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AN ALTERNATIVE PROOF FOR THE SOLUTION EXISTENCE OF FINITE-DIMENSIONAL VARIATIONAL INEQUALITIES

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Abstract. This short note presents a new approach to the proof for the solution existence of finite-dimensional variational inequalities in [2, Proposition 2.2.3]. Instead of using the degree theory like the classical proof, we rely on the fixed-point approach, which is more receptive and easy to follow than the previous one.

1. Introduction

The theorem of Facchinei and Pang [2, Proposition 2.2.3] on the solution existence of finite-dimensional variational inequalities was used by many authors (see, e.g., [2, 5, 6]) to study different questions related to solution methods. This theorem was extended [3] to the case of infinite-dimensional variational inequalities. It was proved basically by using degree-theoretic approach. Firstly, the authors utilize the Tietze extension theorem (also known as the Tietze-Urysohn-Brouwer extension theorem) which says that continuous functions on a closed subset of a normal topological space can be extended to the entire space in order to apply the degree theory. Then, the authors use the homotopy arguments, in which they "homotopize" a suitable map associated to the variational inequalities with the identity map which is known to have a degree of plus one. After that, the degree condition in one of the two statements of [2, Theorem 2.2.1] is used to show the existence of solutions. However, according to the authors of [2], the degree-theoretic approach often lacks an intuitive appeal, especially for those readers who are not well versed in degree arguments, because the concept of degree requires a certain level of familiarity with nonlinear analysis. So, it is natural to raise a question that if there is an alternative way to prove [2, Proposition 2.2.3] which does not require any advanced knowledge. In this note, we will give another approach to the proof of [2, Proposition 2.2.3] which is based on the regularization technique and a well-known result of the solution existence for variational inequalities with bounded constraint sets.

2. The Proof

We begin with the notation in use through this note. Given two vectors x, y in $\mathbb{R}^n$, $\langle x, y\rangle$ denotes their inner product and $\|x\|$ is the Euclidean norm of x. Next,

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we will restate the definition of the variational inequality problem as well as present some well-known results about the existence of solution of the problem.

Definition 2.1. Let $K \subset \mathbb{R}^n$ be a closed convex nonempty set and let $F : K \to \mathbb{R}^n$ be a continuous mapping. The variational inequality defined by K and F, denoted by $\mathrm{VI}(K, F)$, is the problem of finding a vector x in K such that:

$$\langle F(x), y - x\rangle \geq 0, \quad \forall y \in K.$$

The solution set of this problem is denoted by $\mathrm{SOL}(K, F)$.

Next, we recall two results about the existence of solutions. The first one is for $\mathrm{VI}(K, F)$ with the bounded set K and the last one is for the unbounded case.

Theorem 2.2 ([4, Theorem 3.1]). *Let $K \subset \mathbb{R}^n$ be nonempty compact convex and $F : K \to \mathbb{R}^n$ be continuous. Then, $SOL(K, F)$ is nonempty and compact.*

Instead of the compactness assumption in Theorem 2.2, we can assume certain conditions on the function F to establish the same conclusion about $\mathrm{SOL}(K, F)$.

Theorem 2.3 ([4, Corollary 4.3]). *Consider the problem* $\mathrm{VI}(K, F)$ *and assume further that F satisfies the coercive condition on K, i.e., there exists a vector $x^{\mathrm{ref}} \in K$ such that*

$$\lim_{\substack{\|x\|\to+\infty \\ x\in K}} \frac{\left\langle F(x) - F(x^{ref}), x - x^{ref}\right\rangle}{\|x - x^{ref}\|} = \infty.$$

Then, $\mathrm{VI}(K, F)$ *has a solution.*

Theorem 2.2 can be proved by using the Brouwer fixed-point theorem. The proof of Theorem 2.3, which is based on Theorem 2.2, uses the idea of compactification. More specifically, one uses the coercive condition of F to construct a compact set E with nonempty interior, then one shows that the $\mathrm{VI}(E, F)$ has a solution, which belongs to the interior of E. Finally, one uses the convexity of K to show that the solution which was just obtained is also the solution of $\mathrm{VI}(K, F)$. As mentioned earlier in the Introduction, we will borrow that idea and the regularization approach to give an alternative proof for [2, Proposition 2.2.3]. Namely, we will construct a sequence of perturbed problems in order to generate a sequence of perturbed solutions that converges to a solution of the original problem. Although the Brouwer fixed-point theorem is proved by using degree-theoretic approach but it is widely used in lots of mathematical fields, so this theorem is familiar to the readers and is readily accepted. Now, we are ready to state and give a new proof for Proposition 2.2.3 from [2].

