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MEAN VALUE THEOREMS FOR VECTOR-VALUED FUNCTIONS AND SET-VALUED MAPPINGS

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Abstract. This paper is devoted to the study of mean value theorems of the inequality form. Using coderivatives, we establish such theorems for nonsmooth vector-valued functions and set-valued mappings between Banach spaces, which extend the classical mean value theorem of the inequality form for smooth vector-valued functions. By the mean value theorems, we obtain some characterizations of the local Lipschitz continuity property of vector-valued and set-valued mappings via the boundedness of their coderivatives.

1. Introduction

The classical mean value theorem for differentiable functions is probably one of the most powerful theorems in calculus. It has been used to study a great deal of important problems in both theoretical and applied fields; especially, those related to monotonicity and Lipschitzian properties.

During the past years, various substitutes for the concept of differentiability for nondifferentiable functions and even set-valued mappings were suggested, and using such generalized differentiability concepts, one derived many mean value theorems in one form or another. Wegge [33] was the first to obtain a mean value theorem for convex functions via the Fenchel subdifferential. Lebourg [14] established the first mean value theorem for nonsmooth and nonconvex Lipschitzian functions defined on an arbitrary Banach space through the Clarke subdifferential. A mean value theorem in terms of radial upper convex approximations extending the Lebourg mean value theorem was given by Scheffler [26]. Various mean value theorems for nondifferentiable functions and vector-valued mappings via certain kinds of generalized differentiability were presented in [12]. Using prederivatives, Sach [25] gave some generalized versions of the mean value theorem in classical analysis, which allowed him to prove some results on the surjectivity of set-valued mappings. Mean value theorems in terms of directional derivatives were also studied extensively and had some useful applications; see, e.g., [4, 11, 22, 28, 34]. In his landmark paper [35], Zagrodny established the approximate mean value theorem (also, known as the fuzzy mean value theorem in [24]) for the Clarke-Rockafellar subdifferential, which does not have any analogs in the classical differential calculus. After this work, several authors have extended the latter by using various subdifferentials and some refinements of the original proof of Zagrodny; see, e.g., [2, 15, 20, 23, 29, 32]. Among other things, the approximate mean value theorems were used to study the problems of recognizing functions by their subdifferentials [30, 31], characterizing the

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convexity, quasi-convexity, Lipschitzian properties, cone-monotonicty of functions via their subdiferentials [3,6,9,18,20], and characterizing the maximal monotonicity of mappings by their coderivatives [7, 8, 19]. Another noticeable kind of the mean value theorem is the multidirectional mean value inequality introduced by Clarke and Ledyaev [5], which had interesting applications to the calculus of variations, flow invariance, and generalized solutions of partial differential equations. A similar result was given by Luc [16], who called it the strong mean value theorem. For more information on mean value theorems, we refer the reader to [3, 18, 24, 27].

The application potential of nondifferentiable mean value theorems strongly depends on the kind of generalized differentiability used in their formulation. It is therefore more desirable to obtain the mean value theorems in terms of well-developed generalized differentiations. The coderivative introduced by Mordukhovich [17] is such a kind of generalized differentiation with a full calculus, which has been recognized as a convenient tool to study many important issues in variational analysis and optimization [18]. However, to the best of our knowledge, no mean value theorem via the coderivatives has been known so far. This leads us to the natural and interesting question whether we can have some mean value theorems in terms of the coderivatives.

The main aim of this paper is to present some mean value theorems of the inequality form via the regular and limiting coderivatives. Precisely, we establish such theorems for nonsmooth vector-valued functions and set-valued mappings between Banach spaces, which extend the classical mean value theorem of inequality form for smooth vector-valued functions. These results are then applied to deriving some characterizations of Lipschitz single-valued and set-valued mappings via the boundedness of their coderivatives.

The rest of the article is organized as follows. After recalling in Section 2 some well-known notions and facts from variational analysis [18], in Section 3, we establish the mean value theorem for vector-valued functions, which is formulated in term of coderivatives, and then give a characterization of Lipschitz vector-valued functions. Section 4 is devoted to presenting the mean value theorem via coderivatives for set-valued mappings, and a characterization of the set-valued mappings that are locally Lipschitzian in the Pompeiu-Hausdorff distance. Finally, Section 5 contains some open questions in this research direction.

In the sequel, unless otherwise stated, X is assumed to be a Banach space with its dual X^*, and $\mathbb{B}^*$ (or $\mathbb{B}_{X^*}$) and $\mathbb{B}$ (or $\mathbb{B}_X$) are the closed unit balls in X^* and X, respectively. As usual, the symbol $\langle\cdot,\cdot\rangle$ signifies the canonical pairing. The notation $\xrightarrow{w^*}$ indicates for the weak-star convergence in X^*. Given a set-valued mapping $F : X \rightrightarrows X^*$ and a point $\bar{x} \in X$, the symbol

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \Big\{x^* \in X^* | \exists \text{ sequences } x_k \to \bar{x}, x_k^* \xrightarrow{w^*} x^*, \\ \text{with } x_k^* \in F(x_k) \text{ for all } k = 1, 2, ...\Big\}$$

stands for the *sequential Painlevé-Kuratowski outer/upper limit* of F with respect to the norm topology of X and the weak-star topology of X^*.

2. Preliminaries

In this section, we briefly recall some well-known notions and facts from variational analysis, which can be found in, e.g., [18].

Let Ω be a nonempty subset of X. For each $\varepsilon \geq 0$, the set of *ε-normals* to Ω at $\bar{x} \in \Omega$ is defined by

$$\widehat{N}_\varepsilon(\bar{x};\Omega) := \Big\{x^* \in X^* \Big| \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x-\bar{x}\rangle}{\|x-\bar{x}\|} \leq \varepsilon\Big\}, \tag{2.1}$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. When $\varepsilon = 0$, the set $\widehat{N}(\bar{x};\Omega) := \widehat{N}_0(\bar{x};\Omega)$ in (2.1) is a cone called the *regular normal cone* (also, the Fréchet normal cone) to Ω at $\bar{x}$. If $\bar{x} \notin \Omega$, one puts $\widehat{N}_\varepsilon(\bar{x};\Omega) = \emptyset$ for all $\varepsilon \geq 0$.

