_{k+1}\}$ is nonempty. Define a sequence $\{\tau(n)\}_{n \geq n_0}$ of natural numbers as follows:*

$$\tau(n) = \max\{k \leq n : X_k < X_{k+1}\} \quad \textit{for all } n \geq n_0.$$

Then, the following hold:

(a) $\tau(n) \to \infty$ *as* $n \to \infty$;
(b) $X_n \leq X_{\tau(n)+1}$ *for all* $n \geq n_0$;
(c) $X_{\tau(n)} < X_{\tau(n)+1}$ *for all* $n \geq n_0$.

3. Weak approximation

This section describes weak approximation methods for finding common attractive and fixed points of two nonlinear mappings that are not necessarily commutative.

3.1. **Normally Generalized and 2-Generalized Hybrid Mappings.** At the outset, we present a method to approximate the common attractive points of a normally generalized hybrid mapping and a normally 2-generalized hybrid mapping without assuming that the domain of the mappings is closed. An approximation method for finding fixed points can also be obtained by supposing that the domain is closed. The fundamentals of the proof were developed in [6, 15, 19, 24, 32, 35].

Theorem 3.1. *Let C be a nonempty and convex subset of H, let $S : C \to C$ be a normally generalized hybrid mapping, and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Suppose that $A(S) \cap A(T)$ is nonempty. Let $P_{A(S)\cap A(T)}$ be the metric projection from H onto $A(S) \cap A(T)$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $(0,1)$ such that $a_n + b_n + c_n + d_n = 1$ and $0 < a \leq a_n, b_n, c_n, d_n \leq b < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$$x_{n+1} = a_n x_n + b_n S x_n + c_n T x_n + d_n T^2 x_n$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges weakly to a common attractive point $\overline{x} \in A(S) \cap A(T)$, where $\overline{x} \equiv \lim_{n\to\infty} P_{A(S)\cap A(T)} x_n$. Additionally, if C is closed, then $\{x_n\}$ converges weakly to a common fixed point $\widehat{x} \in F(S) \cap F(T)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

Proof. Note that $A(S) \cap A(T)$ is a closed and convex subset of H. As it is assumed that $A(S) \cap A(T) \neq \emptyset$, there exists the metric projection $P_{A(S)\cap A(T)}$ from H onto $A(S) \cap A(T)$.

First, we show that

$$\|x_{n+1} - q\| \leq \|x_n - q\| \tag{3.1}$$

for any $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Indeed, we obtain from $q \in A(S) \cap A(T)$ that

$$\begin{aligned}
\|x_{n+1} - q\| &\equiv \left\|a_n x_n + b_n S x_n + c_n T x_n + d_n T^2 x_n - q\right\| \\
&= \left\|a_n (x_n - q) + b_n (S x_n - q) + c_n (T x_n - q) + d_n \left(T^2 x_n - q\right)\right\| \\
&\leq a_n \|x_n - q\| + b_n \|S x_n - q\| + c_n \|T x_n - q\| + d_n \left\|T^2 x_n - q\right\|
\end{aligned}$$

$$\begin{aligned}
&\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| + d_n \|x_n - q\| \\
&= \|x_n - q\| .
\end{aligned}$$

From (3.1), we obtain the following. (i) The sequence $\{\|x_n - q\|\}$ is convergent in $\mathbb{R}$ for all $q \in A(S) \cap A(T)$. (ii) Using Lemma 2.2, $\{P_{A(S)\cap A(T)} x_n\}$ is also convergent in $A(S) \cap A(T)$. We denote the limit by $\overline{x}$, that is, $\overline{x} \equiv \lim_{n\to\infty} P_{A(S)\cap A(T)} x_n$. (iii) The sequence $\{x_n\}$ is bounded because $\{\|x_n - q\|\}$ is convergent.

Next, we demonstrate that

$$\begin{aligned}
(3.2) \qquad & a_n b_n \|x_n - Sx_n\|^2 + a_n c_n \|x_n - Tx_n\|^2 + a_n d_n \left\|x_n - T^2 x_n\right\|^2 \\
& + b_n c_n \|Sx_n - Tx_n\|^2 + b_n d_n \left\|Sx_n - T^2 x_n\right\|^2 + c_n d_n \left\|Tx_n - T^2 x_n\right\|^2 \\
\leq\ & \|x_n - q\|^2 - \|x_{n+1} - q\|^2
\end{aligned}$$

for any $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Indeed, from Lemma 2.3-(c),

$$\begin{aligned}
& \|x_{n+1} - q\|^2 \\
=\ & \left\|a_n (x_n - q) + b_n (Sx_n - q) + c_n (Tx_n - q) + d_n \left(T^2 x_n - q\right)\right\|^2 \\
=\ & a_n \|x_n - q\|^2 + b_n \|Sx_n - q\|^2 + c_n \|Tx_n - q\|^2 + d_n \left\|T^2 x_n - q\right\|^2 \\
& - a_n b_n \|x_n - Sx_n\|^2 - a_n c_n \|x_n - Tx_n\|^2 - a_n d_n \left\|x_n - T^2 x_n\right\|^2 \\
& - b_n c_n \|Sx_n - Tx_n\|^2 - b_n d_n \left\|Sx_n - T^2 x_n\right\|^2 - c_n d_n \left\|Tx_n - T^2 x_n\right\|^2 \\
\leq\ & a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 + d_n \|x_n - q\|^2 \\
& - a_n b_n \|x_n - Sx_n\|^2 - a_n c_n \|x_n - Tx_n\|^2 - a_n d_n \left\|x_n - T^2 x_n\right\|^2 \\
& - b_n c_n \|Sx_n - Tx_n\|^2 - b_n d_n \left\|Sx_n - T^2 x_n\right\|^2 - c_n d_n \left\|Tx_n - T^2 x_n\right\|^2 \\
=\ & \|x_n - q\|^2 - a_n b_n \|x_n - Sx_n\|^2 - a_n c_n \|x_n - Tx_n\|^2 \\
& - a_n d_n \left\|x_n - T^2 x_n\right\|^2 - b_n c_n \|Sx_n - Tx_n\|^2 \\
& - b_n d_n \left\|Sx_n - T^2 x_n\right\|^2 - c_n d_n \left\|Tx_n - T^2 x_n\right\|^2 .
\end{aligned}$$

Therefore, we obtain (3.2).

