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THE HADAMARD PRODUCT FOR WEIGHTED GEOMETRIC MEANS

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Abstract. H. Lee et al. [10] proposed a family of weighted geometric means $\{\mathfrak{G}(\mathbf{t};\omega;\mathbb{A})\}$ where ω and $\mathbb{A}$ vary over all positive probability vectors in $\mathbb{R}^n$ and n-tuples of positive definite matrices resp. Each of these weighted geometric means interpolates between the weighted ALM($\mathbf{t} = \mathbf{0}_n$) and BMP($\mathbf{t} = \mathbf{1}_n$) geometric means (ALM and BMP geometric means have been defined by Ando-Li-Mathias and Bini-Meini-Poloni, respectively.) Using the well-known connection between the tensor product and the Hadamard product, we show that the Hadamard product of weighted geometric means $\mathfrak{G}(\mathbf{t};\omega;\cdot)$ for permuted tuples with fixed weight is bounded by the Hadamard product of given positive definite matrices. It generalizes the results for the case of two-variable geometric means.

1. Introduction

In 1975 Pusz and Woronowicz [15] have introduced the notion of the geometric mean of two positive definite matrices A and B:

$$A\#B := A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}.$$

Since then, T. Ando [1] has developed the robust definition of the geometric mean $A\#B$ with a variety of properties. One of the interesting properties of the geometric mean is the interaction with the Hadamard product (or called the Schur product). In other words, for two positive definite matrices A and B

$$(A\#B)\circ(A\#B) \le A\circ B, \tag{1.1}$$

where $A\circ B = [a_{ij}b_{ij}]$ for $A = [a_{ij}]$ and $B = [b_{ij}]$. Here, the relation $\le$ is the Löewner order defined as

$$A \le B \text{ if and only if } B - A \text{ is positive semidefinite.}$$

If A and B commute, then the inequality (1.1) reduces to

$$(AB)^{1/2}\circ(AB)^{1/2} \le A\circ B.$$

Using a different method, he also succeeded in generalizing this inequality to the case of several commuting positive definite matrices:

$$\prod_{1}^{m}\circ\left(\prod_{i=1}^{m}A_i\right)^{1/m} \le \prod_{i=1}^{m}\circ A_i = A_1\circ\cdots\circ A_m \tag{1.2}$$

for commuting positive definite matrices $A_1,\ldots,A_m$.

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On the other hand, the inequality (1.2) has not been developed for the multivariable geometric means of several non-commuting positive definite matrices until successfully suggested the multivariable geometric mean. It has been a long-standing problem to extend to n-variables, $n \geq 3$, the two-variable geometric mean of positive definite matrices and a variety of attempts may be found in the literature. Indeed, there is no formal definition of geometric mean of finite number of positive definite matrices and defining multivariable geometric mean is a non-trivial task and is a recent topic of interest in core linear algebra.

For given n-tuple of positive definite matrices $\mathbb{A} = (A_1, A_2, \ldots, A_n)$ and a positive probability vector $\omega = (w_1, \ldots, w_n)$, the definition $G_\omega(\mathbb{A})$ constructed by M. Sagae and K. Tanabe is as follows:

$$G_\omega(\mathbb{A}) := A_n \#_{\omega^{n-1}} (A_{n-1} \#_{\omega^{n-2}} \cdots \#_{\omega^2} (A_2 \#_{\omega^1} A_1)),$$

where for $1 \leq k \leq n-1$

$$\omega^k = 1 - w_{k+1} \left(\sum_{j=1}^{k+1} w_j \right)^{-1}.$$

We call $G_\omega(\mathbb{A})$ the *Sagae-Tanabe geometric mean.*

There is an interesting structure on the open convex cone $\mathbb{P}$ of $m \times m$ positive definite matrices equipped with the Riemannian trace metric δ:

$$\delta(A, B) = \| \log(A^{-1/2} B A^{-1/2}) \|_F$$

for any $A, B \in \mathbb{P}$, where $\|X\|_F$ denotes the Frobenius norm of X. That is, $(\mathbb{P}, \delta)$ is a Hadamard space (or a non-positive curvature space), which is a complete metric space satisfying the semi-parallelogram law. For given n points on the Hadamard space, in general, there exists a unique minimizer of the weighted sum of squares of the distances to each point. So in the setting of positive definite matrices,

$$\underset{X \in \mathbb{P}}{\arg\min} \sum_{j=1}^{n} w_j \delta^2(X, A_j)$$

exists uniquely for $\mathbb{A} = (A_1, A_2, \ldots, A_n) \in \mathbb{P}^n$ and a positive probability vector $\omega = (w_1, \ldots, w_n)$. We call it the *weighted Karcher mean* or the *least squares mean*, and we denote as $\Lambda_\omega(A_1, \ldots, A_n)$. See [6, 9, 12–14]

