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THE CONVEXITY OF CIRCULAR CONE TRACE FUNCTIONS

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Abstract. In this short paper, we establish the convexity of some functions associated with circular cones, called circular cone trace functions. As illustrated, these functions play a key role in the development of penalty and barrier function methods for cone programs. With this method one may offer much simpler proofs to some useful inequalities.

1. Introduction

The second-order cone (SOC) in $\mathbb{R}^n$, also called Lorentz cone, is the set defined as

$$\mathcal{K}^n := \Big\{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geq \|x_2\|\Big\}, \tag{1.1}$$

where $\|\cdot\|$ denotes the Euclidean norm. When $n = 1$, $\mathcal{K}^n$ reduces to the set of nonnegative real numbers $\mathbb{R}_+$.

As shown in [14], $\mathcal{K}^n$ is also a set composed of the squared elements from Jordan algebra $(\mathbb{R}^n, \circ)$, where the Jordan product "$\circ$" is a binary operation defined by

$$x \circ y := (\langle x, y\rangle, x_1 y_2 + y_1 x_2) \tag{1.2}$$

for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R} \times \mathbb{R}^{n-1}$.

Here for any $x \in \mathbb{R}^n$, we use x_1 to denote the first component of x, and x_2 to denote the vector consisting of the rest $n-1$ components.

From [13, 14], we recall that each $x \in \mathbb{R}^n$ admits a spectral decomposition associated with $\mathcal{K}^n$ of the following form

$$x = \lambda_1(x) u_x^{(1)} + \lambda_2(x) u_x^{(2)}, \tag{1.3}$$

where $\lambda_i(x)$ and $u_x^{(i)}$ for $i = 1, 2$ are the spectral values and the associated spectral vectors of x, respectively, defined by

$$\lambda_i(x) = x_1 + (-1)^i \|x_2\|, \quad u_x^{(i)} = \frac{1}{2}\Big(1, (-1)^i \bar{x}_2\Big), \tag{1.4}$$

with $\bar{x}_2 = \frac{x_2}{\|x_2\|}$ if $x_2 \neq 0$, and otherwise $\bar{x}_2$ being any vector in $\mathbb{R}^{n-1}$ such that $\|\bar{x}_2\| = 1$.

When $x_2 \neq 0$, the spectral decomposition is unique. The determinant and trace of x are defined as $\det(x) := \lambda_1(x)\lambda_2(x)$ and $\mathrm{tr}(x) := \lambda_1(x) + \lambda_2(x)$, respectively.

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With the spectral decomposition above, for any given scalar function $\phi : J \subseteq \mathbb{R} \to \mathbb{R}$, we may define a vector-valued function $\phi^{\text{soc}} : S \subseteq \mathbb{R}^n \to \mathbb{R}^n$ by

$$\phi^{\text{soc}}(x) \ := \ \phi(\lambda_1(x))u_x^{(1)} + \phi(\lambda_2(x))u_x^{(2)} \tag{1.5}$$

where J is an interval (finite or infinite, open or closed) of $\mathbb{R}$, and S is the domain of ϕ^{soc} determined by ϕ.

Then, we can define the SOC trace function associated with ϕ

$$\phi^{\text{tr}}(x) := \phi(\lambda_1(x)) + \phi(\lambda_2(x)) = \text{tr}(\phi^{\text{soc}}(x)) \quad \forall x \in S. \tag{1.6}$$

Chen, Liao and Pan [12] give the following relation between ϕ^{tr} and ϕ^{soc}

$$\nabla \phi^{\text{tr}}(x) = 2(\phi^{'})^{\text{soc}}(x) \quad \text{and} \quad \nabla^2 \phi^{\text{tr}}(x) = 2\nabla(\phi^{'})^{\text{soc}}(x) \qquad \forall x \in \textbf{int}S. \tag{1.7}$$

By using Schur Complement Theorem, they establish the convexity of SOC trace functions and the compounds of SOC trace functions. Some of these functions are the key of penalty and barrier function methods for second-order cone programs (SOCPs), as well as the establishment of some important inequalities associated with SOCs, for which the proof of convexity of these functions is a necessity.