The *limiting normal cone* (also, the Mordukhovich normal cone) to Ω at $\bar{x}$ is the set $N(\bar{x};\Omega)$ defined by

$$N(\bar{x};\Omega) := \operatorname*{Lim\,sup}_{\substack{x \xrightarrow{\Omega} \bar{x} \\ \varepsilon \downarrow 0}} \widehat{N}_\varepsilon(x;\Omega),$$

where one can put $\varepsilon = 0$ when Ω is closed and the space X is *Asplund*, i.e., a Banach space whose separable subspaces have separable duals.

Let $F : X \rightrightarrows Y$ be a set-valued mapping between two Banach spaces X and Y. The *domain* $\operatorname{dom} F$ and the *graph* $\operatorname{gph} F$ of F are given, respectively, by

$$\operatorname{dom} F := \big\{x \in X \big|\ F(x) \neq \emptyset\big\} \ \text{ and } \ \operatorname{gph} F := \big\{(x,y) \in X \times Y \big|\ y \in F(x)\big\}.$$

In the sequel, we always assume that F is proper in the sense that $\operatorname{dom} F \neq \emptyset$.

The *limiting coderivative* or *the Mordukhovich coderivative* of F at $(\bar{x},\bar{y}) \in X \times Y$ is the set-valued mapping $D^*F(\bar{x},\bar{y})\colon Y^* \rightrightarrows X^*$ defined by

$$D^*F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*,-y^*) \in N((\bar{x},\bar{y}); \operatorname{gph} F)\big\},$$

i.e.,

$$\begin{aligned} D^*F(\bar{x},\bar{y})(y^*) = \{x^* \in X^* \quad | \quad & \exists \varepsilon_k \downarrow 0, (x_k,y_k) \to (\bar{x},\bar{y}), (x_k^*,y_k^*) \xrightarrow{w^*} (x^*,y^*) \text{ with} \\ & (x_k^*,-y_k^*) \in \widehat{N}_{\varepsilon_k}((x_k,y_k); \operatorname{gph} F), k \to \infty\}. \end{aligned}$$

The *regular coderivative* or *the Fréchet coderivative* of F at $(\bar{x},\bar{y}) \in X \times Y$ is the set-valued mapping $\widehat{D}^*F(\bar{x},\bar{y}) : Y^* \rightrightarrows X^*$ defined by

$$\widehat{D}^*F(\bar{x},\bar{y})(y^*) := \{x^* \in X^* \mid (x^*,-y^*) \in \widehat{N}((\bar{x},\bar{y}); \operatorname{gph} F))\} \quad \forall y^* \in Y^*.$$

One drops $\bar{y}$ in the notations of the coderivatives if $F(\bar{x}) = \{\bar{y}\}$. Note that, when F is single-valued,

$$\widehat{D}^*F(\bar{x})(y^*) = \big\{\nabla F(\bar{x})^* y^*\big\} \text{ and resp. } D^*F(\bar{x})(y^*) = \big\{\nabla F(\bar{x})^* y^*\big\} \ \text{ for all } \ y^* \in Y^*$$

if F is Fréchet differentiable and strictly differentiable at $\bar{x}$, respectively, where $\nabla F(\bar{x})^*$ stands for the adjoint derivative operator; see, e.g., [18, Theorem 1.38].

Recall that a set-valued mapping $F : X \rightrightarrows Y$ is said to be positively homogeneous if $0 \in F(0)$ and $F(\lambda x) \supset \lambda F(x)$ for all $x \in X$ and $\lambda > 0$, or equivalently, when

$\operatorname{gph} F$ is a cone in $X \times Y$. The norm of a positively homogeneous mapping F is defined by

$$\|F\| := \sup \big\{ \|y\| \,|\, y \in F(x),\ \|x\| \le 1 \big\}.$$

Note that if $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ then the set-valued mappings $\widehat{D}^* F(\bar{x}, \bar{y})$ and $D^* F(\bar{x}, \bar{y})$ are positively homogeneous.

Let $\varphi : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ be a proper extended-real-valued function (i.e., it is not identically equal to infinity) and $\bar{x} \in \operatorname{dom} \varphi := \{x \in X | \varphi(x) < \infty\}$. The *regular subdifferential*(also, the Fréchet subdifferential) of φ at $\bar{x}$ is the set $\widehat{\partial}\varphi(\bar{x})$ defined by

$$\widehat{\partial}\varphi(\bar{x}) := \Big\{ x^* \in X^* |\ \liminf_{x \to \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \ge 0 \Big\}.$$

If $\bar{x} \notin \operatorname{dom} \varphi$, put $\widehat{\partial}\varphi(\bar{x}) := \emptyset$.

Proposition 2.1. ([18, Corollary 3.50]) *Let $\varphi : X \to \overline{\mathbb{R}}$ be a lower semicontinuous function defined on an Asplund space X, with $a \in \operatorname{dom} \varphi$. Then, for any $b \in X$ and $\varepsilon > 0$, one has*

$$|\varphi(b) - \varphi(a)| \le \|b - a\| \sup \big\{ \|x^*\| \ \big|\ x^* \in \widehat{\partial}\varphi(x),\ x \in [a, b] + \varepsilon \mathbb{B} \big\}, \tag{2.2}$$

where $[a, b] := \{\lambda a + (1 - \lambda) b |\ \lambda \in [0, 1]\}$.

Note that in Proposition 2.1 we cannot replace $\varepsilon > 0$ by 0.

Example 2.2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be defined by

$$\varphi(x) := \begin{cases} 1 & \text{if } x > 0, \\ -\sqrt{-x} & \text{if } x \le 0. \end{cases}$$

It holds that

$$\widehat{\partial}\varphi(x) = \begin{cases} \{0\} & \text{if } x \in (0, 1], \\ \emptyset & \text{if } x = 0, \end{cases} \quad \text{and } \sup \big\{ \|x^*\| \ \big|\ x^* \in \widehat{\partial}\varphi(x),\ x \in [0, 1] \big\} = 0.$$

Note that $|\varphi(1) - \varphi(0)| = 1$, we have

$$|\varphi(1) - \varphi(0)| > |1 - 0| \sup \big\{ \|x^*\| \ \big|\ x^* \in \widehat{\partial}\varphi(x),\ x \in [0, 1] \big\}.$$

Recall that a given set-valued mapping $F : X \rightrightarrows Y$ is said to be Lipschitz-like around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $\kappa \ge 0$ if there exists $\delta > 0$ such that

$$F(x) \cap \mathbb{B}_\delta(\bar{y}) \subset F(u) + \kappa \|x - u\| \mathbb{B}_Y \text{ for all } x, u \in \mathbb{B}_\delta(\bar{x}).$$

The infimum of all such moduli κ is called the exact Lipschitzian bound of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{lip} F(\bar{x}, \bar{y})$. If F is not Lipschitz-like around $(\bar{x}, \bar{y})$, put $\operatorname{lip} F(\bar{x}, \bar{y}) = +\infty$.