As $\{\|x_n - q\|\}$ is convergent and it is assumed that $0 < a \leq a_n, b_n, c_n, d_n \leq b < 1$ for all $n \in \mathbb{N}$, we obtain from (3.2) that

$$\begin{aligned}
(3.3) \qquad x_n - Sx_n &\to 0, \quad x_n - Tx_n \to 0, \quad x_n - T^2 x_n \to 0, \\
Sx_n - Tx_n &\to 0, \quad Sx_n - T^2 x_n \to 0, \quad Tx_n - T^2 x_n \to 0.
\end{aligned}$$

Because $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $u \in H$ such that $x_{n_i} \rightharpoonup u$. As the mapping S is normally generalized hybrid and T is normally 2-generalized hybrid, we have from Lemma 2.4-(a), (b) and (3.3) that $u \in A(S) \cap A(T)$.

We prove that $x_n \rightharpoonup u$. Let $x_{n_j} \rightharpoonup u_1$ and $x_{n_k} \rightharpoonup u_2$, where $\{x_{n_j}\}$ and $\{x_{n_k}\}$ are subsequences of $\{x_n\}$. It suffices to show that $u_1 = u_2$. From (3.3) and Lemma 2.4, we have that $u_1, u_2 \in A(S) \cap A(T)$. From (3.1), the two sequences $\{\|x_n - u_1\|\}$ and $\{\|x_n - u_2\|\}$ are convergent. Define $a \equiv \lim_{n\to\infty} (\|x_n - u_1\| - \|x_n - u_2\|) \in \mathbb{R}$. As

$$\|x_n - u_1\| - \|x_n - u_2\| = 2 \langle x_n,\ u_2 - u_1 \rangle + \|u_1\| - \|u_2\|$$

for all $n \in \mathbb{N}$, we obtain

$$\begin{aligned} a &= 2\langle u_1,\ u_2 - u_1\rangle + \|u_1\| - \|u_2\| \text{ and} \\ a &= 2\langle u_2,\ u_2 - u_1\rangle + \|u_1\| - \|u_2\|. \end{aligned}$$

Thus, $2\langle u_1 - u_2,\ u_2 - u_1\rangle = 0$, which means that $u_1 = u_2$.

Next, we show that $u = \overline{x}\left(\equiv \lim_{n\to\infty} P_{A(S)\cap A(T)}x_n\right)$. As $u \in A(S) \cap A(T)$, it holds that

$$\left\langle x_n - P_{A(S)\cap A(T)}x_n,\ P_{A(S)\cap A(T)}x_n - u\right\rangle \geq 0$$

for all $n \in \mathbb{N}$. Since $x_n \rightharpoonup u$ and $P_{A(S)\cap A(T)}x_n \to \overline{x}$, we get that $\langle u - \overline{x},\ \overline{x} - u\rangle \geq 0$. This means that $u = \overline{x}$.

Additionally, suppose that C is closed in H. As C is weakly closed, we have that $\overline{x} \in C \cap A(S) \cap A(T)$, where $\overline{x} \equiv \lim_{n\to\infty} P_{A(S)\cap A(T)}x_n$. From Lemma 2.5, $\overline{x} \in F(S) \cap F(T)$. Thus, $F(S) \cap F(T)$ is nonempty. As S and T are quasi-nonexpansive, $F(S) \cap F(T)$ is closed and convex. Hence, there exists the metric projection $P_{F(S)\cap F(T)}$ from H onto $F(S) \cap F(T)$. In much the same way as for the proof of (3.1), we can obtain

$$\|x_{n+1} - q\| \leq \|x_n - q\|$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Thus, we have from Lemma 2.2 that $\{P_{F(S)\cap F(T)}x_n\}$ converges strongly to an element $\widehat{x}$ of $F(S) \cap F(T)$, that is, $\widehat{x} \equiv \lim_{n\to\infty} P_{F(S)\cap F(T)}x_n$. We show that

$$\overline{x}\left(\equiv \lim_{n\to\infty} P_{A(S)\cap A(T)}x_n\right) = \widehat{x}\left(\equiv \lim_{n\to\infty} P_{F(S)\cap F(T)}x_n\right).$$

From a property of the metric projection, we have that

$$\langle x_n - P_{F(S)\cap F(T)}x_n,\ P_{F(S)\cap F(T)}x_n - \overline{x}\rangle \geq 0$$

for all $n \in \mathbb{N}$. As $x_n \rightharpoonup \overline{x}$ and $P_{F(S)\cap F(T)}x_n \to \widehat{x}$, we have that $\langle \overline{x} - \widehat{x},\ \widehat{x} - \overline{x}\rangle \geq 0$, which means that $\widehat{x} = \overline{x}$. This implies that $\{x_n\}$ converges weakly to $\widehat{x} = \lim_{n\to\infty} P_{F(S)\cap F(T)}x_n \in F(S) \cap F(T)$. This completes the proof. $\square$

Theorem 3.1 offers alternative approximation methods to those of Takahashi's Theorem 3.2 [32], which is reproduced below as Theorem 3.3. These methods find common attractive and fixed points for two noncommutative generalized hybrid mappings, because the classes of normally generalized hybrid mappings and normally 2-generalized hybrid mappings include generalized hybrid mappings as special cases.

3.2. **Two Normally Generalized Hybrid Mappings.** Let us refocus the proof of Theorem 3.1. By using Lemma 2.3-(b) instead of part (c), as well as Lemma 2.4-(a), we can obtain the following theorem, which presents weak approximation methods for finding common attractive and fixed points of two noncommutative normally generalized hybrid mappings.