As a generalization of second-order cone, Zhou and Chen[23] begin to study a new cone. For any angle $\theta \in (0^\circ, 90^\circ)$. they define the circular cone $\mathcal{L}_\theta$ as

$$\begin{aligned} \mathcal{L}_\theta : &= \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geq \|x\| \cos\theta\} \\ &= \{x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 \geq \|x_2\| \cot\theta\}. \end{aligned} \tag{1.8}$$

Although $\mathcal{L}_\theta$ is not a symmetric cone (except for $\theta = 45^\circ$), one can still, due to its special structure, give an explicit form of orthogonal decomposition (or spectral decomposition) as

$$x = \lambda_1(x) \cdot u_x^{(1)} + \lambda_2(x) \cdot u_x^{(2)}, \tag{1.9}$$

where

$$\left\{ \begin{array}{l} \lambda_1(x) = x_1 - \|x_2\| \cot\theta, \\ \lambda_2(x) = x_1 + \|x_2\| \tan\theta, \end{array} \right. \qquad \text{and} \qquad \left\{ \begin{array}{l} u_x^{(1)} = \begin{bmatrix} \sin^2\theta \\ -(\sin\theta\cos\theta)\tilde{x_2} \end{bmatrix}, \\ u_x^{(2)} = \begin{bmatrix} \cos^2\theta \\ (\sin\theta\cos\theta)\tilde{x_2} \end{bmatrix}, \end{array} \right. \tag{1.10}$$

with $\tilde{x_2} = x_2/\|x_2\|$ if $x_2 \neq 0$, and $\tilde{x_2}$ being any unit vector $w \in \mathbb{R}^{n-1}$ if $x_2 = 0$.

Follow the same trick, given a twice differentiable function $f : \mathbb{R} \to \mathbb{R}$, we may define the *trace function* of f associated to $\mathcal{L}_\theta$ as

$$f^{\text{tr}}(x) := f\big(\lambda_1(x)\big) + f\big(\lambda_2(x)\big). \tag{1.11}$$

When f is non-constant, f^{tr} is differentiable on $\mathcal{L}_\theta$ except for the set

$$E := \{(x_1, 0) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid x_1 > 0\} \subset \mathcal{L}_\theta. \tag{1.12}$$

Nonetheless, the similar computation in [12], in $\mathcal{L}_\theta \setminus E$ the gradient of f^{tr} can be expressed by

$$\nabla f^{\mathrm{tr}}(x) = f'\big(\lambda_1(x)\big) \begin{bmatrix} 1 \\ -(\cot\theta)\tilde{x_2} \end{bmatrix} + f'\big(\lambda_2(x)\big) \begin{bmatrix} 1 \\ (\tan\theta)\tilde{x_2} \end{bmatrix}, \tag{1.13}$$

and the Hessian of f^{tr} is given by

$$\begin{aligned} \nabla^2 f^{\mathrm{tr}}(x) &= \begin{bmatrix} b(x) & c(x)\tilde{x_2}^T \\ c(x)\tilde{x_2} & a(x)I + \big(d(x) - a(x)\big)\tilde{x_2}\tilde{x_2}^T \end{bmatrix}, \\ \text{where} \quad a(x) &= \frac{f'\big(\lambda_2(x)\big)\tan\theta - f'\big(\lambda_1(x)\big)\cot\theta}{\|x_2\|}, \\ b(x) &= f''\big(\lambda_1(x)\big) + f''\big(\lambda_2(x)\big), \\ c(x) &= f''\big(\lambda_2(x)\big)\tan\theta - f''\big(\lambda_1(x)\big)\cot\theta, \\ d(x) &= f''\big(\lambda_1(x)\big)\cot^2\theta + f''\big(\lambda_2(x)\big)\tan^2\theta. \end{aligned} \tag{1.14}$$

In the following section, we will show the convexity of the trace function $f^{\mathrm{tr}}(x)$.

2. The monotone condition

All notations in Section 1 are kept, and we denote $M \succeq O$ (resp. $M \succ O$) when a symmetric matrix M is semipositive definite (resp. positive definite). The Schur Complement Theorem gives a condition for the positive definiteness(semidefiniteness) with respect that to a block partition of the matrix, which is stated as below.

Lemma 2.1 (Schur Complement Theorem [15])**.** *Let $A \in \mathbb{R}^{m\times m}$ be a symmetric positive definite matrix, $C \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and $B \in \mathbb{R}^{m\times n}$. Then,*

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq O \Leftrightarrow C - B^T A^{-1} B \succeq O; \tag{2.1}$$

and

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ O \Leftrightarrow C - B^T A^{-1} B \succ O. \tag{2.2}$$

In this section, we will find sufficient conditions that imply the convexity of f^{tr} in $\mathcal{L}_\theta$. We start with the following theorem.