Note that the Lipschitz-like property, which is known also the Aubin property [10], was introduced by Aubin [1] who called it the pseudo-Lipschitz property. Here we follow Mordukhovich [18] in using the terminology "the Lipschitz-like property".

Proposition 2.3. ([18, Theorems 1.43, 4.7 & 4.10]) *Let $F : X \rightrightarrows Y$ be a set-valued mapping between two Banach spaces X and Y. The following assertions hold.*

(i) *If F is Lipschitz-like around some $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $\kappa \geq 0$, then there exists $\eta > 0$ such that*

$$\sup \left\{ \|x^*\| \,|\, x^* \in \widehat{D}^* F(x, y)(y^*) \right\} \leq \kappa \|y^*\|$$

whenever $(x, y) \in \operatorname{gph} F \cap [\mathbb{B}_\eta(\bar{x}) \times \mathbb{B}_\eta(\bar{y})]$ and $y^* \in Y^*$.

(ii) *If X and Y are Asplund spaces, and* gphF *is closed, then the following properties are equivalent.*

(a) *F is Lipschitz-like around $(\bar{x}, \bar{y}) \in gphF$.*

(b) *There are positive number κ and η such that*

$$\sup \left\{ \|x^*\| \,|\, x^* \in \widehat{D}^* F(x, y)(y^*) \right\} \leq \kappa \|y^*\|$$

whenever $(x, y) \in \operatorname{gph} F \cap [\mathbb{B}_\eta(\bar{x}) \times \mathbb{B}_\eta(\bar{y})]$ and $y^* \in Y^*$. Moreover, the exact Lipschitzian bound of F around $(\bar{x}, \bar{y})$ is computed by

$$\operatorname{lip} F(\bar{x}, \bar{y}) = \inf_{\eta > 0} \sup \left\{ \|\widehat{D}^* F(x, y)\| \,|\, (x, y) \in \operatorname{gph} F \cap [\mathbb{B}_\eta(\bar{x}) \times \mathbb{B}_\eta(\bar{y})] \right\}.$$

(iii) *If X and Y are finite-dimensional, and* gphF *is closed, then*

$$\operatorname{lip} F(\bar{x}, \bar{y}) = \|D^* F(\bar{x}, \bar{y})\|.$$

3. Mean value theorem for vector-valued functions

In this section, we establish an mean value theorem via the regular and limiting coderivatives and use it to prove a characterization of locally Lipschitz vector-valued functions, which is expressed in terms of the boundedness of the regular coderivative.

The following is our first main result, which is the mean value theorem in a inequality form, reducing to the classical one [13, p. 27] when the mapping under consideration is a continuously differentiable mapping defined on an Asplund space.

Theorem 3.1. *Let X be an Asplund space, Y a Banach space, and $F : X \to Y$ a mapping continuous on an open convex set U of X. Then, for any $a, b \in U$, one has*

$$\|F(b) - F(a)\| \leq \|b - a\| \inf_{\varepsilon > 0} \sup \left\{ \|\widehat{D}^* F(x)\| \;\middle|\; x \in [a, b] + \varepsilon \mathbb{B} \right\}, \tag{3.1}$$

where $\|\widehat{D}^* F(x)\| := \sup\{\|x^*\| \mid x^* \in \widehat{D}^* F(x)(y^*),\ \|y^*\| \leq 1\}$ and $\sup \emptyset := 0$ by convention. Consequently, if X and Y are finite-dimensional, then

$$\|F(b) - F(a)\| \leq \|b - a\| \sup \left\{ \|D^* F(x)\| \;\middle|\; x \in [a, b] \right\}. \tag{3.2}$$

Proof. Obviously, (3.1) holds if $F(b) = F(a)$. Suppose now that $F(b) \neq F(a)$. By the Hahn-Banach theorem, there exists $y_0^* \in Y^*$ with $\|y_0^*\| = 1$ such that

$$\langle y_0^*, F(b) - F(a) \rangle = \|F(b) - F(a)\|. \tag{3.3}$$

Define the function $\varphi : X \to \mathbb{R}$ by $\varphi(x) := \langle y_0^*, F(x) \rangle$ for all $x \in X$. On one hand, since F is continuous, so is φ. On the other hand, X is an Asplund space. Thus, for each $\varepsilon > 0$, by Proposition 2.2,

$$|\varphi(b) - \varphi(a)| \leq \|b - a\| \sup \left\{ \|x^*\| \;\middle|\; x^* \in \widehat{\partial} \varphi(x),\ x \in [a, b] + \varepsilon \mathbb{B} \right\}. \tag{3.4}$$

Note that

$$|\varphi(b) - \varphi(a)| = \|F(b) - F(a)\|, \tag{3.5}$$

due to (3.3). We have $\widehat{\partial}\varphi(x) \subset \widehat{D}^*F(x)(y_0^*)$. Indeed, take any $x^* \in \widehat{\partial}\varphi(x)$. By the definition of Fréchet subgradient,

$$\liminf_{u \to x} \frac{\varphi(u) - \varphi(x) - \langle x^*, u - x\rangle}{\|u - x\|} \geq 0,$$

or, equivalently,

$$\liminf_{u \to x} \frac{\langle y_0^*, F(u)\rangle - \langle y_0^*, F(x)\rangle - \langle x^*, u - x\rangle}{\|u - x\|} \geq 0.$$

The latter implies that

$$\limsup_{(u,v) \overset{\mathrm{gph}\,F}{\to} (x,F(x))} \frac{-\langle y_0^*, v - F(x)\rangle + \langle x^*, u - x\rangle}{\|u - x\| + \|v - F(x)\|} \leq 0.$$

Thus,

$$(x^*, -y_0^*) \in \widehat{N}\big((x, F(x)); \mathrm{gph}\, F\big),$$

i.e., $x^* \in \widehat{D}^*F(x)(y_0^*)$, which implies that $\widehat{\partial}\varphi(x) \subset \widehat{D}^*F(x)(y_0^*)$. Since $\|y_0^*\| = 1$, we have

$$\|x^*\| \leq \|\widehat{D}^*F(x)\| \quad \forall x^* \in \widehat{\partial}\varphi(x),\ x \in X.$$

Thus,

$$\sup\big\{\|x^*\| \;\big|\; x^* \in \widehat{\partial}\varphi(x),\ x \in [a,b] + \varepsilon\mathbb{B}\big\} \leq \sup\big\{\|\widehat{D}^*F(x)\| \;\big|\; x \in [a,b] + \varepsilon\mathbb{B}\big\}. \tag{3.6}$$

Combining (3.4), (3.5) with (3.6), we obtain (3.1).