The inequality (1.2) has been extended to the Sagae-Tanabe geometric means $G_\omega(\mathbb{A})$ by B. Feng and A. Tonge [5], and to the weighted Karcher means $\Lambda(\omega; \mathbb{A})$ by H. Lee and S. Kim [11], respectively.

Theorem 1.1 ([5,11])**.** *Let ω be a positive probability vector in $\mathbb{R}^n$, and let $A_1, \ldots, A_n$ be positive definite matrices. If $\sigma_1, \sigma_2, \ldots, \sigma_n$ are permutations on $\{1, 2, \ldots, n\}$ satisfying $\{\sigma_1(j), \sigma_2(j), \ldots, \sigma_n(j)\} = \{1, 2, \ldots, n\}$ for each $j = 1, 2, \ldots, n$, then*

$$\begin{aligned} G_\omega(A_{\sigma_1(1)}, \ldots, A_{\sigma_1(n)}) \circ \cdots \circ G_\omega(A_{\sigma_n(1)}, \ldots, A_{\sigma_n(n)}) &\leq A_1 \circ \cdots \circ A_n \\ \Lambda_\omega(A_{\sigma_1(1)}, \ldots, A_{\sigma_1(n)}) \circ \cdots \circ \Lambda_\omega(A_{\sigma_n(1)}, \ldots, A_{\sigma_n(n)}) &\leq A_1 \circ \cdots \circ A_n. \end{aligned}$$

Two approaches to extend to multi-variables the two-variable geometric mean of positive definite matrices have been given by Ando-Li-Mathias [2] and Bini-Meini-Poloni [4] via " symmetrization procedures " and induction, which are called by ALM and BMP geometric means, respectively. H. Lee, Y. Lim and T. Yamazaki [10] proposed weighted geometric means based on a weighted version of the generalized symmetrization procedure of [4] depending on the parameter $\mathbf{t} \in [0,1]^n$. For a three dimensional positive probability vector $\omega = (w_1, w_2, w_3)$, positive definite matrices $A_i, i = 1,2,3$, and $(t_1, t_2, t_3) \in [0,1]^3$, the generalized symmetrization method is given by

$$\begin{aligned} A_i^{(0)} &= A_i, \\ A_1^{(r)} &= \left(A_2^{(r-1)} \#_{\frac{w_3}{1-w_1}} A_3^{(r-1)}\right) \#_{t_3 w_1} A_1^{(r-1)}, \\ A_2^{(r)} &= \left(A_1^{(r-1)} \#_{\frac{w_3}{1-w_2}} A_3^{(r-1)}\right) \#_{t_3 w_2} A_2^{(r-1)}, \\ A_3^{(r)} &= \left(A_1^{(r-1)} \#_{\frac{w_2}{1-w_3}} A_2^{(r-1)}\right) \#_{t_3 w_3} A_3^{(r-1)}. \end{aligned}$$

Then the sequences $\{A_i^{(r)}\}_{r=0}^{\infty}$, $i = 1,2,3$, converge to a common limit, yielding a weighted geometric mean $\mathfrak{G}_3(t_1, t_2, t_3; \omega; A_1, A_2, A_3)$. Inductively we obtain higher-order weighted geometric means $\mathfrak{G}_n(\mathbf{t}; \omega; A_1, \ldots, A_n)$ via the generalized symmetrization procedure and induction, where $\mathbf{t} = (t_1, \ldots, t_n) \in [0,1]^n$ (the first two variables t_1, t_2 do not effect) and n-dimensional positive probability vector ω. From the fact that $A \#_t B = B \#_{1-t} A$, one can see that this is indeed an extension of the ALM (resp. BMP) geometric mean by taking $\omega = (1/n, ..., 1/n)$ and $t_i = 0$ (resp. $t_i = 1$) for all i.