Theorem 2.2. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable convex function. Then in $\mathcal{L}_\theta \setminus E$,*

(1) *If $b(x) = 0$, then $\nabla^2 f^{\mathrm{tr}}(x) \succeq O$ if and only if $a(x)\big(I - \tilde{x_2}\tilde{x_2}^T\big) \succeq O$.*
(2) *If $b(x) > 0$, then*

$$\nabla^2 f^{\mathrm{tr}}(x) \succeq O \quad \Longleftrightarrow \quad a(x)I + \big(d(x) - a(x) - \frac{(c(x))^2}{b(x)}\big)\tilde{x_2}\tilde{x_2}^T \succeq O;$$

$$\nabla^2 f^{\mathrm{tr}}(x) \succ O \quad \Longleftrightarrow \quad a(x)I + \big(d(x) - a(x) - \frac{(c(x))^2}{b(x)}\big)\tilde{x_2}\tilde{x_2}^T \succ O.$$

Proof. (1) We assume that $b(x) = 0$ first. Since $b(x) = f''\big(\lambda_1(x)\big) + f''\big(\lambda_2(x)\big)$, $b(x) = 0$ implies that $f''\big(\lambda_1(x)\big) = f''\big(\lambda_2(x)\big) = 0$. From (1.14), it immediately follows that $c(x) = d(x) = 0$ as well. Therefore the Hessian matrix $\nabla^2 f^{\mathrm{tr}}(x)$ has the form

$$\nabla^2 f^{\mathrm{tr}}(x) = \begin{bmatrix} 0 & 0 \\ 0 & a(x)\big(I - \tilde{x_2}\tilde{x_2}^T\big) \end{bmatrix}.$$

It is then clear that $\nabla^2 f^{\mathrm{tr}}(x)$ is semi-positive definite if and only if $a(x)\big(I - \tilde{x_2}\tilde{x_2}^T\big)$ is semi-positive definite.

(2) For the case $b(x) > 0$, because $\nabla^2 f^{\mathrm{tr}}(x)$ is a symmetric matrix, the Schur Complement Theorem directly applies here, that is,

$$\begin{aligned} \nabla^2 f^{\mathrm{tr}}(x) \succeq O \quad &\Longleftrightarrow \quad a(x)I + \big(d(x) - a(x) - \frac{(c(x))^2}{b(x)}\big)\tilde{x_2}\tilde{x_2}^T \succeq O; \\ \nabla^2 f^{\mathrm{tr}}(x) \succ O \quad &\Longleftrightarrow \quad a(x)I + \big(d(x) - a(x) - \frac{(c(x))^2}{b(x)}\big)\tilde{x_2}\tilde{x_2}^T \succ O. \end{aligned}$$

□

Theorem 2.3. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable convex function. If either*

(1) $0 < \theta \leq 45°$ *and $f(x)$ is decreasing; or,*
(2) $45° \leq \theta < 90°$ *and $f(x)$ is increasing,*

then the Hessian $\nabla^2 f^{\mathrm{tr}}$ is semi-positive definite in $\mathcal{L}_\theta \setminus E$.

Proof. Again we examine the case $b(x) = 0$ first. By Theorem 2.2,

$$\nabla^2 f^{\mathrm{tr}}(x) \succeq O \quad \Longleftrightarrow \quad a(x)\big(I - \tilde{x_2}\tilde{x_2}^T\big) \succeq O.$$

Take any $v \in \mathbb{R}^{n-1}$, then

$$v^T a(x)\big(I - \tilde{x_2}\tilde{x_2}^T\big)v = a(x)\big(\|v\|^2 - \langle v, \tilde{x_2}\rangle^2\big).$$

Because $\tilde{x_2}$ is a unit vector, $\|v\| \geq |\langle v, \tilde{x_2}\rangle|$. Therefore $a(x)\big(I - \tilde{x_2}\tilde{x_2}^T\big)$ is semi-positive definite if and only if $a(x) \geq 0$. Recall from (1.14) that

$$a(x) = \frac{f'\big(\lambda_2(x)\big)\tan\theta - f'\big(\lambda_1(x)\big)\cot\theta}{\|x_2\|}.$$

Note that $\lambda_1(x) = x_1 - \|x_2\|\cot\theta \leq x_1 + \|x_2\|\tan\theta = \lambda_2(x)$. When $0 < \theta \leq 45°$ and f is decreasing, both $f'\big(\lambda_2(x)\big), f'\big(\lambda_1(x)\big) \leq 0$ and $\tan\theta \leq \cot\theta$; when $45° < \theta \leq 90°$ and f is increasing, both $f'\big(\lambda_2(x)\big), f'\big(\lambda_1(x)\big) \geq 0$ and $\tan\theta \geq \cot\theta$. In both cases, we can always conclude that $a(x) \geq 0$, which finishes the case $b(x) = 0$.