Theorem 2.4. *Let $K \subset \mathbb{R}^n$ be nonempty closed convex and $F : K \to \mathbb{R}^n$ be continuous. Consider the following statements:*

(a) *There exists a vector $x^{\mathrm{ref}} \in K$ such that the set*

$$L_< := \left\{x \in K \,:\, \left\langle F(x), x - x^{\mathrm{ref}}\right\rangle < 0\right\}$$

is bounded (possibly empty).

(b) *There exist a bounded convex open set* Ω *and a vector* $x^{\mathrm{ref}} \in K \cap \Omega$ *such that*

$$\left\langle F(x), x - x^{\mathrm{ref}} \right\rangle \geq 0, \quad \forall x \in K \cap \partial\Omega,$$

where $\partial\Omega$ *is the boundary of* Ω.

(c) $\mathrm{VI}(K, F)$ *has a solution.*

It holds that (a)$\Rightarrow$(b)$\Rightarrow$(c). *Moreover, if the set*

$$L_{\leq} := \left\{ x \in K \ : \ \left\langle F(x), x - x^{\mathrm{ref}} \right\rangle \leq 0 \right\},$$

which is nonempty and larger than $L_{<}$, *is bounded, then* $\mathrm{SOL}(K, F)$ *is nonempty and compact.*

Proof. The proof of the implication (a)$\Rightarrow$(b) as well as the last assertion which is relevant to the boundedness of $L_{\leq}$ is easily to obtain and the readers can refer the original proof in [2, Proposition 2.2.3]. In this note, we only prove the part (b)$\Rightarrow$(c).

Assume that (b) holds. Consider the set K_Ω which is defined by

$$K_\Omega = K \cap \overline{\Omega},$$

where $\overline{\Omega}$ is the closure of Ω. Then, K_Ω is a nonempty convex compact set in $\mathbb{R}^n$. For each $k \in \mathbb{N}$, consider the mapping $F_k : K \to \mathbb{R}^n$ given by

$$F_k(x) = F(x) + \frac{1}{k}(x - x^{\mathrm{ref}}), \ x \in K.$$

It is clear that F_k is continuous on K for every natural number k. By applying Theorem 2.2, we can assert that the $\mathrm{VI}(K_\Omega, F_k)$ has a solution x_k for all $k \in \mathbb{N}$. We will next show that x_k belongs to Ω for each $k \in \mathbb{N}$. Suppose by contradiction that there exists an index k such that x_k belongs to $\partial\Omega$, then, in view of condition (b), we have that

$$\left\langle F(x_k), x_k - x^{\mathrm{ref}} \right\rangle \geq 0.$$

Since x_k is a solution of $\mathrm{VI}(K_\Omega, F_k)$, we have $\langle F_k(x_k), y - x_k \rangle \geq 0$ for all $y \in K_\Omega$ or, equivalently,

$$\left\langle F(x_k) + \frac{1}{k}(x_k - x^{\mathrm{ref}}), y - x_k \right\rangle \geq 0.$$

On the other hand, it is easy to see that $x^{\mathrm{ref}} \in K_\Omega$, then we can substitute y by x^{ref} in the above inequality and obtain

$$\left\langle F(x_k) + \frac{1}{k}(x_k - x^{\mathrm{ref}}), x^{\mathrm{ref}} - x_k \right\rangle \geq 0,$$

which leads to

$$\left\langle F(x_k), x^{\mathrm{ref}} - x_k \right\rangle \geq \frac{1}{k}\|x_k - x^{\mathrm{ref}}\|^2.$$

It follows from the assumption $x_k \in \partial\Omega$ that $x_k \neq x^{\mathrm{ref}}$, therefore we obtain the following inequalities

$$0 \geq \left\langle F(x_k), x^{\mathrm{ref}} - x_k \right\rangle \geq \frac{1}{k}\|x_k - x^{\mathrm{ref}}\|^2 > 0,$$

which is a contradiction. So we must have $x_k \in \Omega$ for each k.