In this paper our main purpose is to extend the inequality (1.1) to the weighted geometric means $\mathfrak{G}_n(\mathbf{t}; \omega; \cdot)$ based on a weighted version of the generalized symmetrization procedure depending on the parameter $\mathbf{t} \in [0,1]^n$.

2. Hadamard product and tensor product

Let $M_{m,n}$ be the set of all $m \times n$ matrices. For $A = [a_{ij}]$ and $B = [b_{ij}]$ in $M_{m,n}$, the *Hadamard product* (or the *Schur product*) $A \circ B$ is the $m \times n$ matrix of entry-wise products:

$$A \circ B := [a_{ij} b_{ij}].$$

One can see easily that the Hadamard product is commutative, associative, and bilinear. Moreover, the Hadamard product of two positive semidefinite (positive definite) matrices is again positive semidefinite (positive definite, respectively). This is known as the Schur product theorem. Moreover, for positive semidefinite matrices A and B

$$\det(A \circ B) \geq \det(A) \det(B).$$

Let $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{s,t}$. Then the *tensor product* (or the *Kronecker product*) $A \otimes B$ of A and B is the $ms \times nt$ matrix given by

$$A \otimes B := \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

Note that the tensor product is associative and bilinear, but not commutative. Furthermore, the tensor product of two positive semidefinite (positive definite) matrices is positive semidefinite (positive definite, respectively). To prove our main result, we list more useful properties for the tensor product.

Lemma 2.1 ([17, Section 4.3]). *The tensor product $\otimes$ satisfies the following.*

(1) *For $A \in M_{m,n}, B \in M_{r,s}, C \in M_{n,k}$, and $D \in M_{s,t}$*

$$(A \otimes B)(C \otimes D) = AC \otimes BD.$$

(2) *For any invertible matrices A and B*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

(3) *For positive definite matrices A, B and any real number t*

$$(A \otimes B)^t = A^t \otimes B^t.$$

(4) *The function $(A, B) \mapsto A \otimes B$ is continuous.*

In many situations properties of tensor products transfer to Hadamard products, due to an important connection between the Hadamard product and the tensor product (see Lemma 4 in [1]). One can see that the Hadamard product is a principal submatrix of the tensor product. Indeed, there is a positive linear map Φ such that

$$\Phi(A_1 \otimes \cdots \otimes A_n) = A_1 \circ \cdots \circ A_n \tag{2.1}$$

for all $m \times m$ matrices $A_1, \ldots, A_n$. Note that this map Φ is also strictly positive: a linear map $\Psi : M_n := M_{n,n} \to M_k$ is positive if $\Psi(A) \geq O$ whenever $A \geq O$, and strictly positive if $\Psi(A) > O$ whenever $A > O$. Also the map Φ is unital, i.e., $\Phi(I) = I$, where I is the identity matrix.

The positive linear map including its related properties is an important tool in operator algebra and quantum information theory. See [3] and its bibliographies. We introduce one of the properties for a strictly positive unital map provided by M.-D. Choi.

Theorem 2.2 ([3, Theorem 2.3.6]). *Let Ψ be a strictly positive and unital map. Then for every positive definite matrix A*

$$\Psi(A)^{-1} \leq \Psi(A^{-1}).$$

The open convex cone $\mathbb{P}$ equipped with the Riemannian trace metric δ is the Hadamard space. Moreover,

$$\gamma : [0, 1] \to \mathbb{P}, \quad \gamma(t) := A\#_t B = A^{1/2} \left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2}$$

is a unique Riemannian geodesic in $\mathbb{P}$ connecting from A to B in $\mathbb{P}$. It is called the *weighted geometric mean* of A and B. The following lemma shows the properties of tensor product and Hadamard product for two-variable weighted geometric means on $\mathbb{P}$.

Lemma 2.3 ([1,11]). *For $A, B, C, D \in \mathbb{P}$ and $t \in [0,1]$*

$$\begin{aligned}(A\#_t B)\otimes(C\#_t D) &= (A\otimes C)\#_t(B\otimes D),\\ (A\#_t B)\circ(C\#_t D) &\leq (A\circ C)\#_t(B\circ D).\end{aligned}$$

The Hadamard and tensor products of finitely many matrices can be defined by induction. We denote as

$$\prod_{j=1}^{n} \circ A_j = A_1 \circ \cdots \circ A_n,\ \prod_{j=1}^{n} \otimes A_j = A_1 \otimes \cdots \otimes A_n$$

for matrices $A_1, \ldots, A_n$ with appropriate sizes.