For the other case $b(x) > 0$, we perform the following computation. For sake of simplicity, we write $k_j = f''\big(\lambda_j(x)\big)$ for $j = 1, 2$. Again from Theorem 2.2, we need to check the (semi-)positive definiteness for

$$a(x)\big(I - \tilde{x_2}\tilde{x_2}^T\big) + \big(d(x) - \frac{(c(x))^2}{b(x)}\big)\tilde{x_2}\tilde{x_2}^T.$$

Note that $d(x) - \frac{(c(x))^2}{b(x)} = \frac{b(x)d(x) - (c(x))^2}{b(x)}$, and

$$\begin{aligned} b(x)d(x) - (c(x))^2 &= (k_1 + k_2)(k_1 \cot^2\theta + k_2 \tan^2\theta) - (k_2 \tan\theta - k_1 \cot\theta)^2 \\ &= k_1 k_2 (\cot\theta + \tan\theta)^2, \end{aligned}$$

which will be non-negative when f is convex. Assuming the monotonicity of f, then for $v \in \mathbb{R}^{n-1}$,

$$\begin{aligned} & v^T\Big[a(x)\big(I - \tilde{x_2}\tilde{x_2}^T\big) + \big(d(x) - \frac{(c(x))^2}{b(x)}\big)\tilde{x_2}\tilde{x_2}^T\Big]v \\ = {} & a(x)\big(\|v\|^2 - \langle v, \tilde{x_2}\rangle^2\big) + \frac{b(x)d(x) - (c(x))^2}{b(x)}\langle v, \tilde{x_2}\rangle^2. \end{aligned}$$

As in the previous case $b(x) = 0$, we see that $\nabla^2 f^{\mathrm{tr}}(x)$ is semi-positive definite (resp. positive definite) if $a(x) \geq 0$ (resp. $a(x) > 0$ and $b(x) > 0$). The argument is the same as before, and the proof is finished. □

With aid of Theorem 2.3, we are in position to prove the convexity of the trace function f^{tr} in the whole circular cone $\mathcal{L}_\theta$ subject to some sufficient conditions stated as below.

Theorem 2.4. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable convex function. Then f^{tr} is a convex function in $\mathcal{L}_\theta$ if either $0 < \theta \leq 45^\circ$ and f is decreasing, or $45^\circ \leq \theta < 90^\circ$ and f is increasing.*

Proof. For sake of brevity, we refer the condition "either $0 < \theta \leq 45^\circ$ and f is decreasing, or $45^\circ \leq \theta < 90^\circ$ and f is increasing" as *Hypothesis K*. We separate our discussions into 4 cases ($[x, y]$ denotes the line segment whose endpoints are $x, y \in \mathcal{L}_\theta$):

Case 1. $[x, y] \cap E = \varnothing$. In this situation, we can conclude that $f^{\mathrm{tr}}\big(\alpha x + (1 - \alpha)y\big) \leq \alpha f^{\mathrm{tr}}(x) + (1 - \alpha) f^{\mathrm{tr}}(y)$ for all $\alpha \in [0, 1]$, thanks to Theorem 2.3.

Case 2. Suppose $x \in \mathcal{L}_\theta$, $y \in E$. For $t \in [0, 1]$, define $u(t) = y + t(x - y)$. Since f^{tr} is continuous on $\mathcal{L}_\theta$, we have

$$\lim_{t \to 0^+} f^{\mathrm{tr}}\big(u(t)\big) = f^{\mathrm{tr}}(y).$$

For $t > 0$, we know that $[x, u(t)] \subset \mathcal{L}_\theta \setminus E$. Hence if Hypothesis K holds, by Case 1 we have: for each $\lambda \in [0, 1]$,

$$\begin{aligned}
\lambda f^{\mathrm{tr}}(x) + (1-\lambda) f^{\mathrm{tr}}(y) &= \lim_{t \to 0+} \big(\lambda f^{\mathrm{tr}}(x) + (1-\lambda) f^{\mathrm{tr}}(u(t))\big) \\
&\geq \lim_{t \to 0+} f^{\mathrm{tr}}\big(\lambda x + (1-\lambda) u(t)\big) \\
&= f^{\mathrm{tr}}\big(\lambda x + (1-\lambda) y\big).
\end{aligned}$$

Case 3. Suppose $x, y \in \mathcal{L}_\theta \setminus E$, and the segment $[x, y]$ intersects with E at some interior point, that is, there is a $t_0 \in (0, 1)$ such that $z := t_0 x + (1 - t_0) y \in E$. Set $x = (x_1, x_2)$, $y = (y_1, y_2)$ in $\mathbb{R} \times \mathbb{R}^{n-1}$. Then we have

$$z = (t_0 x_1 + (1-t_0) y_1, 0), \qquad y_2 = \frac{-t_0}{1-t_0} x_2, \qquad \|y_2\| = \frac{t_0}{1-t_0}\|x_2\|. \tag{2.3}$$