Theorem 3.2. *Let C be a nonempty and convex subset of H, and let S and T be normally generalized hybrid mappings from C into itself. Suppose that $A(S) \cap A(T)$ is nonempty. Let $P_{A(S)\cap A(T)}$ be the metric projection from H onto $A(S) \cap A(T)$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $(0,1)$ such* *that* $a_n+b_n+c_n=1$ *and* $0<a\leq a_n,b_n,c_n\leq b<1$ *for all* $n\in\mathbb{N}$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_{n+1}=a_nx_n+b_nSx_n+c_nTx_n \quad \textit{for all } n\in\mathbb{N},$$

where $x_1\in C$ *is given. Then, the sequence* $\{x_n\}$ *converges weakly to a common attractive point* $\overline{x}\in A(S)\cap A(T)$, *where* $\overline{x}\equiv\lim_{n\to\infty}P_{A(S)\cap A(T)}x_n$. *Additionally, if* C *is closed, then* $\{x_n\}$ *converges weakly to a common fixed point* $\widehat{x}\in F(S)\cap F(T)$, *where* $\widehat{x}\equiv\lim_{n\to\infty}P_{F(S)\cap F(T)}x_n$.

As an application of Theorem 3.2, we can obtain Theorem 3.2 of Takahashi [32] as follows.

Theorem 3.3 (Theorem 3.2 of Takahashi [32]). *Let* C *be a nonempty and convex subset of* H, *let* S *and* T *be generalized hybrid mappings from* C *into itself with* $A(S)\cap A(T)\neq\emptyset$, *and let* $P_{A(S)\cap A(T)}$ *be the metric projection from* H *onto* $A(S)\cap A(T)$. *Let* $\{\alpha_n\}$ *and* $\{\gamma_n\}$ *be sequences of real numbers in the interval* $(0,1)$ *such that* $0<a\leq\alpha_n,\gamma_n\leq b<1$ *for all* $n\in\mathbb{N}$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_{n+1}=\alpha_nx_n+(1-\alpha_n)(\gamma_nSx_n+(1-\gamma_n)Tx_n) \quad \textit{for all } n\in\mathbb{N},$$

where $x_1\in C$ *is given. Then, the sequence* $\{x_n\}$ *converges weakly to a common attractive point* $\overline{x}\in A(S)\cap A(T)$, *where* $\overline{x}\equiv\lim_{n\to\infty}P_{A(S)\cap A(T)}x_n$. *Additionally, if* C *is closed, then* $\{x_n\}$ *converges weakly to a common fixed point* $\widehat{x}\in F(S)\cap F(T)$, *where* $\widehat{x}\equiv\lim_{n\to\infty}P_{F(S)\cap F(T)}x_n$.

Proof. We know that a generalized hybrid mapping is normally generalized hybrid. Define $a_n\equiv\alpha_n$, $b_n\equiv(1-\alpha_n)\gamma_n$, and $c_n\equiv(1-\alpha_n)(1-\gamma_n)$. Then, it holds that $a_n+b_n+c_n=1$. Furthermore, as there exist real numbers $a,b\in(0,1)$ such that $a\leq\alpha_n,\gamma_n\leq b$ for all $n\in\mathbb{N}$, we have that $c,d\in(0,1)$ exist such that $c\leq a_n,b_n,c_n\leq d$ for all $n\in\mathbb{N}$. Hence, from Theorem 3.2, we obtain the desired result. □

3.3. **Two Normally 2-Generalized Hybrid Mappings.** In the proof of Theorem 3.1, by using Lemma 2.3-(d) instead of (c), as well as Lemma 2.4-(b), we can obtain the following theorem which presents weak approximation methods concerning two noncommutative normally 2-generalized hybrid mappings.

Theorem 3.4. *Let* C *be a nonempty and convex subset of* H, *and let* S *and* T *be normally* 2*-generalized hybrid mappings from* C *into itself. Suppose that* $A(S)\cap A(T)$ *is nonempty. Let* $P_{A(S)\cap A(T)}$ *be the metric projection from* H *onto* $A(S)\cap A(T)$. *Let* $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, *and* $\{e_n\}$ *be sequences of real numbers in the interval* $(0,1)$ *such that* $a_n+b_n+c_n+d_n+e_n=1$ *and* $0<a\leq a_n,b_n,c_n,d_n,e_n\leq b<1$ *for all* $n\in\mathbb{N}$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_{n+1}=a_nx_n+b_nSx_n+c_nS^2x_n+d_nTx_n+e_nT^2x_n \quad \textit{for all } n\in\mathbb{N},$$

where $x_1\in C$ *is given. Then, the sequence* $\{x_n\}$ *converges weakly to a common attractive point* $\overline{x}\in A(S)\cap A(T)$, *where* $\overline{x}\equiv\lim_{n\to\infty}P_{A(S)\cap A(T)}x_n$. *Additionally, if* C *is closed, then* $\{x_n\}$ *converges weakly to a common fixed point* $\widehat{x}\in F(S)\cap F(T)$, *where* $\widehat{x}\equiv\lim_{n\to\infty}P_{F(S)\cap F(T)}x_n$.

Theorem 3.4 offers alternative approximation methods to those of Takahashi's Theorem 3.2 [32] for finding common attractive and fixed points for two noncommutative generalized hybrid mappings. This is because the class of normally 2-generalized hybrid mappings contains generalized hybrid mappings.

4. Strong approximation

This section presents strong approximation methods for finding common attractive and fixed points of two nonlinear mappings.

4.1. **Normally Generalized and 2-Generalized Hybrid Mappings.** In the first theorem of this section, we deal with a normally generalized hybrid mapping and a normally 2-generalized hybrid mapping. The fundamentals of the proof were developed in [8, 20, 28, 36].