To justify (3.2), assume further that both X and Y are finite-dimensional. Put

$$\alpha := \inf_{\varepsilon > 0} \sup\big\{\|\widehat{D}^*F(x)\| \;\big|\; x \in [a,b] + \varepsilon\mathbb{B}\big\}.$$

If $\alpha = 0$ then (3.2) holds, due to (3.1). Let us now consider the case where $\alpha \in (0, \infty)$. Then we can find $\varepsilon_k \downarrow 0$, $x_k \in [a,b] + \varepsilon_k\mathbb{B}$, $y_k^* \in Y^*$ with $\|y_k^*\| \leq 1$, and $x_k^* \in \widehat{D}^*F(x_k)(y_k^*)$ such that $\|x_k^*\| \to \alpha$ as $k \to \infty$. Since X and Y are finite-dimensional and $\alpha \in \mathbb{R}$, assume without loss of generality that $x_k \to \bar{x} \in [a,b]$, $y_k^* \to \bar{y}^* \in Y^*$ with $\|\bar{y}^*\| \leq 1$ and $x_k^* \to \bar{x}^* \in X^*$ with $\|\bar{x}^*\| = \alpha$. Note that $\bar{x}^* \in D^*F(\bar{x})(\bar{y}^*)$ and $\|\bar{y}^*\| \leq 1$. We have

$$\alpha = \|\bar{x}^*\| \leq \|D^*F(\bar{x})\| \leq \sup\big\{\|D^*F(x)\| \;\big|\; x \in [a,b]\big\}.$$

Thus, by (3.1), (3.2) holds. Finally, consider the case where $\alpha = +\infty$. We will prove that $\ell := \sup\big\{\|D^*F(x)\| \,\big|\, x \in [a,b]\big\} = +\infty$. Indeed, if this is not true, then $\ell < +\infty$. By Proposition 2.3 (iii), for each $x \in [a,b]$, F is Lipschitz-like around $(x, F(x))$ with $\mathrm{lip}F\big(x, F(x)\big) \leq \ell$. Thus, by Proposition 2.3 (i), for each $x \in [a,b]$, there exists $\varepsilon_x > 0$ such that

$$\sup\big\{\|\widehat{D}^*F(u)\| \,\big|\, u \in \mathbb{B}_{\varepsilon_x}(x)\big\} \leq \ell + 1.$$

Since $[a,b]$ is compact, one can find $\varepsilon_0 > 0$ with $[a,b] + \varepsilon_0\mathbb{B} \subset \bigcup_{x \in [a,b]} \mathbb{B}_{\varepsilon_x}(x)$. This implies that

$$\alpha \leq \sup\big\{\|\widehat{D}^*F(x)\| \;\big|\; x \in [a,b] + \varepsilon_0\mathbb{B}\big\} \leq \ell + 1 < +\infty,$$

which is a contradiction. Thus $\ell = +\infty$ and (3.2) holds. The proof is complete. $\square$

By the above result, we can obtain the following characterization of Lipschitz vector-valued mappings, which improves [18, Theorem 4.7] (Proposition 2.3 (ii)) in the single-valued settings by omitting the assumption of the Asplund property of Y.

Corollary 3.2. *Let X be an Asplund space, Y a Banach space, and $F : X \to Y$ a mapping continuous on an open convex set U of X. Then, the following assertions are equivalent.*

(i) *The mapping F is locally Lipschitz around $\bar{x} \in U$ with modulus κ.*

(ii) *There exists $r > 0$ such that $\|\widehat{D}^*F(x)\| \leq \kappa$ for all $x \in \mathbb{B}_r(\bar{x})$.*

Proof. The implication **(i)** $\Rightarrow$ **(ii)** is valid without the Asplund property of X, due to Proposition 2.3 (i). To prove the inverse, suppose that for some $r > 0$, $\|\widehat{D}^*F(x)\| \leq \kappa$ for all $x \in \mathbb{B}_r(\bar{x})$. Take any $a, b \in \mathbb{B}_{\frac{r}{2}}(\bar{x})$ and $\varepsilon \in (0, \frac{r}{2})$. By Theorem 3.1,

$$\|F(b) - F(a)\| \leq \|b - a\| \sup \left\{ \|\widehat{D}^*F(x)\| \;\middle|\; x \in [a, b] + \varepsilon\mathbb{B} \right\}.$$

Since $[a, b] + \varepsilon\mathbb{B} \subset \mathbb{B}_r(\bar{x})$, it holds that $\|\widehat{D}^*F(x)\| \leq \kappa$ for all $x \in [a, b] + \varepsilon\mathbb{B}$. Thus,

$$\|F(b) - F(a)\| \leq \kappa\|b - a\| \quad \forall a, b \in \mathbb{B}_{\frac{r}{2}}(\bar{x}).$$

The proof is complete. $\square$

4. Mean value theorem for set-valued mappings

The aim of this section is to establish some set-valued counterparts of the obtained results. Note that, in contrast to the single-valued case, the classical scalarization technique seems not to be suitable for the set-valued case. Thus, to achieve the purpose, we can use some ideas given by Penot [22].

Let C be a subset of X and $\alpha > 0$. Put $\mathbb{B}_\alpha(C) := C + \alpha\mathbb{B}_X$. Recall that the excess of C over a set D of X is given by

$$e(C, D) := \inf \left\{ \varepsilon > 0 \,|\, C \subset \mathbb{B}_\varepsilon(D) \right\} = \sup_{x \in C} d(x, D),$$

and the Pompeiu-Hausdorff distance is defined by

$$h(C, D) := \max \left\{ e(C, D), e(D, C) \right\}.$$

The Pompeiu-Hausdorff distance h possesses the following properties:

(d_1) $h(A, B) \geq 0$, (d_2) $h(A, B) = h(B, A)$, and

(d_3) $h(A, C) + h(C, B) \geq h(A, B)$ for every $A, B, C \subset X$.