3. Weighted geometric means $\mathfrak{G}_n$

Hosoo et al. [10] proposed a family of weighted geometric means $\{\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A})\}$ where ω and $\mathbb{A}$ vary over all positive probability vectors in $\mathbb{R}^n$ and n-tuples of positive definite matrices resp. In this section, we shall introduce the definition a family of weighted geometric means $\{\mathfrak{G}(\mathbf{t};\omega;\mathbb{A})\}$.

Let $\Delta_n = \{(w_1, w_2, \ldots, w_n) \in (0,1)^n : \sum_{i=1}^n w_i = 1\}$ be the set of n ($n \geq 2$)-dimensional positive probability vectors and let $\Delta_{n,t} = \{\omega = (w_1, \ldots, w_n) \in \Delta_n : w_i < \frac{1}{t},\ i = 1, 2, \ldots, n\}$ for $t \geq 1$. For $\omega = (w_1, \ldots, w_n) \in \Delta_n$ and $\mathbf{t} = (t_1, \ldots, t_n) \in [0,1]^n$, we denote

$$\begin{aligned}\omega_{\neq j} &= (w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_n),\\ \hat{\omega}_{\neq j} &= \tfrac{1}{1-w_j}\omega_{\neq j} \ \in \Delta_{n-1}, \quad (n \geq 3)\\ \mathbf{t}_{\neq n} &= (t_1, \ldots, t_{n-1}).\end{aligned}$$

Definition 3.1 ([10]). Let $\mathfrak{G}_2 : [0,1]^2 \times \Delta_2 \times \mathbb{P}^2 \to \mathbb{P}$ be defined by

$$\mathfrak{G}_2(t_1, t_2; w_1, w_2; A_1, A_2) = A_1 \#_{w_2} A_2,$$

the $w_2 = (1 - w_1)$-weighted geometric mean of A_1 and A_2. For $n \geq 3$, we define $\mathfrak{G}_n : [0,1]^n \times \Delta_n \times \mathbb{P}^n \to \mathbb{P}$ by induction as follows. Assume that $\mathfrak{G}_{n-1}$ is defined. Let $\mathbf{t} = (t_1, \ldots, t_n) \in [0,1]^n$, $\omega = (w_1, \ldots, w_n) \in \Delta_n$, and let $\mathbb{A} = (A_1, \ldots, A_n) \in \mathbb{P}^n$. Let $\{A_i^{(r)}\}_{r=0}^{\infty}$ be the positive definite operator sequence defined by

$$A_i^{(0)} = A_i, \quad A_i^{(r+1)} = \mathfrak{G}_{n-1}\left(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq i}; \mathbb{A}_{\neq i}^{(r)}\right) \#_{t_n w_i} A_i^{(r)} \tag{3.1}$$

where $\mathbb{A}_{\neq i}^{(r)} = (A_1^{(r)}, \ldots, A_{i-1}^{(r)}, A_{i+1}^{(r)}, \ldots, A_n^{(r)}) \in \mathbb{P}^{n-1}$. Then there exists $\lim_{r\to\infty} A_i^{(r)}$ and it does not depend on i. We define the common limit by $\lim_{r\to\infty} A_i^{(r)} = \mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A})$.

We denote $\mathbf{0}_n = (0, 0, \ldots, 0), \mathbf{1}_n = (1, 1, \ldots, 1) \in \mathbb{R}^n$.