Theorem 4.1. *Let C be a nonempty and convex subset of H, let $S : C \to C$ be a normally generalized hybrid mapping, and let $T : C \to C$ be a normally 2-generalized hybrid mapping. Suppose that $A(S) \cap A(T)$ is nonempty. Let $P_{A(S)\cap A(T)}$ be the metric projection from H onto $A(S) \cap A(T)$. Let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be sequences of real numbers in the interval $(0, 1)$ such that*

$$\lambda_n \to 0, \qquad \sum_{n=1}^{\infty} \lambda_n = \infty, \tag{4.1}$$

$$a_n + b_n + c_n + d_n = 1, \tag{4.2}$$

$$0 < a \le a_n, b_n, c_n, d_n \le b < 1 \quad \text{for all } n \in \mathbb{N}. \tag{4.3}$$

Let $\{z_n\}$ be a sequence in C such that $z_n \to z\,(\in H)$. Define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n z_n + (1 - \lambda_n) \left(a_n x_n + b_n S x_n + c_n T x_n + d_n T^2 x_n\right)$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges strongly to a common attractive point $\overline{z} \in A(S) \cap A(T)$, where $\overline{z} \equiv P_{A(S)\cap A(T)} z$. Additionally, if C is closed, then $\{x_n\}$ converges strongly to a common fixed point $\widehat{z} \in F(S) \cap F(T)$, where $\widehat{z} \equiv P_{F(S)\cap F(T)} z$.

Proof. Define $y_n \equiv a_n x_n + b_n S x_n + c_n T x_n + d_n T^2 x_n \in C$ for all $n \in \mathbb{N}$. Then, $x_{n+1} = \lambda_n z_n + (1 - \lambda_n) y_n \in C$.

First, we will show that $x_n \to \overline{z} \equiv P_{A(S)\cap A(T)} z$. It is easy to show that

$$\|y_n - q\| \le \|x_n - q\| \tag{4.4}$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. Indeed, because $q \in A(S) \cap A(T)$, we have from (4.2) that

$$\begin{aligned}
\|y_n - q\| &\equiv \left\|a_n x_n + b_n S x_n + c_n T x_n + d_n T^2 x_n - q\right\| \\
&\le a_n \|x_n - q\| + b_n \|S x_n - q\| + c_n \|T x_n - q\| + d_n \left\|T^2 x_n - q\right\| \\
&\le a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| + d_n \|x_n - q\| \\
&= \|x_n - q\| .
\end{aligned}$$

From (4.4), we can demonstrate that the sequence $\{x_n\}$ is bounded using mathematical induction. Indeed, let $q \in A(S) \cap A(T)$, and define

$$M \equiv \max\left\{\sup_{k\in\mathbb{N}} \|z_k - q\|, \ \ \|x_1 - q\|\right\}.$$

As $\{z_n\}$ is bounded, M is a real number. We will prove that $\|x_n - q\| \leq M$ for all $n \in \mathbb{N}$. (i) It is obvious for the case of $n = 1$. (ii) Assume that $\|x_k - q\| \leq M$ for some $k \in \mathbb{N}$. We have from (4.4) that

$$\begin{aligned} \|x_{k+1} - q\| &\leq \|\lambda_k z_k + (1-\lambda_k) y_k - q\| \\ &\leq \lambda_k \|z_k - q\| + (1-\lambda_k) \|y_k - q\| \\ &\leq \lambda_k \|z_k - q\| + (1-\lambda_k) \|x_k - q\| \\ &\leq \lambda_k M + (1-\lambda_k) M = M. \end{aligned}$$

Hence, $\{x_n\}$ is bounded.

Let us show that

$$\begin{aligned} &a_n b_n \|x_n - Sx_n\|^2 + a_n c_n \|x_n - Tx_n\|^2 + a_n d_n \left\|x_n - T^2 x_n\right\|^2 \\ &+ b_n c_n \|Sx_n - Tx_n\|^2 + b_n d_n \left\|Sx_n - T^2 x_n\right\|^2 + c_n d_n \left\|Tx_n - T^2 x_n\right\|^2 \\ &\leq \lambda_n \|z_n - q\|^2 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \end{aligned} \tag{4.5}$$

for all $q \in A(S) \cap A(T)$ and $n \in \mathbb{N}$. By using Lemma 2.3-(a), (c), we obtain

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ = \ &\|\lambda_n (z_n - q) + (1-\lambda_n)(y_n - q)\|^2 \\ \leq \ &\lambda_n \|z_n - q\|^2 + (1-\lambda_n) \|y_n - q\|^2 \\ \leq \ &\lambda_n \|z_n - q\|^2 \\ &+ \left\|a_n (x_n - q) + b_n (Sx_n - q) + c_n (Tx_n - q) + d_n \left(T^2 x_n - q\right)\right\|^2 \\ = \ &\lambda_n \|z_n - q\|^2 + a_n \|x_n - q\|^2 + b_n \|Sx_n - q\|^2 \\ &+ c_n \|Tx_n - q\|^2 + d_n \left\|T^2 x_n - q\right\|^2 \\ &- a_n b_n \|x_n - Sx_n\|^2 - a_n c_n \|x_n - Tx_n\|^2 - a_n d_n \left\|x_n - T^2 x_n\right\|^2 \\ &- b_n c_n \|Sx_n - Tx_n\|^2 - b_n d_n \left\|Sx_n - T^2 x_n\right\|^2 - c_n d_n \left\|Tx_n - T^2 x_n\right\|^2 \\ \leq \ &\lambda_n \|z_n - q\|^2 + a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 \\ &+ c_n \|x_n - q\|^2 + d_n \|x_n - q\|^2 \\ &- a_n b_n \|x_n - Sx_n\|^2 - a_n c_n \|x_n - Tx_n\|^2 - a_n d_n \left\|x_n - T^2 x_n\right\|^2 \\ &- b_n c_n \|Sx_n - Tx_n\|^2 - b_n d_n \left\|Sx_n - T^2 x_n\right\|^2 - c_n d_n \left\|Tx_n - T^2 x_n\right\|^2 \\ = \ &\lambda_n \|z_n - q\|^2 + \|x_n - q\|^2 \\ &- a_n b_n \|x_n - Sx_n\|^2 - a_n c_n \|x_n - Tx_n\|^2 - a_n d_n \left\|x_n - T^2 x_n\right\|^2 \\ &- b_n c_n \|Sx_n - Tx_n\|^2 - b_n d_n \left\|Sx_n - T^2 x_n\right\|^2 - c_n d_n \left\|Tx_n - T^2 x_n\right\|^2. \end{aligned}$$

This implies that (4.5) holds.