Since f is convex, so for $0 < t < 1$,

$$\begin{aligned}
t f^{\mathrm{tr}}(x) + (1-t) f^{\mathrm{tr}}(y) &= t\big(f(\lambda_1(x)) + f(\lambda_2(x))\big) + (1-t)\big(f(\lambda_1(y)) + f(\lambda_2(y))\big) \\
&= t f(\lambda_1(x)) + (1-t) f(\lambda_2(y)) + t f(\lambda_2(x)) + (1-t) f(\lambda_1(y)) \\
&\geq f\big(t\lambda_1(x) + (1-t)\lambda_2(y)\big) + f\big(t\lambda_2(x) + (1-t)\lambda_1(y)\big) \\
&= f(t x_1 + (1-t) y_1 - t\|x_2\| \cot\theta + (1-t)\|y_2\| \tan\theta) \\
&\quad + f(t x_1 + (1-t) y_1 + t\|x_2\| \tan\theta - (1-t)\|y_2\| \cot\theta).
\end{aligned}$$

When we put t_0 in place of t in the last expression and use (2.3), we see that

$$\begin{aligned}
&f(t_0 x_1 + (1-t_0) y_1 - t_0\|x_2\| \cot\theta + (1-t_0)\|y_2\| \tan\theta) \\
&\quad + f(t_0 x_1 + (1-t_0) y_1 + t_0\|x_2\| \tan\theta - (1-t_0)\|y_2\| \cot\theta) \\
&= 2f\big(t_0 x_1 + (1-t_0) y_1 + t_0\|x_2\|(\tan\theta - \cot\theta)\big).
\end{aligned}$$

To recap, at $t = t_0$ we have

$$t_0 f^{\mathrm{tr}}(x) + (1-t_0) f^{\mathrm{tr}}(y) \geq 2f\big(t_0 x_1 + (1-t_0) y_1 + t_0\|x_2\|(\tan\theta - \cot\theta)\big).$$

When $0 < \theta \leq 45°$, we have $t_0\|x_2\|(\tan\theta - \cot\theta) \leq 0$. Therefore if f is decreasing, then

$$2f\big(t_0 x_1 + (1-t_0) y_1 + t_0\|x_2\|(\tan\theta - \cot\theta)\big) \geq 2f\big(t_0 x_1 + (1-t_0) y_1\big).$$

On the other hand, when $45° \leq \theta < 90°$, we have $t_0\|x_2\|(\tan\theta - \cot\theta) \geq 0$. Therefore if f is increasing, then

$$2f\big(t_0 x_1 + (1-t_0) y_1 + t_0\|x_2\|(\tan\theta - \cot\theta)\big) \geq 2f\big(t_0 x_1 + (1-t_0) y_1\big).$$

Combining these results, we see that under Hypothesis K, we always have

$$\begin{aligned}
t_0 f^{\mathrm{tr}}(x) + (1-t_0) f^{\mathrm{tr}}(y) &\geq 2f\big(t_0 x_1 + (1-t_0) y_1 + t_0\|x_2\|(\tan\theta - \cot\theta)\big) \\
&\geq 2f\big(t_0 x_1 + (1-t_0) y_1\big) = f^{\mathrm{tr}}(z).
\end{aligned}$$

Now, when $t \in (0, t_0)$, we have $tx + (1-t)y \in [z, y]$. Set $\mu = \dfrac{t}{t_0}$ and

$$tx + (1-t)y = \mu z + (1-\mu) y.$$

Under Hypothesis K, we have from Case 2,

$$\begin{aligned} tf^{\mathrm{tr}}(x) + (1-t)f^{\mathrm{tr}}(y) &= \mu\big(t_0 f^{\mathrm{tr}}(x) + (1-t_0)f^{\mathrm{tr}}(y)\big) + (1-\mu)f^{\mathrm{tr}}(y) \\ &\geq \mu f^{\mathrm{tr}}(z) + (1-\mu)f^{\mathrm{tr}}(y) \\ &\geq f^{\mathrm{tr}}(\mu z + (1-\mu)y) \\ &= f^{\mathrm{tr}}(tx + (1-t)y). \end{aligned}$$

By symmetry, the same conclusion can be reached when $t \in (t_0, 1)$.