One says that a set-valued mapping $F : X \rightrightarrows Y$ is locally Lipschitz around $\bar{x} \in X$ in the Pompeiu-Hausdorff distance if there exist $\kappa > 0$ and $r > 0$ such that

$$h\big(F(x_1), F(x_2)\big) \leq \kappa pa\|x_1 - x_2\| \text{ for all } x_1, x_2 \in \mathbb{B}_r(\bar{x}).$$

To proceed, we need the following result, which is similar to [22, Theorem 3.1] where the sprout coderivative rather than coderivative was used. Our proof here is somewhat different from the one of Penot [22].

Lemma 4.1. *Let X and Y be Asplund spaces, and let $F : X \rightrightarrows Y$ be a set-valued mapping with closed graph. Suppose that there exist $c, \alpha, \beta > 0$, $x_0 \in X$, and a nonempty subset V of Y such that*

$$\|\widehat{D}^*F(x,y)\| \le c \text{ for any } x \in \mathbb{B}_\alpha(x_0),\ y \in F(x) \cap (V + \beta\mathbb{B}_Y). \tag{4.1}$$

Then there exists $\rho \in (0, \alpha)$ such that

$$e\big(F(x) \cap V, F(x')\big) \le c\|x - x'\| \ \ \forall x, x' \in \mathbb{B}_\rho(x_0).$$

Proof. Suppose to the contrary that the conclusion of lemma is false. Then there exist $\widehat{c}_k > c$, $\widehat{c}_k \to c$, $x_k \to x_0$, $x'_k \to x_0$ and $y_k \in F(x_k) \cap V$ such that

$$d\big(y_k, F(x'_k)\big) > \widehat{c}_k\|x_k - x'_k\| \quad \text{for all } k. \tag{4.2}$$

Note that $x_k \ne x'_k$ for every k. Put $f(x,y) := \widehat{c}_k\|x - x'_k\| + \delta\big((x,y); \mathrm{gphF}\big)$, where $\delta\big(\cdot; \mathrm{gphF}\big)$ is the indicator of the graph of F, i.e., $\delta\big((x,y); \mathrm{gphF}\big) = 0$ if $(x,y) \in \mathrm{gphF}$ and $\delta\big((x,y); \mathrm{gphF}\big) = \infty$ otherwise. Since $\mathrm{gph}F$ is closed, f is lower semicontinuous on $X \times Y$. Here $X \times Y$ is equipped with the norm $\|(x,y)\| := \gamma_k\|x\| + (1-\gamma_k)\|y\|$, where $\gamma_k \in (0,1)$ is chosen such that $\widehat{c}_k - 2\gamma_k > c$ and

$$d\big(y_k, F(x'_k)\big) > \frac{\widehat{c}_k}{1-\gamma_k}\|x_k - x'_k\| \quad \text{for all } k. \tag{4.3}$$

Obviously, it holds that $0 < f(x_k, y_k) \le \inf f + \varepsilon$ with $\varepsilon := f(x_k, y_k)$. By the Ekeland variational principle ($\lambda := \varepsilon$), there exists $(u_k, v_k) \in \mathrm{gphF}$ such that

$$\widehat{c}_k\|u_k - x'_k\| \le \widehat{c}_k\|x - x'_k\| + \gamma_k\|x - u_k\| + (1-\gamma_k)\|y - v_k\| \quad \forall (x,y) \in \mathrm{gph}F \tag{4.4}$$

and

$$\widehat{c}_k\|u_k - x'_k\| + \gamma_k\|x_k - u_k\| + (1-\gamma_k)\|y_k - v_k\| \le \widehat{c}_k\|x_k - x'_k\|. \tag{4.5}$$

If $u_k = x'_k$ then (4.5) would yield

$$d\big(y_k, F(x'_k)\big) \le \|y_k - v_k\| \le \frac{\widehat{c}_k}{1-\gamma_k}\|x_k - x'_k\|,$$

which contradicts (4.3) and hence $u_k \ne x'_k$ for every k. We next use the sum norm in the product space $X \times Y$. Then, the norm in the dual space $(X \times Y)^* \equiv X^* \times Y^*$ is $\|(x^*, y^*)\| = \max\{\|x^*\|, \|y^*\|\}$, and $\mathbb{B}_{X^* \times Y^*} = \mathbb{B}_{X^*} \times \mathbb{B}_{Y^*}$. Let

$$g(x,y) := \widehat{c}_k\|x - x'_k\| + \gamma_k\|x - u_k\| + (1-\gamma_k)\|y - v_k\|.$$

Then it holds that $g(u_k, v_k) + \delta\big((u_k, v_k); \mathrm{gph}F\big) = \min\limits_{(x,y) \in X \times Y} \big[g(x,y) + \delta\big((x,y); \mathrm{gph}F\big)\big]$, and thus, according to the Fermat rule,

$$(0,0) \in \widehat{\partial}\big[g + \delta(\cdot; \mathrm{gph}F)\big](u_k, v_k).$$

By the fuzzy sum rule, there exist $(\widetilde{u}_k, \widetilde{v}_k)$, $\widetilde{u}_k \ne x'_k$, and $(\widetilde{x}_k, \widetilde{y}_k) \in \mathrm{gph}F$ such that

$$(0,0) \in \widehat{\partial}g(\widetilde{u}_k, \widetilde{v}_k) + \widehat{N}\big((\widetilde{x}_k, \widetilde{y}_k), \mathrm{gph}F\big) + \gamma_k\mathbb{B}_{X^* \times Y^*},$$

$$\|(\widetilde{u}_k, \widetilde{v}_k) - (u_k, v_k)\| < \gamma_k,$$

and

$$\|(\widetilde{x}_k, \widetilde{y}_k) - (u_k, v_k)\| < \gamma_k.$$

Note that g is convex and $\widehat{\partial} g(\widetilde{u}_k, \widetilde{v}_k) \subset \big(\widehat{c}_k S_{X^*} + \gamma_k \mathbb{B}_{X^*}\big) \times (1-\gamma_k)\mathbb{B}_{Y^*}$. Thus

$$(0,0) \in \big(\widehat{c}_k S_{X^*} + 2\gamma_k \mathbb{B}_{X^*}\big) \times \mathbb{B}_{Y^*} + \widehat{N}\big((\widetilde{x}_k, \widetilde{y}_k), \mathrm{gph} F\big).$$