Furthermore, it holds that

$$\|x_{n+1}-x_n\| \le \lambda_n \|z_n-x_n\| + \|Sx_n-x_n\| + \|Tx_n-x_n\| + \left\|T^2x_n-x_n\right\| \tag{4.6}$$

for all $n \in \mathbb{N}$. This inequality can be ascertained as follows:

$$\begin{aligned}
& \|x_{n+1}-x_n\| \\
= \; & \|\lambda_n z_n + (1-\lambda_n)\, y_n - x_n\| \\
\le \; & \lambda_n \|z_n-x_n\| + (1-\lambda_n)\|y_n-x_n\| \\
\le \; & \lambda_n \|z_n-x_n\| \\
& + \left\|a_n x_n + b_n Sx_n + c_n Tx_n + d_n T^2 x_n - (a_n+b_n+c_n+d_n)\, x_n\right\| \\
\le \; & \lambda_n \|z_n-x_n\| + b_n \|Sx_n-x_n\| + c_n \|Tx_n-x_n\| + d_n \left\|T^2x_n-x_n\right\| \\
\le \; & \lambda_n \|z_n-x_n\| + \|Sx_n-x_n\| + \|Tx_n-x_n\| + \left\|T^2x_n-x_n\right\|.
\end{aligned}$$

Define $X_n \equiv \|x_n-\overline{z}\|^2\,(\ge 0)$, where $\overline{z} \equiv P_{A(S)\cap A(T)}z$. Our purpose is to demonstrate that $X_n \to 0$ as $n \to \infty$. The rest of the proof is divided into two cases.

Case (A). Suppose that there exists a natural number n' such that $X_{n+1} \le X_n$ for all $n \ge n'$. In this case, the sequence $\{X_n\}$ is convergent. As $\overline{z} \in A(S) \cap A(T)$, it holds from (4.5) that

$$\begin{aligned}
& a_n b_n \|x_n-Sx_n\|^2 + a_n c_n \|x_n-Tx_n\|^2 + a_n d_n \left\|x_n-T^2x_n\right\|^2 \\
& +b_n c_n \|Sx_n-Tx_n\|^2 + b_n d_n \left\|Sx_n-T^2x_n\right\|^2 + c_n d_n \left\|Tx_n-T^2x_n\right\|^2 \\
\le \; & \lambda_n \|z_n-\overline{z}\|^2 + \|x_n-\overline{z}\|^2 - \|x_{n+1}-\overline{z}\|^2 \\
\equiv \; & \lambda_n \|z_n-\overline{z}\|^2 + X_n - X_{n+1}
\end{aligned} \tag{4.7}$$

for all $n \in \mathbb{N}$. Because $\{z_n\}$ is bounded, we have from (4.7), (4.1) and (4.3) that

$$\begin{aligned}
x_n - Sx_n & \to 0, \quad x_n - Tx_n \to 0, \quad x_n - T^2x_n \to 0, \\
Sx_n - Tx_n & \to 0, \quad Sx_n - T^2x_n \to 0, \quad Tx_n - T^2x_n \to 0.
\end{aligned} \tag{4.8}$$

Then, it holds from (4.6) that

$$x_{n+1} - x_n \to 0. \tag{4.9}$$

By using (2.1) and (4.4), we obtain

$$\begin{aligned}
X_{n+1} \equiv \; & \|x_{n+1}-\overline{z}\|^2 \\
= \; & \|\lambda_n (z_n-\overline{z}) + (1-\lambda_n)(y_n-\overline{z})\|^2 \\
\le \; & (1-\lambda_n)^2 \|y_n-\overline{z}\|^2 + 2\lambda_n \langle x_{n+1}-\overline{z},\ z_n-\overline{z}\rangle \\
\le \; & (1-\lambda_n)\|x_n-\overline{z}\|^2 + 2\lambda_n \left(\langle x_{n+1}-x_n,\ z_n-\overline{z}\rangle + \langle x_n-\overline{z},\ z_n-\overline{z}\rangle\right) \\
\equiv \; & (1-\lambda_n) X_n + 2\lambda_n \left(\langle x_{n+1}-x_n,\ z_n-\overline{z}\rangle + \langle x_n-\overline{z},\ z_n-\overline{z}\rangle\right)
\end{aligned}$$

for all $n \in \mathbb{N}$. As $\{z_n\}$ is bounded, it holds from (4.9) that $\langle x_{n+1}-x_n, z_n-\overline{z}\rangle \to 0$. Hence, from (4.1) and Lemma 2.6, it suffices to prove that

$$\limsup_{n\to\infty} \langle x_n-\overline{z},\ z_n-\overline{z}\rangle \le 0.$$

As the sequences $\{x_n\}$ and $\{z_n\}$ are bounded, we can assume, without loss of generality, that there exist subsequences $\{x_{n_i}\}$ of $\{x_n\}$ and $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty} \langle x_n - \overline{z},\ z_n - \overline{z}\rangle = \lim_{i\to\infty} \langle x_{n_i} - \overline{z},\ z_{n_i} - \overline{z}\rangle$$

and $x_{n_i} \rightharpoonup u$ for some $u \in H$. Lemma 2.4-(a), (b) and (4.8) imply that $u \in A(S) \cap A(T)$. As $z_n \to z$ and $\overline{z} \equiv P_{A(S)\cap A(T)} z$, we have that

$$\begin{aligned}\limsup_{n\to\infty} \langle x_n - \overline{z},\ z_n - \overline{z}\rangle &= \lim_{i\to\infty} \langle x_{n_i} - \overline{z},\ z_{n_i} - \overline{z}\rangle \\ &= \langle u - \overline{z},\ z - \overline{z}\rangle \le 0.\end{aligned}$$

This completes the proof for Case (A).