Case 4. Suppose $x, y \in E$. Then $x = (x_1, 0)$ and $y = (y_1, 0)$. Since f is convex, hence for $t \in (0, 1)$,

$$\begin{aligned} tf^{\mathrm{tr}}(x) + (1-t)f^{\mathrm{tr}}(y) &= 2\big(tf(x_1) + (1-t)f(y_1)\big) \\ &\geq 2f(tx_1 + (1-t)y_1)) \\ &= f^{\mathrm{tr}}(tx + (1-t)y). \end{aligned}$$

These conclude the proof of Theorem 2.4. □

By examining the inequalities more closely, the following statement follows immediately and its proof is omitted.

Corollary 2.5. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function with $f''(x) > 0$ for all $x \in \mathbb{R}$. If either $0 < \theta \leq 45^\circ$ and f is decreasing, or $45^\circ \leq \theta < 90^\circ$ and f is increasing, then f^{tr} is strictly convex.*

3. Counterexamples out of conditions

In this section we will give a few examples out of conditions to show that the trace function from a convex function may not be convex.

Example 3.1. $f(t) = \mathrm{e}^t$, $0 < \theta < \frac{\pi}{4}$.

Here $f'(t) = \mathrm{e}^t > 0$ for all $t \in \mathbb{R}$, hence from Equation (1.14) we have

$$\begin{aligned} a(x) &= \frac{f'\big(\lambda_2(x)\big)\tan\theta - f'\big(\lambda_1(x)\big)\cot\theta}{\|x_2\|} \\ &= \frac{\mathrm{e}^{\lambda_1(x)}\tan\theta}{\|x_2\|}\Big(\mathrm{e}^{\lambda_2(x)-\lambda_1(x)} - \cot^2\theta\Big) \\ &= \frac{\mathrm{e}^{\lambda_1(x)}\tan\theta}{\|x_2\|}\Big(\exp\big(\|x_2\|(\tan\theta + \cot\theta)\big) - \cot^2\theta\Big). \end{aligned}$$

As $0 < \theta < \frac{\pi}{4}$, $\cot^2\theta > 1$. Hence, when $\|x_2\| < \frac{\ln(\cot^2\theta)}{\tan\theta + \cot\theta}$, we have $\exp(\|x_2\|(\tan\theta + \cot\theta)) - \cot^2\theta < 0$, and $a(x) < 0$.

For example, we take $x = (10, \frac{\ln 2}{2}, 0)$, $y = (10, 0, \frac{\ln 2}{2}) \in \mathcal{L}_\theta \subset \mathbb{R}^3$, with $\cot\theta = 2$ and $t_0 = \frac{3}{7}$. We have

$$f^{\mathrm{tr}}(x) = f^{\mathrm{tr}}(y) = (2^{\frac{1}{4}} + 2^{-1})\,\mathrm{e}^{10};$$

$$t_0 f^{\mathrm{tr}}(x) + (1-t_0) f^{\mathrm{tr}}(y) = (2^{\frac{1}{4}} + 2^{-1})\,\mathrm{e}^{10} \approx 37207.2627;$$
$$f^{\mathrm{tr}}(t_0 x + (1-t_0)y) = (2^{\frac{5}{28}} + 2^{-\frac{5}{7}})\,\mathrm{e}^{10} \approx 38354.01251.$$

As $t_0 f^{\mathrm{tr}}(x) + (1-t_0) f^{\mathrm{tr}}(y) < f^{\mathrm{tr}}(t_0 x + (1-t_0)y)$, f^{tr} is not convex.

Example 3.2. $f(t) = \mathrm{e}^{-t}$, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

Here $f'(t) = -\mathrm{e}^{-t} < 0$ for all $t \in \mathbb{R}$. again from Equation (1.14) we have

$$\begin{aligned} a(x) &= \frac{f'\big(\lambda_2(x)\big)\tan\theta - f'\big(\lambda_1(x)\big)}{\|x_2\|} \\ &= \frac{-\mathrm{e}^{-\lambda_1(x)}\tan\theta}{\|x_2\|}\Big(\mathrm{e}^{\lambda_1(x)-\lambda_2(x)} - \cot^2\theta\Big) \\ &= \frac{-\mathrm{e}^{-\lambda_1(x)}\tan\theta}{\|x_2\|}\Big(\exp\big(-\|x_2\|(\tan\theta+\cot\theta)\big) - \cot^2\theta\Big). \end{aligned}$$

As $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, $\cot^2\theta < 1$. Hence, when $\|x_2\| < \frac{-\ln(\cot^2\theta)}{\tan\theta+\cot\theta}$, we have $\exp(-\|x_2\|(\tan\theta + \cot\theta)) - \cot^2\theta > 0$, and $a(x) < 0$.