Remark 3.2. The first two variables t_1, t_2 do not effect in the definition of $\mathfrak{G}_n(\mathbf{t}, \omega; \mathbb{A})$; in fact the family of weighted geometric means $\mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A})$ depend only on $\mathbf{t} \in [0,1]^{n-2}$. By definition and $A\#_t B = B\#_{1-t}A$, we have

$$\begin{aligned} \mathfrak{G}_n(\mathbf{0}_n; (1/n)\mathbf{1}_n; \mathbb{A}) &= Alm_n(\mathbb{A}), \\ \mathfrak{G}_n(\mathbf{1}_n; (1/n)\mathbf{1}_n; \mathbb{A}) &= Bmp_n(\mathbb{A}). \end{aligned}$$

Let $n \geq 2$ be a natural number. The n-variable elementary symmetric polynomial σ_k of order k is defined

$$\begin{aligned} \sigma_0(x_1, \ldots, x_n) &= 1 \\ \sigma_k(x_1, \ldots, x_n) &= \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq n} \prod_{i=1}^{k} x_{j_i} \quad (k = 1, \ldots, n-1). \end{aligned}$$

Definition 3.3. For $n \geq 2$, define $\mathbf{q}_n : \Delta_n \to \Delta_{n,n-1}$ by $\mathbf{q}_n(\omega) = (q_n(\omega)_1, \ldots, q_n(\omega)_n)$, where

$$q_n(\omega)_i := \frac{w_i \sigma_{n-2}(\omega_{\neq i})}{(n-1)\sigma_{n-1}(\omega)}.$$

A probability vector consists of non-negative elements whose sum equals to 1. A square matrix is called a stochastic matrix if its columns are probability vectors. A stochastic matrix A is called *regular* if there is a positive integer m such that all elements of A^m are positive. The following theorem is an application of Perron-Frobenius theorem.

Theorem 3.4. *Let A be a regular stochastic matrix. Then 1 is an eigenvalue of A, and there is a unique positive probability vector z such that $Az = z$. Furthermore, the sequence $\{A_k\}_k$ converges to the matrix S whose columns are all equal to z.*

In [10], a family of self-maps $\{\Gamma_n(\mathbf{t}, \cdot)\}$ on Δ_n is constructed varying over $[0,1]^n$ and showed that each of these self-maps interpolates between $\mathbf{q}_n$ and $\mathbf{id}_{\Delta_n}$ and leaves invariant the probability vector $\frac{1}{n}\mathbf{q}_n$.

Definition 3.5 ([10])**.** Let $\Gamma_2 : [0,1]^2 \times \Delta_2 \to \Delta_2$ be defined by $\Gamma_2(t_1, t_2; \omega) = \omega$. For $n \geq 3$,we define $\Gamma_n : [0,1]^n \times \Delta_n \to \Delta_n$ by induction as follows. Assume that Γ_{n-1} is defined. Define an $n \times n$ matrix $U = [u_{ij}]$ depending on $\mathbf{t}$, ω and Γ_{n-1} by

$$u_{ij} = \begin{cases} \Gamma_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq j})_{i-1}(1 - t_n w_j), & i > j; \\ t_n w_j, & i = j; \\ \Gamma_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq j})_i(1 - t_n w_j), & i < j. \end{cases}$$

Then U is a regular stochastic matrix, because all columns of U are probability vectors and all elements of U^2 are positive (all entries of its off-diagonal are positive). By Theorem 3.4, there exists a unique positive probability vector z such that $Uz = z$. We define $\Gamma_n(\mathbf{t}; \omega) := z$.

Theorem 3.6 ([10])**.** *For each $n \geq 2$, the map $\Gamma_n : [0,1]^n \times \Delta_n \to \Delta_n$ defined in Definition 3.5 satisfies*

(1) $\Gamma_n(\mathbf{0}_n;\omega) = \mathbf{q}_n(\omega)$ *for all* $\omega \in \Delta_n$;
(2) $\Gamma_n(\mathbf{1}_n;\omega) = \omega$ *for all* $\omega \in \Delta_n$;
(3) $\Gamma_n(\mathbf{t};\frac{1}{n}\mathbf{1}_n) = \frac{1}{n}\mathbf{1}_n$.