This implies that there exist $\widetilde{y}_k^* \in \mathbb{B}_{Y^*}$, $\widetilde{u}_k^* \in S_{X^*}$ and $b_k^* \in \mathbb{B}_{X^*}$ such that

$$(\widetilde{x}_k^*, -\widetilde{y}_k^*) \in \widehat{N}\big((\widetilde{x}_k, \widetilde{y}_k), \mathrm{gph} F\big)$$

or, equivalently,

$$\widetilde{x}_k^* \in \widehat{D}^* F(\widetilde{x}_k, \widetilde{y}_k)(\widetilde{y}_k^*),$$

where $\widetilde{x}_k^* := \widehat{c}_k \widetilde{u}_k^* + 2\gamma_k b_k^*$. Thus

$$\|\widehat{D}^* F(\widetilde{x}_k, \widetilde{y}_k)\| \geq \|\widetilde{x}_k^*\| \geq \widehat{c}_k - 2\gamma_k > c.$$

Since $\widetilde{x}_k \to x_0$, $\|\widetilde{y}_k - y_k\| \to 0$ and $y_k \in V$, it follows that $\widetilde{x}_k \in \mathbb{B}_\alpha(x_0)$ and $\widetilde{y}_k \in F(\widetilde{x}_k) \cap (V + \beta\mathbb{B}_Y)$ for all k sufficiently large. This contradicts (4.1). Thus we have the desired conclusion. ☐

We are now ready to formulate and prove the main result of this section, which is a set-valued counterpart of Theorem 3.1.

Theorem 4.2. *Let $F : X \rightrightarrows Y$ be a set-valued mapping with closed graph between two Asplund spaces X and Y, and $a, b \in X$ with $a \neq b$. Then, one has*

$$h\big(F(a), F(b)\big) \leq \|b-a\| \inf_{\varepsilon>0} \sup\big\{\|\widehat{D}^* F(x,y)\| \mid x \in [a,b] + \varepsilon\mathbb{B},\, y \in F(x)\big\}. \tag{4.6}$$

If one suppose further that X and Y are finite-dimensional, and F is locally uniformly bounded at each $x \in [a,b]$, that is, there exist positive real numbers r_x and $M_x > 0$ such that $F\big(\mathbb{B}_{r_x}(x)\big) \subset M_x \mathbb{B}_Y$, then

$$h\big(F(a), F(b)\big) \leq \|b-a\| \sup\big\{\|D^* F(x,y)\| \mid x \in [a,b],\, y \in F(x)\big\}. \tag{4.7}$$

Proof. Let $\alpha := \inf_{\varepsilon>0} \sup\big\{\|\widehat{D}^* F(x,y)\| \mid x \in [a,b] + \varepsilon\mathbb{B},\, y \in F(x)\big\}$. Obviously, (4.6) is valid if $\alpha = \infty$. Otherwise, let us take any $c \in (\alpha, +\infty)$. One can find $\varepsilon > 0$ such that

$$c \geq \sup\big\{\|\widehat{D}^* F(x,y)\| \mid x \in [a,b] + \varepsilon\mathbb{B},\, y \in F(x)\big\}.$$

Put $x_0 := a$, $x_1 := a + \varepsilon\frac{b-a}{\|b-a\|}, \ldots, x_k := a + k\varepsilon\frac{b-a}{\|b-a\|}, \ldots, x_n := a + n\varepsilon\frac{b-a}{\|b-a\|}$, $x_{n+1} := b$, where n is the integer part of $\frac{\|b-a\|}{\varepsilon}$. Then $\big\{\mathbb{B}_\varepsilon(x_i) \,|\, i = 0, 1, ..., n+1\big\}$ satisfies the condition: $x_i \in \mathbb{B}_\varepsilon(x_{i-1})$ for all $i \in \{1, ..., n+1\}$. Note that, for each $i \in \{0, 1, ..., n+1\}$, $x_i \in [a,b]$ and $\mathbb{B}_\varepsilon(x_i) \subset [a,b] + \varepsilon\mathbb{B}_X$. Thus, it holds

$$\sup\big\{\|\widehat{D}^* F(x,y)\| : x \in \mathbb{B}_\varepsilon(x_i),\, y \in F(x)\big\} \leq c.$$

By Lemma 4.1 (with $V := Y$), we obtain

$$h\big(F(x_i), F(x_{i-1})\big) \leq c\|x_i - x_{i-1}\|, \ \forall i = 1, ..., n+1.$$

This implies that

$$\begin{aligned} h\big(F(a), F(b)\big) &\leq h\big(F(a), F(x_1)\big) + h\big(F(x_1), F(x_2)\big) + ... + h\big(F(x_n), F(b)\big) \\ &\leq c(\|x_1 - a\| + \|x_2 - x_1\| + ... + \|b - x_n\|) \\ &= c\|b-a\|. \end{aligned}$$

Since $c > \alpha$ is taken arbitrarily, we get

$$h\big(F(a), F(b)\big) \leq \|b-a\| \inf_{\varepsilon>0} \sup\big\{\|\widehat{D}^*F(x,y)\| \mid x \in [a,b]+\varepsilon\mathbb{B},\, y \in F(x)\big\},$$

that is, (4.6) holds.