For example, we take $x = (10, \frac{\ln 2}{2}, 0)$, $y = (10, 0, \frac{\ln 2}{2}) \in \mathcal{L}_\theta \subset \mathbb{R}^3$, with $\cot\theta = \frac{1}{2}$ and $t_0 = \frac{3}{7}$. We have

$$f^{\mathrm{tr}}(x) = f^{\mathrm{tr}}(y) = (2^{\frac{1}{4}} + 2^{-1})\,\mathrm{e}^{-10};$$
$$t_0 f^{\mathrm{tr}}(x) + (1-t_0) f^{\mathrm{tr}}(y) = (2^{\frac{1}{4}} + 2^{-1})\,\mathrm{e}^{-10} \approx 7.668988 \times 10^{-5};$$
$$f^{\mathrm{tr}}(t_0 x + (1-t_0)y) = (2^{\frac{5}{28}} + 2^{-\frac{5}{7}})\,\mathrm{e}^{10} \approx 7.905351 \times 10^{-5}.$$

As $t_0 f^{\mathrm{tr}}(x) + (1-t_0) f^{\mathrm{tr}}(y) < f^{\mathrm{tr}}(t_0 x + (1-t_0)y)$, f^{tr} is not convex.

Example 3.3. $f(t) = -\ln t$ *for* $t > 0$, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

Here $f'(t) = -\frac{1}{t} < 0$ for all $x > 0$. From the definitions in (1.8) and (1.10) we know that both $\lambda_1(x)$ and $\lambda_2(x)$ are non-negative for all $x \in \mathcal{L}_\theta$.

For those $x \in \mathcal{L}_\theta$ with $\lambda_1(x) > 0$ and $\lambda_2(x) > 0$,

$$\begin{aligned} a(x) &= \frac{f'(\lambda_2(x))\tan\theta - f'(\lambda_1(x))\cot\theta}{\|x_2\|} \\ &= \frac{f'(\lambda_1(x))\tan\theta}{\|x_2\|}\Big(\frac{f'(\lambda_2(x))}{f'(\lambda_1(x)} - \cot^2\theta\Big) \\ &= -\frac{\tan\theta}{\lambda_1(x)\|x_2\|}\Big(\frac{\lambda_1(x)}{\lambda_2(x)} - \cot^2\theta\Big). \end{aligned}$$

We know from the definition that $\lambda_2(x) \geq \lambda_1(x) > 0$, hence $\frac{\lambda_1(x)}{\lambda_2(x)} \leq 1$. For $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, $\cot^2\theta > 1$. Hence we conclude that $a(x) < 0$ in this case.

On the other hand,

$$\begin{aligned} f^{\mathrm{tr}}(x) &= f(\lambda_1(x)) + f(\lambda_2(x)) = -\ln\lambda_1(x) - \ln\lambda_2(x) = -\ln(\lambda_1(x)\lambda_2(x)) \\ &= -\ln(x_1^2 + x_1\,\|x_2\|(\tan\theta - \cot\theta) - \|x_2\|^2). \end{aligned}$$

For example, we consider $\theta = \operatorname{arccot}\frac{1}{2}$, $x = (10,1,0)$, $y = (10,0,1) \in \mathcal{L}_\theta$, and $t_0 = \frac{1}{2}$. We have

$$\begin{aligned} f^{\mathrm{tr}}(x) &= -\ln(100+15-1) = -\ln 114;\\ f^{\mathrm{tr}}(y) &= -\ln(100+15-1) = -\ln 114;\\ t_0 f^{\mathrm{tr}}(x) + (1-t_0) f^{\mathrm{tr}}(y) &= -\ln 114. \end{aligned}$$

On the other hand, $t_0 x + (1-t_0)y = (10, \frac{1}{2}, \frac{1}{2})$, so

$$f^{\mathrm{tr}}(t_0 x + (1-t_0)y) = -\ln\big(100 + \sqrt{\frac{225}{2}} - \frac{1}{2}\big) \approx -\ln 110.1066.$$

As $t_0 f^{\mathrm{tr}}(x) + (1-t_0) f^{\mathrm{tr}}(y) < f^{\mathrm{tr}}(t_0 x + (1-t_0)y)$, f^{tr} is not convex.

Example 3.4. $f(t) = (t-1)^2$.