Case (B). Suppose that there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let n_0 be a natural number such that $\{k \le n_0 : X_k < X_{k+1}\}$ is nonempty. Define

$$\tau(n) = \max\{k \le n : X_k < X_{k+1}\} \quad \text{for all } n \ge n_0.$$

From Lemma 2.7, the following hold:

$$\tau(n) \to \infty \quad \text{as } n \to \infty; \tag{4.10}$$

$$X_n \le X_{\tau(n)+1} \quad \text{for all } n \ge n_0; \tag{4.11}$$

$$X_{\tau(n)} < X_{\tau(n)+1} \quad \text{for all } n \ge n_0. \tag{4.12}$$

From (4.11), it is sufficient to show that $X_{\tau(n)+1} \to 0$. From (4.1), (4.10), (4.2), and (4.3), we have

$$\lambda_{\tau(n)} \to 0 \quad \text{as } n \to \infty, \tag{4.13}$$

$$a_{\tau(n)} + b_{\tau(n)} + c_{\tau(n)} + d_{\tau(n)} = 1 \text{ and} \tag{4.14}$$

$$0 < a \le a_{\tau(n)},\ b_{\tau(n)},\ c_{\tau(n)},\ d_{\tau(n)} \le b < 1 \tag{4.15}$$

for all $n \ge n_0$. As $\overline{z} \in A(S) \cap A(T)$, inequalities (4.4)–(4.6) yield

$$\left\|y_{\tau(n)} - \overline{z}\right\| \le \left\|x_{\tau(n)} - \overline{z}\right\|, \tag{4.16}$$

$$\begin{aligned} &a_{\tau(n)}b_{\tau(n)} \left\|x_{\tau(n)} - Sx_{\tau(n)}\right\|^2 + a_{\tau(n)}c_{\tau(n)} \left\|x_{\tau(n)} - Tx_{\tau(n)}\right\|^2 \\ &+ a_{\tau(n)}d_{\tau(n)} \left\|x_{\tau(n)} - T^2x_{\tau(n)}\right\|^2 + b_{\tau(n)}c_{\tau(n)} \left\|Sx_{\tau(n)} - Tx_{\tau(n)}\right\|^2 \\ &+ b_{\tau(n)}d_{\tau(n)} \left\|Sx_{\tau(n)} - T^2x_{\tau(n)}\right\|^2 + c_{\tau(n)}d_{\tau(n)} \left\|Tx_{\tau(n)} - T^2x_{\tau(n)}\right\|^2 \\ &\le \lambda_{\tau(n)} \left\|z_{\tau(n)} - \overline{z}\right\|^2 + \left\|x_{\tau(n)} - \overline{z}\right\|^2 - \left\|x_{\tau(n)+1} - \overline{z}\right\|^2 \\ &\equiv \lambda_{\tau(n)} \left\|z_{\tau(n)} - \overline{z}\right\|^2 + X_{\tau(n)} - X_{\tau(n)+1}, \text{ and}\end{aligned} \tag{4.17}$$

$$\begin{aligned}\left\|x_{\tau(n)+1} - x_{\tau(n)}\right\| &\le \lambda_{\tau(n)} \left\|z_{\tau(n)} - x_{\tau(n)}\right\| + \left\|Sx_{\tau(n)} - x_{\tau(n)}\right\| \\ &+ \left\|Tx_{\tau(n)} - x_{\tau(n)}\right\| + \left\|T^2x_{\tau(n)} - x_{\tau(n)}\right\|,\end{aligned} \tag{4.18}$$

respectively. From (4.12) and (4.17), it holds that

$$\begin{aligned}&a_{\tau(n)}b_{\tau(n)} \left\|x_{\tau(n)} - Sx_{\tau(n)}\right\|^2 + a_{\tau(n)}c_{\tau(n)} \left\|x_{\tau(n)} - Tx_{\tau(n)}\right\|^2 \\ &+a_{\tau(n)}d_{\tau(n)} \left\|x_{\tau(n)} - T^2x_{\tau(n)}\right\|^2 + b_{\tau(n)}c_{\tau(n)} \left\|Sx_{\tau(n)} - Tx_{\tau(n)}\right\|^2\end{aligned}$$

$$+b_{\tau(n)}d_{\tau(n)}\left\|Sx_{\tau(n)}-T^2x_{\tau(n)}\right\|^2+c_{\tau(n)}d_{\tau(n)}\left\|Tx_{\tau(n)}-T^2x_{\tau(n)}\right\|^2$$
$$\leq \lambda_{\tau(n)}\left\|z_{\tau(n)}-\overline{z}\right\|^2.$$

As $\{z_{\tau(n)}\}$ is bounded, we obtain from (4.10), (4.13) and (4.15) that

$$\begin{aligned} x_{\tau(n)}-Sx_{\tau(n)} &\to 0, \quad x_{\tau(n)}-Tx_{\tau(n)}\to 0,\\ x_{\tau(n)}-T^2x_{\tau(n)} &\to 0,\\ Sx_{\tau(n)}-Tx_{\tau(n)} &\to 0, \quad Sx_{\tau(n)}-T^2x_{\tau(n)}\to 0,\\ Tx_{\tau(n)}-T^2x_{\tau(n)} &\to 0 \end{aligned} \tag{4.19}$$

as $n\to\infty$. Thus, (4.13), (4.18), and (4.19) imply that

$$x_{\tau(n)+1}-x_{\tau(n)}\to 0. \tag{4.20}$$

As $\{x_{\tau(n)}\}$ and $\{x_{\tau(n)+1}\}$ are bounded, we have that

$$X_{\tau(n)+1}-X_{\tau(n)}\to 0. \tag{4.21}$$

Thus, our goal is to prove that $X_{\tau(n)}\to 0$. Using (2.1) and (4.16), we obtain