To justify (4.7), assume that X and Y are finite-dimensional and F is locally uniformly bounded at each $x \in [a,b]$. If $\alpha = 0$ then, by (4.6), $h\big(F(a), F(b)\big) = 0$ and thus (4.7) holds. Suppose now that $\alpha \in (0, \infty)$. Then we can find $\varepsilon_k \downarrow 0$, $x_k \in [a,b] + \varepsilon_k\mathbb{B}$, $y_k \in F(x_k)$, $y_k^* \in Y^*$ with $\|y_k^*\| \leq 1$ and $x_k^* \in \widehat{D}^*F(x_k, y_k)(y_k^*)$ such that $\|x_k^*\| \to \alpha$ as $k \to \infty$. Since X and Y are finite-dimensional and $\alpha \in \mathbb{R}$, we can assume without loss of generality that $x_k \to \bar{x} \in [a,b]$, $y_k^* \to \bar{y}^* \in Y^*$ with $\|\bar{y}^*\| \leq 1$ and $x_k^* \to \bar{x}^* \in X^*$ with $\|\bar{x}^*\| = \alpha$. By the locally uniform boundedness of F at each point of $[a,b]$ and $\bar{x} \in [a,b]$, there exist positive real numbers $r_{\bar{x}}$ and $M_{\bar{x}} > 0$ such that $F\big(\mathbb{B}_{r_{\bar{x}}}(\bar{x})\big) \subset M_{\bar{x}}\mathbb{B}_Y$. Hence $y_k \in M_{\bar{x}}\mathbb{B}_Y$ for all k sufficiently large. Since $M_{\bar{x}}\mathbb{B}_Y$ is compact, by replacing a subsequence if necessary, we can assume that $y_k \to \bar{y} \in Y$. Note that $(\bar{x}, \bar{y}) \in \mathrm{gph}F$, due to the closedness of $\mathrm{gph}F$. Thus $\bar{x}^* \in D^*F(\bar{x}, \bar{y})(\bar{y}^*)$ and $\|\bar{y}^*\| \leq 1$. We have

$$\alpha = \|\bar{x}^*\| \leq \|D^*F(\bar{x}, \bar{y})\| \leq \sup\big\{\|D^*F(x,y)\| \;\big|\; x \in [a,b],\; y \in F(x)\big\}.$$

Thus, by (4.6), (4.7) holds. Finally, consider the case where $\alpha = +\infty$. We claim that

$$\sup\big\{\|D^*F(x,y)\| \mid x \in [a,b],\, y \in F(x)\big\} = +\infty.$$

Indeed, if this is not true, then $\ell := \sup\big\{\|D^*F(x,y)\| \mid x \in [a,b],\, y \in F(x)\big\} < +\infty$. By Proposition 2.3 (iii), for each $x \in [a,b]$ and $y \in F(x)$, F is Lipschitz-like around (x,y) with $\mathrm{lip}F\big(x,y\big) \leq \ell$. Thus, by Proposition 2.3 (ii), for each $x \in [a,b]$ and $y \in F(x)$, there exists $\varepsilon_{x,y} > 0$ such that

$$\sup\big\{\|\widehat{D}^*F(u,v)\| \;\big|\; (u,v) \in \mathrm{gph}F \cap \big[\mathbb{B}_{\varepsilon_{x,y}}(x) \times \mathbb{B}_{\varepsilon_{x,y}}(y)\big]\big\} \leq \ell + 1. \tag{4.8}$$

Note that, due to the boundedness assumption of F, $([a,b]\times Y) \cap \mathrm{gph}F$ is a compact subset of the open set $\bigcup_{x\in[a,b],\; y\in F(x)} \big[\mathrm{int}\mathbb{B}_{\varepsilon_{x,y}}(x) \times \mathrm{int}\mathbb{B}_{\varepsilon_{x,y}}(y)\big]$. Thus, there exists $\varepsilon_0 > 0$ such that

$$\big[([a,b]+\varepsilon_0\mathbb{B}) \times Y\big] \cap \mathrm{gph}F \subset \bigcup_{x\in[a,b],\; y\in F(x)} \big[\mathbb{B}_{\varepsilon_{x,y}}(x) \times \mathbb{B}_{\varepsilon_{x,y}}(y)\big] \cap \mathrm{gph}F.$$

By (4.8), for each $(u,v) \in \big[([a,b]+\varepsilon_0\mathbb{B}) \times Y\big] \cap \mathrm{gph}F$, we have $\|\widehat{D}^*F(u,v)\| \leq \ell + 1$. This implies that

$$\sup\big\{\|\widehat{D}^*F(u,v)\| \;\big|\; u \in [a,b]+\varepsilon_0\mathbb{B},\; v \in F(u)\big\} \leq \ell + 1 < +\infty.$$

Thus $\alpha < +\infty$, which is a contradiction, and the claim is justified. Hence (4.7) holds. The proof is complete. □

Remark 4.3. If $F : X \rightrightarrows Y$ is a continuous single-valued mapping, then Theorem 4.2 reduces to Theorem 3.1, provided that Y is an Asplund space.

The next result gives us a characterization of the set-valued mappings that are locally Lipschitzian in the Pompeiu-Hausdorff distance.

Corollary 4.4. *Let $F : X \rightrightarrows Y$ be a set-valued mapping with closed graph between two Asplund spaces X and Y. Then, the following assertions are equivalent.*

(i) *The mapping F is locally Lipschitz around $\bar{x}$ with modulus κ.*

(ii) *There exists $r > 0$ such that $\|\widehat{D}^* F(x, y)\| \le \kappa$ for all $x \in \mathbb{B}_r(\bar{x})$ and $y \in F(x)$.*

Proof. To justify (**i**) $\Rightarrow$ (**ii**), suppose that F is locally Lipschitz around $\bar{x}$ with modulus κ. Then there exists $r > 0$ such that F is Lipschitz-like around (x, y) with modulus κ for any $x \in \mathbb{B}_r(\bar{x})$ and $y \in F(x)$. By Proposition 2.3 (i),

$$\|\widehat{D}^* F(x, y)\| \le \kappa \text{ for all } x \in \mathbb{B}_r(\bar{x}),\ y \in F(x).$$

We now prove (**ii**) $\Rightarrow$ (**i**). Suppose that, for some $r > 0$, it holds that

$$\|\widehat{D}^* F(x, y)\| \le \kappa \text{ for all } x \in \mathbb{B}_r(\bar{x}),\ y \in F(x).$$

Take any $a, b \in \mathbb{B}_{\frac{r}{2}}(\bar{x})$ and $\varepsilon \in (0, \frac{r}{2})$. By Theorem 4.2,

$$h\big(F(b), F(a)\big) \le \|b - a\| \sup\big\{\|\widehat{D}^* F(x, y)\| \;\big|\; x \in [a, b] + \varepsilon\mathbb{B},\ y \in F(x)\big\}.$$