Here $f'(t) = 2t-2$, so $f'(t) \neq 0$ if and only if $t \neq 1$. When $x \in \mathcal{L}_\theta \setminus E$, $\lambda_2(x) > \lambda_1(x)$. If $\lambda_1(x) \neq 1$,

$$\begin{aligned} a(x) &= \frac{f'(\lambda_2(x))\tan\theta - f'(\lambda_1(x))\cot\theta}{\|x_2\|}\\ &= \frac{f'(\lambda_1(x))\tan\theta}{\|x_2\|}\Big(\frac{f'(\lambda_2(x))}{f'(\lambda_1(x))} - \cot^2\theta\Big)\\ &= \frac{2(\lambda_1(x)-1)\tan\theta}{\|x_2\|}\Big(\frac{\lambda_2(x)-1}{\lambda_1(x)-1} - \cot^2\theta\Big). \end{aligned}$$

Case 1. When $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, we have $0 < \cot^2\theta < 1$. Thus in the case $\lambda_1(x) < \lambda_2(x) \leq 1$, we have

$$0 \leq \frac{\lambda_2(x)-1}{\lambda_1(x)-1} < 1.$$

Nevertheless, if $\dfrac{\lambda_2(x)-1}{\lambda_1(x)-1} > \cot^2\theta$, then $a(x) < 0$, that is, f^{tr} would not be in this case.

For example, let us consider $\theta = \operatorname{arccot}\frac{1}{2}$, $x = (\frac{1}{2}, \frac{1}{8}, 0)$, $y = (\frac{1}{2}, 0, \frac{1}{8}) \in \mathcal{L}_\theta$, and $t_0 = \frac{3}{7}$.

$$\begin{aligned} f^{\mathrm{tr}}(x) = f^{\mathrm{tr}}(y) &= \Big(\frac{-9}{16}\Big)^2 + \Big(\frac{-1}{4}\Big)^2 = \frac{97}{256} \approx 0.3789;\\ t_0 f^{\mathrm{tr}}(x) + (1-t_0) f^{\mathrm{tr}}(y) &= \frac{97}{256} \approx 0.37891. \end{aligned}$$

On the other hand, $t_0 x + (1-t_0)y = (\frac{1}{2}, \frac{3}{56}, \frac{4}{56})$, hence

$$f^{\mathrm{tr}}(t_0 x + (1-t_0)y) = \Big(\frac{-61}{112}\Big)^2 + \Big(\frac{-9}{28}\Big)^2 = \frac{5017}{12544} \approx 0.39995.$$

As $t_0 f^{\mathrm{tr}}(x) + (1-t_0) f^{\mathrm{tr}}(y) < f^{\mathrm{tr}}(t_0 x + (1-t_0)y)$, f^{tr} is not convex.

Case 2. When $0 < \theta < \frac{\pi}{4}$, we have $\cot^2\theta > 1$. Thus in the case $\lambda_1(x) > 1$, we have

$$\frac{\lambda_2(x)-1}{\lambda_1(x)-1} > 1$$

since $\lambda_2(x) > \lambda_1(x) > 1$. However if $\dfrac{\lambda_2(x)-1}{\lambda_1(x)-1} < \cot^2\theta$, then $a(x) < 0$, that is, $f^{\rm tr}$ would not be in this case.

For example, let us consider $\theta = \operatorname{arccot} 2$, $x = (10, 2, 0)$, $y = (10, 0, 2) \in \mathcal{L}_\theta$, and $t_0 = \frac{3}{7}$.

$$f^{\rm tr}(x) = f^{\rm tr}(y) = 5^2 + 10^2 = 125;$$
$$t_0 f^{\rm tr}(x) + (1-t_0) f^{\rm tr}(y) = 125$$

On the other hand, $t_0 x + (1-t_0)y = (10, \frac{6}{7}, \frac{8}{7})$, hence

$$f^{\rm tr}(t_0x + (1-t_0)y) = \left(\frac{43}{7}\right)^2 + \left(\frac{68}{7}\right)^2 = \frac{6473}{49} \approx 132.1020.$$

As $t_0 f^{\rm tr}(x) + (1-t_0) f^{\rm tr}(y) < f^{\rm tr}(t_0x + (1-t_0)y)$, $f^{\rm tr}$ is not convex.

From these two cases, we see that $f^{\rm tr}$ is convex only if $\theta = \frac{\pi}{4}$.

4. Conclusion

This short paper is an extension to [12]. We only focus on the convexity of the trace functions. If one is interested in more inequatilities related to the convexity or penalty and barrier function methods [1, 2, 3], he can go to check the details in [12]. A similar result about the general symmetric cone is shown in [9]. It is obvious to see the key point to show the convexity of the trace function heavily depends on the decomposition (1.9). This inspires us if we want to study another nonsymmetric cone, we need to get a good decomposition formula. Once we have a way to construct lots of convex functions, every thing goes easily.

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