The weighted geometric means $\mathfrak{G}(\mathbf{t};\omega;\mathbb{A})$ fulfill the weighted version of Ando-Li-Mathias 10 properties. For $\mathbf{a} = (a_1,\dots,a_n) \in (0,\infty)^n$, $\omega = (w_1,\dots,w_n) \in \Delta_n$, $\mathbb{A} = (A_1,\dots,A_n) \in \mathbb{P}^n$, a nonsingular matrix M, and a permutation $\sigma \in S^n$ on n-letters, we denote

$$\begin{aligned}
\mathbf{a}\cdot\mathbb{A} &= (a_1A_1, a_2A_2\dots,a_nA_n)\\
\mathbb{A}^{-1} &= (A_1^{-1}, A_2^{-1}\dots,A_n^{-1})\\
\omega_\sigma &= (w_{\sigma(1)}, w_{\sigma(2)}\dots,w_{\sigma(n)})\\
\mathbb{A}_\sigma &= (A_{\sigma(1)}, A_{\sigma(2)}\dots,A_{\sigma(n)})\\
M^*\mathbb{A}M &= (M^*A_1M, M^*A_2M,\dots,M^*A_nM).
\end{aligned}$$

Theorem 3.7 ([10]). *The map* $\mathfrak{G}_n : [0,1]^n \times \Delta_n \times \mathbb{P}^n \to \mathbb{P}$ *satisfies the following properties*

(P1) $\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}) = \prod_{i=1}^n A_i^{\Gamma_n(\mathbf{t};\omega)_i}$ *for commuting* A_i*'s;*
(P2) *(Joint homogeneity)* $\mathfrak{G}_n(\mathbf{t};\omega;\mathbf{a}\cdot\mathbb{A}) = \left(\prod_{i=1}^n a_i^{\Gamma_n(\mathbf{t};\omega)_i}\right)\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A})$;
(P3) *(Permutation invariance)* $\mathfrak{G}_n(\mathbf{t};\omega_\sigma;\mathbb{A}_\sigma) = \mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A})$;
(P4) *(Monotonicity) If* $B_i \le A_i$ *for all* $1 \le i \le n$, *then* $\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{B}) \le \mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A})$;
(P5) *(Continuity) The map* $\mathfrak{G}_n(\mathbf{t};\omega;\cdot)$ *is continuous;*
(P6) *(Congruence invariance)* $\mathfrak{G}_n(\mathbf{t};\omega;M^*\mathbb{A}M) = M^*\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A})M$;
(P7) *(Joint concavity)* $\mathfrak{G}_n(\mathbf{t};\omega;\cdot)$ *is jointly concave;*
(P8) *(Self-duality)* $\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}^{-1})^{-1} = \mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A})$;
(P9) *(Determinantal identity)* $\det\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}) = \prod_{i=1}^n(\det A_i)^{\Gamma_n(\mathbf{t};\omega)_i}$;
(P10) *(AGH mean inequalities)* $\left(\sum_{i=1}^n \Gamma(\mathbf{t};\omega)_i A_i^{-1}\right)^{-1} \le \mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}) \le \sum_{i=1}^n \Gamma(\mathbf{t};\omega)_i A_i$.

By Remark 3.2, the following definition for weighted ALM and BMP geometric means looks natural.

Definition 3.8 (Weighted ALM and BMP geometric means). For $\omega = (w_1,\dots,w_n) \in \Delta_n$, we define

$$\begin{aligned}
Alm_n(\omega;\mathbb{A}) &= \mathfrak{G}_n(\mathbf{0}_n;\omega;\mathbb{A}),\\
Bmp_n(\omega;\mathbb{A}) &= \mathfrak{G}_n(\mathbf{1}_n;\omega;\mathbb{A})
\end{aligned}$$

.

4. Hadamard and tensor products for the weighted geometric means $\mathfrak{G}_n$

We show the property of weighted geometric means $\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A})$ for tensor product $\otimes$. For convenience, we denote

$$\mathbb{A}\otimes\mathbb{B} := (A_1\otimes B_1,\dots,A_n\otimes B_n),$$

where $\mathbb{A} = (A_1, \ldots, A_n), \mathbb{B} = (B_1, \ldots, B_n) \in \mathbb{P}^n$.