$$\begin{aligned} X_{\tau(n)+1} &\equiv \left\|x_{\tau(n)+1}-\overline{z}\right\|^2\\ &= \left\|\lambda_{\tau(n)}\left(z_{\tau(n)}-\overline{z}\right)+\left(1-\lambda_{\tau(n)}\right)\left(y_{\tau(n)}-\overline{z}\right)\right\|^2\\ &\leq \left(1-\lambda_{\tau(n)}\right)^2\left\|y_{\tau(n)}-\overline{z}\right\|^2+2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1}-\overline{z},\ z_{\tau(n)}-\overline{z}\right\rangle\\ &\leq \left(1-\lambda_{\tau(n)}\right)\left\|x_{\tau(n)}-\overline{z}\right\|^2+2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1}-\overline{z},\ z_{\tau(n)}-\overline{z}\right\rangle\\ &\equiv \left(1-\lambda_{\tau(n)}\right)X_{\tau(n)}+2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1}-\overline{z},\ z_{\tau(n)}-\overline{z}\right\rangle, \end{aligned}$$

and hence,

$$\lambda_{\tau(n)}X_{\tau(n)}\leq X_{\tau(n)}-X_{\tau(n)+1}+2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1}-\overline{z},\ z_{\tau(n)}-\overline{z}\right\rangle.$$

From (4.12),

$$\lambda_{\tau(n)}X_{\tau(n)}\leq 2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1}-\overline{z},\ z_{\tau(n)}-\overline{z}\right\rangle.$$

As $\lambda_{\tau(n)}>0$, we have that

$$\begin{aligned} X_{\tau(n)} &\leq 2\left\langle x_{\tau(n)+1}-\overline{z},\ z_{\tau(n)}-\overline{z}\right\rangle\\ &= 2\left\langle x_{\tau(n)+1}-x_{\tau(n)},\ z_{\tau(n)}-\overline{z}\right\rangle+2\left\langle x_{\tau(n)}-\overline{z},\ z_{\tau(n)}-\overline{z}\right\rangle\\ &= 2\left\langle x_{\tau(n)+1}-x_{\tau(n)},\ z_{\tau(n)}-\overline{z}\right\rangle\\ &\quad +2\left\langle x_{\tau(n)}-\overline{z},\ z_{\tau(n)}-z\right\rangle+2\left\langle x_{\tau(n)}-\overline{z},\ z-\overline{z}\right\rangle \end{aligned}$$

Because $\{x_{\tau(n)}\}$ is bounded and $z_{\tau(n)}\to z$, we have from (4.20) that

$$2\left\langle x_{\tau(n)+1}-x_{\tau(n)},\ z_{\tau(n)}-\overline{z}\right\rangle+2\left\langle x_{\tau(n)}-\overline{z},\ z_{\tau(n)}-z\right\rangle\to 0$$

as $n\to\infty$. Hence, it suffices to prove that

$$\limsup_{n\to\infty}\left\langle x_{\tau(n)}-\overline{z},\ z-\overline{z}\right\rangle.$$

As $\{x_{\tau(n)}\}$ is bounded, we can assume, without loss of generality, that there is a subsequence $\{x_{\tau(n_i)}\}$ of $\{x_{\tau(n)}\}$ such that

$$\limsup_{n\to\infty} \langle x_{\tau(n)} - \overline{z},\ z - \overline{z}\rangle = \lim_{i\to\infty} \langle x_{\tau(n_i)} - \overline{z},\ z - \overline{z}\rangle$$

and $x_{\tau(n_i)} \rightharpoonup u$ for some $u \in H$. From (4.19), it holds that $u \in A(S) \cap A(T)$. As $\overline{z} \equiv P_{A(S)\cap A(T)}z$, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle x_{\tau(n)} - \overline{z},\ z - \overline{z}\rangle &= \lim_{i\to\infty} \langle x_{\tau(n_i)} - \overline{z},\ z - \overline{z}\rangle \\ &= \langle u - \overline{z},\ z - \overline{z}\rangle \le 0. \end{aligned}$$

This completes the proof for Case (B), and we have shown that $x_n \to \overline{z} \equiv P_{A(S)\cap A(T)}z$.

Next, suppose that, in addition to the other assumptions, C is closed in H. Our target is to show that $x_n \to \widehat{z} \equiv P_{F(S)\cap F(T)}z$. As $x_n \to \overline{z} \equiv P_{A(S)\cap A(T)}z$ and C is closed, we have that $\overline{z} \in C \cap A(S) \cap A(T)$. From Lemma 2.5, $\overline{z} \in F(S) \cap F(T)$. Thus, $F(S) \cap F(T)$ is nonempty. As S and T are quasi-nonexpansive, $F(S) \cap F(T)$ is closed and convex. Hence, there exists the metric projection $P_{F(S)\cap F(T)}$ from H onto $F(S) \cap F(T)$. It is sufficient to prove that

$$(\widehat{z} \equiv)\, P_{F(S)\cap F(T)}z = \overline{z}\left(\equiv P_{A(S)\cap A(T)}z\right).$$

Because $\overline{z} \in F(S) \cap F(T)$, it suffices to prove that $\|z - \overline{z}\| \le \|z - v\|$ for all $v \in F(S) \cap F(T)$. Let $v \in F(S) \cap F(T)$. Because $F(S) \cap F(T) \subset A(S) \cap A(T)$, we have that

$$\begin{aligned} \|z - \overline{z}\| &= \inf\{\|z - q\| : q \in A(S) \cap A(T)\} \\ &\le \inf\{\|z - q\| : q \in F(S) \cap F(T)\} \\ &\le \|z - v\|. \end{aligned}$$

This means that $\overline{z} = P_{F(S)\cap F(T)}z\,(\equiv \widehat{z})$. This completes the proof. □

Theorem 4.1 offers alternative approximation methods for Theorem 4.1 of Takahashi [32], which is reproduced below as Theorem 4.3. These methods find common attractive and fixed points of two generalized hybrid mappings, because the classes of normally generalized hybrid mappings and normally 2-generalized hybrid mappings include generalized hybrid mappings as special cases.

4.2. **Two Normally Generalized Hybrid Mappings.** Recall the proof of Theorem 4.1. By using Lemma 2.3-(b) instead of (c), as well as Lemma 2.4-(a), we can obtain the following theorem which presents strong approximation methods concerning two normally generalized hybrid mappings.