Since $[a, b] + \varepsilon\mathbb{B} \subset \mathbb{B}_r(\bar{x})$, it holds that $\|\widehat{D}^* F(x, y)\| \le \kappa$ for all $x \in [a, b] + \varepsilon\mathbb{B}$ and $y \in F(x)$. Thus,

$$h\big(F(b), F(a)\big) \le \kappa\|b - a\| \quad \text{for all}\, a, b \in \mathbb{B}_{\frac{r}{2}}(\bar{x}).$$

This means F is locally Lipschitz around $\bar{x}$. □

5. Concluding Remarks

The main results of this paper are some mean value theorems via coderivatives, which allow us to derive some characterizations of Lipschitz mappings. For the regular coderivative, only the "fuzzy" mean value inequalities (3.1) and (4.6) have been established. Until now, we have not known whether the "exact" mean value inequalities (3.2) and (4.7) hold if the limiting coderivative is replaced by the regular counterpart. The validity of the mean value inequalities (3.2) and (4.7) in the infinite-dimensional settings has been also unclear to us. Finally, it would be interesting if we could remove the assumption that Y is an Asplund space in Theorem 4.2.

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References

[1] J.-P. Aubin, *Lipschitz behavior of solutions to convex minimization problems*, Math. Oper. Res. **9** (1984), 87–111.

[2] D. Aussel, J.-N. Corvellec and M. Lassonde, *Mean value property and subdifferential criterial for lower semicontinuous functions*, Trans. Amer. Math. Soc. **347** (1995), 4147–4161.

[3] J. M. Borwein and Q. Zhu, *Techniques of variational analysis*, Springer, New York, 2005.

[4] M. Castellani and M. Pappalardo, *On the mean value theorem for semidifferentiable functions,* J. Glob. Optim. **46** (2010), 503–508.

[5] F. H. Clarke and Y. S. Ledyaev, *Mean value inequalities in Hilbert space*, Trans. Amer. Math. Soc. **344** (1994), 307–324.
[6] N. H. Chieu and N. Q. Huy, *Second-order subdifferentials and convexity of real-valued functions*, Nonlinear Anal. **74** (2011), 154–160.
[7] N. H. Chieu, G. M. Lee, B. S. Mordukhovich and T. T. A. Nghia, *Coderivative characterizations of maximal monotonicity for set-valued mappings*, J. Convex Anal. **23** (2016), 461–480.
[8] N. H. Chieu and N. T. Q. Trang, *Coderivative and monotonicity of continuous mappings*, Tawanese J. Math. **16** (2012), 353–365.
[9] R. Correa, A. Jofré and L. Thibault, *Characterization of lower semicontinuous convex functions*, Proceding Amer. Math. Soc. **116** (1992), 67–72.
[10] A. L. Dontchev and R. T. Rockafellar, *Characterizations of strong regularity for variational inequalities over polyhedral convex sets*, SIAM J. Optim. **6** (1996), 1087–1105.
[11] L. Gajek and D. Zagrodny, *Geometric mean value theorems for the Dini derivative*, J. Math. Anal. Appl. **191** (1995), 56–76.
[12] J.-B. Hiriart-Urruty, *Mean value theorems in nonsmooth analysis*, Numer. Funct. Anal. Optim. **2** (1980), 1–30.
[13] A. D. Ioffe and V. M. Tikhomirov, *Theory of extremal problems*, North-Holland, Netherlands, 1979.
[14] G. Lebourg, *Valeur moyenne pour gradient généraliseé*, C. R. Acad. Sci. Paris **281** (1975), 795–798.
[15] P. D. Loewen, *A mean value theorem for Fréchet subgradients*, Nonlinear Anal. **23**(1994), 1365–1381.
[16] D. T. Luc, *A strong mean value theorem and applications*, Nonlinear Anal. **26** (1996), 915–923
[17] B. S. Mordukhovich, *Metric approximations and necessary optimality conditions for general classes of extremal problems*, Soviet Math. Dokl. **22** (1980), 526–530.
[18] B. S. Mordukhovich, *Variational analysis and generalized differentiation, Vol. I: Basic theory*, Springer, Berlin, 2006.
[19] B. S. Mordukhovich, T. T. A. Nghia, *Local strong maximal monotonicity and full stability for parametric variational systems*, SIAM J. Optim. **23** (2016), 1032–1059.
[20] B. S. Mordukhovich and Y. Shao, *Nonsmooth sequential analysis in Asplund spaces*, Trans. Amer. Math. Soc. **348** (1996), 1235–1280.
[21] J.-P. Penot, *On the mean value theorem*, Optimization **19** (1988), 147–156.
[22] J.-P. Penot, *Mean value theorems for correspondences*, Acta Math. Vietnam. **26** (2001), 365–376.
[23] J.-P. Penot, *Mean value theorem with small subdifferentials*, J. Optim. Theory Appl. **94** (1997), 209–221.
[24] J.-P. Penot, *Calculus without derivatives*, Springer, New York, 2013.
[25] P. H. Sach, *Differentiability of set-valued maps in Banach spaces*, Math. Nachr. **139** (1988), 215–235.
[26] H.-P. Scheffler, *Mean value properties of nondifferentiable functions and their application in nonsmooth analysis*, Optimization **20** (1989), 743–759.
[27] W. Schirotzek, *Nonsmooth Analysis,* Springer, Berlin, 2007.
[28] M. Studniarski, *Mean value theorems and sufficient optimality conditions for nonsmooth functions*, J. Math. Anal. Appl. **111** (1985), 313–326.
[29] L. Thibault, *A note on the Zagrodny mean value theorem*, Optimization **35** (1995), 127–130.
[30] L. Thibault and D. Zagrodny, *Integration of subdifferentials of lower semicontinuous functions on Banach spaces*, J. Math. Anal. Appl. **189** (1995), 33–58.
[31] L. Thibault and D. Zagrodny, *Enlarged inclusion of subdifferentials*, Canadian Math. Bull. **48** (2005), 283–301.
[32] N. T. Q. Trang, *A note on an approximate mean value theorem for Fréchet subgradients*, Nonlinear Anal. **75** (2012), 380–383.
[33] L. Wegge, *Mean value theorems for convex functions,* J. Math. Econ. **1** (1974), 207–208.
[34] N. D. Yen, *A mean value theorem for semidifferentiable functions,* Vietnam J. Math. **23** (1996), 221–228.

[35] D. Zagrodny, *Approximate mean value theorem for upper subderivatives*, Nonlinear Anal. **12** (1988), 1413–1428.

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