We claim that x_k belongs to $\mathrm{SOL}(K, F_k)$ for each k. Indeed, for each k in $\mathbb{N}$ we have that x_k belongs to Ω, then there is a positive number r_k such that

$$B(x_k, r_k) := \{y \in \mathbb{R}^n : \|y - x_k\| < r_k\} \subset \Omega.$$

Fix an arbitrarily natural number k, for each vector y in K which differs from x_k, we can find a positive number δ_k satisfying the following conditions:

$$\delta_k < \frac{r_k}{\|x_k - y\|} \quad \text{and} \quad \delta_k < 1.$$

From the first condition, we have that

$$\|x_k - (x_k + \delta_k(y - x_k))\| = \delta_k \|y - x_k| < r_k,$$

therefore, we obtain

$$x_k + \delta_k(y - x_k) \in B(x_k, r_k) \subset \Omega.$$

The last condition of δ_k asserts that $x_k + \delta_k(y - x_k)$ belongs to K since K is convex. Combine the above arguments, we have that

$$x_k + \delta_k(y - x_k) \in K \cap \Omega = K_\Omega.$$

Because x_k is a solution of $\mathrm{VI}(K_\Omega, F_k)$, we obtain the following inequality:

$$\langle F_k(x_k), x_k + \delta_k(y - x_k) - x_k\rangle \geq 0,$$

or, equivalently,

$$\langle F_k(x_k), y - x_k\rangle \geq 0,$$

Since the vector y is considered arbitrarily in K then we have that the last inequality holds for all y in K. In other word, x_k is a solution of $\mathrm{VI}(K, F_k)$ for all k.

We now end the proof by showing the existence of solution of the original problem. Because the sequence $\{x_k\}$ lies in Ω, it is bounded. This leads to the existence of a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ which converges to a vector $\overline{x}$ in $\mathbb{R}^n$. The closedness of K implies that this vector must belong to K. Since x_{k_i} is a solution of $\mathrm{VI}(K, F_{k_i})$ for each i, for all $y \in K$,

$$\langle F_{k_i}(x_{k_i}), y - x_{k_i}\rangle \geq 0, \ \forall i \in \mathbb{N}.$$

This leads to

$$\left\langle F(x_{k_i}) + \frac{1}{k_i}(x_{k_i} - x^{\mathrm{ref}}), y - x_{k_i}\right\rangle \geq 0, \ \forall i \in \mathbb{N}, \ \forall y \in K.$$

Letting $k_i \to +\infty$, we obtain

$$\langle F(\overline{x}), y - \overline{x}\rangle \geq 0, \ \forall y \in K.$$

The last statement shows that $\overline{x}$ is a solution of $\mathrm{VI}(K, F)$. In other word, we obtain the nonemptiness of $\mathrm{SOL}(K, F)$; thus $(b) \Rightarrow (c)$. □

Remark 2.5. In the part of the above proof where we show that x_k belongs to $\mathrm{SOL}(K, F_k)$ for all k, we have relied on the technique used in the proof of [2, Proposition 2.2.8].

Remark 2.6. We can see that the first advantage of this alternative proof is that we do not need to use the Tietze extension theorem. Second, this proof does not require much knowledge about the degree theory as well as some relevant results and techniques. Another advantage of this proof is that it is algorithmic. More precisely, this proof opens a way to solve the original problem, in which a sequence of auxiliary problems with better properties is solved. One of these properties which is easy to see is that we will treat the subproblem on a compact set, while the set K in the initial problem can be unbounded. Finally, any limit point of the sequence of solutions of these subproblems is a solution of the initial problem.

Remark 2.7. In [1], the author provides an elementary proof for the solution existence of the $\mathrm{VI}(K, F)$ problem where K is a bounded closed set and F is continuous and monotone. Combining this result with our alternative approach, we obtain an elementary way to prove the solution existence in the foregoing case.

3. Conclusions

In this note, we established a new proof for the solution existence for the finite-dimensional variational inequalities which does not rely much on the concept of degree theory. This proof also provides a way to solve the original problem by solving the sequence of subproblems with better properties. One property which can be outlined is that the domain in which we solve the problem is a compact set.

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