Theorem 4.2. *Let C be a nonempty and convex subset of H, let S and T be normally generalized hybrid mappings from C into itself with $A(S) \cap A(T) \neq \emptyset$, and let $P_{A(S)\cap A(T)}$ be the metric projection from H onto $A(S) \cap A(T)$. Let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $(0,1)$ such that*

$$\lambda_n \to 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$

$$a_n + b_n + c_n = 1, \quad 0 < a \le a_n, b_n, c_n \le b < 1 \quad \textit{for all } n \in \mathbb{N}.$$

Let $\{z_n\}$ *be a sequence in* C *such that* $z_n \to z\,(\in H)$*. Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_{n+1} = \lambda_n z_n + (1-\lambda_n)\,(a_n x_n + b_n S x_n + c_n T x_n) \in C \quad \text{for all } n \in \mathbb{N},$$

where $x_1 \in C$ *is given. Then, the sequence* $\{x_n\}$ *converges strongly to a common attractive point* $\overline{z} \in A(S) \cap A(T)$*, where* $\overline{z} \equiv P_{A(S)\cap A(T)} z$*. Additionally, if* C *is closed, then* $\{x_n\}$ *converges strongly to a common fixed point* $\widehat{z} \in F(S) \cap F(T)$*, where* $\widehat{z} \equiv P_{F(S)\cap F(T)} z$*.*

As an application of Theorem 4.2, we can obtain Theorem 4.1 of Takahashi [32] as follows.

Theorem 4.3 (Theorem 4.1 of Takahashi [32])**.** *Let* C *be a nonempty and convex subset of* H*, let* S *and* T *be generalized hybrid mappings from* C *into itself with* $A(S) \cap A(T) \neq \emptyset$*, and let* $P_{A(S)\cap A(T)}$ *be the metric projection from* H *onto* $A(S) \cap A(T)$*. Let* $\{\alpha_n\}$*,* $\{\beta_n\}$*, and* $\{\gamma_n\}$ *be sequences of real numbers in the interval* $(0,1)$ *such that*

$$\alpha_n \to 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty,$$
$$0 < a \le \beta_n, \gamma_n \le b < 1 \quad \text{for all } n \in \mathbb{N}.$$

Let $\{z_n\}$ *be a sequence in* C *such that* $z_n \to z\,(\in H)$*. Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_{n+1} = \alpha_n z_n + (1-\alpha_n)\,(\beta_n x_n + (1-\beta_n)\,(\gamma_n S x_n + (1-\gamma_n)\,T x_n))$$

for all $n \in \mathbb{N}$*, where* $x_1 \in C$ *is given. Then, the sequence* $\{x_n\}$ *converges strongly to a common attractive point* $\overline{z} \in A(S)\cap A(T)$*, where* $\overline{z} \equiv P_{A(S)\cap A(T)} z$*. Additionally, if* C *is closed, then* $\{x_n\}$ *converges strongly to a common fixed point* $\widehat{z} \in F(S)\cap F(T)$*, where* $\widehat{z} \equiv P_{F(S)\cap F(T)} z$*.*

Proof. We know that a generalized hybrid mapping is normally generalized hybrid. Define $\lambda_n \equiv \alpha_n$, $a_n \equiv \beta_n$, $b_n \equiv (1-\beta_n)\,\gamma_n$, and $c_n \equiv (1-\beta_n)\,(1-\gamma_n)$. Then, it holds that $a_n + b_n + c_n = 1$ and that there exist real numbers $c, d \in \mathbb{R}$ such that $0 < c \le a_n, b_n, c_n \le d < 1$ for all $n \in \mathbb{N}$. From Theorem 4.2, we obtain the desired results. ☐

4.3. **Two Normally 2-Generalized Hybrid Mappings.** We now refocus the proof of Theorem 4.1. By using Lemma 2.3-(d) instead of (c), as well as Lemma 2.4-(b), we can obtain the following theorem, which presents strong approximation methods for finding common attractive and fixed points of two normally 2-generalized hybrid mappings.

Theorem 4.4. *Let* C *be a nonempty and convex subset of* H*, let* S *and* T *be normally* 2*-generalized hybrid mappings from* C *into itself with* $A(S) \cap A(T) \neq \emptyset$*, and let* $P_{A(S)\cap A(T)}$ *be the metric projection from* H *onto* $A(S) \cap A(T)$*. Let* $\{\lambda_n\}$*,* $\{a_n\}$*,* $\{b_n\}$*,* $\{c_n\}$*,* $\{d_n\}$*, and* $\{e_n\}$ *be sequences of real numbers in the interval* $(0,1)$ *such that*

$$\lambda_n \to 0, \qquad \sum_{n=1}^{\infty} \lambda_n = \infty,$$

$$a_n + b_n + c_n + d_n + e_n = 1,$$
$$0 < a \le a_n, b_n, c_n, d_n, e_n \le b < 1 \quad \textit{for all } n \in \mathbb{N}.$$

Let $\{z_n\}$ *be a sequence in* C *such that* $z_n \to z\,(\in H)$*. Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_{n+1} = \lambda_n z_n + (1-\lambda_n)\left(a_n x_n + b_n S x_n + c_n S^2 x_n + d_n T x_n + e_n T^2 x_n\right)$$

for all $n \in \mathbb{N}$*, where* $x_1 \in C$ *is given. Then, the sequence* $\{x_n\}$ *converges strongly to a common attractive point* $\overline{z} \in A(S) \cap A(T)$*, where* $\overline{z} \equiv P_{A(S)\cap A(T)} z$*. Additionally, if* C *is closed, then* $\{x_n\}$ *converges strongly to a common fixed point* $\widehat{z} \in F(S) \cap F(T)$*, where* $\widehat{z} \equiv P_{F(S)\cap F(T)} z$*.*

Theorem 4.4 offers alternative approximation methods for Theorem 4.1 of Takahashi [32], because the class of normally 2-generalized hybrid mappings contains generalized hybrid mappings.

As a final remark, all results in this paper can be extended to the case of finite families of normally 2-generalized hybrid mappings.

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