Proposition 4.1. *Let* $\mathbb{A} = (A_1, \ldots, A_n), \mathbb{B} = (B_1, \ldots, B_n) \in \mathbb{P}^n$ *and* $\omega = (w_1, w_2, \ldots, w_n)$ *a positive probability vector. Then*

$$\mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}) \otimes \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{B}) = \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A} \otimes \mathbb{B}). \tag{4.1}$$

Proof. We first show that the identity (4.1) hold for $n = 2$. Let $\mathbf{t} \in [0, 1]^2$, $\omega = (w_1, w_2) \in \Delta_2$ and let $\mathbb{A} = (A_1, A_2), \mathbb{B} = (B_1, B_2) \in \mathbb{P}^2$. By Lemma 2.3, we have

$$\begin{aligned} \mathfrak{G}_2(\mathbf{t}; \omega; \mathbb{A}) \otimes \mathfrak{G}_2(\mathbf{t}; \omega; \mathbb{B}) &= (A_1 \#_{w_2} A_2) \otimes (B_1 \#_{w_2} B_2) \\ &= (A_1 \otimes B_1) \#_{w_2} (A_2 \otimes B_2) \\ &= \mathfrak{G}_2(\mathbf{t}; \omega; \mathbb{A} \otimes \mathbb{B}). \end{aligned}$$

We complete the proof of (4.1) by induction on n. Suppose that $\mathfrak{G}_{n-1}(\hat{\mathbf{t}}; \hat{\omega}; \cdot)$ satisfies (4.1) for any $\hat{\mathbf{t}} \in [0, 1]^{n-1}$ and $\hat{\omega} \in \Delta_{n-1}$.

Let $\{A_i^{(r)}\}_{r=0}^{\infty}$ be the positive definite matrix sequences defined in (3.1) and let $\{B_i^{(r)}\}_{r=0}^{\infty}$ (resp. $\{C_i^{(r)}\}_{r=0}^{\infty} = \{(A_i \otimes B_i)^{(r)}\}_{r=0}^{\infty}$) be defined in the same fashion as $\{A_i^{(r)}\}_{r=0}^{\infty}$, but starting from B_i (resp.$A_i \otimes B_i$). Suppose that for some r, $A_i^{(r)} \otimes B_i^{(r)} = C_i^{(r)}$ for all i. Then for each $j = 1, 2, \ldots, n$

$$\mathbb{A}^{(r)} \otimes \mathbb{B}^{(r)} = \mathbb{C}^{(r)} \quad \text{and} \quad (\mathbb{A}^{(r)})_{\neq j} \otimes (\mathbb{B}^{(r)})_{\neq j} = (\mathbb{C}^{(r)})_{\neq j} \tag{4.2}$$

and by the inductive assumption

$$\begin{aligned} C_j^{(r+1)} &= \mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq j}; (\mathbb{C}^{(r)})_{\neq j}) \\ &= \mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq j}; (\mathbb{A}^{(r)})_{\neq j} \otimes (\mathbb{B}^{(r)})_{\neq j}) \\ &= \mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq j}; (\mathbb{A}^{(r)})_{\neq j}) \otimes \mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq j}; (\mathbb{B}^{(r)})_{\neq j}) \\ &= A_j^{(r+1)} \otimes B_j^{(r+1)}. \end{aligned}$$

Since $C_i^{(0)} = A_i \otimes B_i = A_i^{(0)} \otimes B_i^{(0)}$ for all i, (4.2) holds for any $j = 1, \ldots, n$ and $r \in \mathbb{N}$.

From the inductive assumption, the continuity of weighted geometric mean (P5) and (4.2), we have

$$\begin{aligned} & \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}) \otimes \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{B}) \\ = & \left(\lim_{r\to\infty} \mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq i}; (\mathbb{A}^{(r)})_{\neq i})\right) \otimes \left(\lim_{r\to\infty} \mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq i}; (\mathbb{B}^{(r)})_{\neq i})\right) \\ = & \lim_{r\to\infty} \left(\mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq i}; (\mathbb{A}^{(r)})_{\neq i}) \otimes \mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq i}; (\mathbb{B}^{(r)})_{\neq i})\right) \\ = & \lim_{r\to\infty} \left(\mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq i}; (\mathbb{A}^{(r)})_{\neq i} \otimes (\mathbb{B}^{(r)})_{\neq i})\right) \\ = & \lim_{r\to\infty} \left(\mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq i}; (\mathbb{A}^{(r)} \otimes \mathbb{B}^{(r)})_{\neq i})\right) \\ = & \lim_{r\to\infty} \left(\mathfrak{G}_{n-1}(\mathbf{t}_{\neq n}; \hat{\omega}_{\neq i}; (\mathbb{C}^{(r)})_{\neq i})\right) \\ = & \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A} \otimes \mathbb{B}). \end{aligned}$$

□

We simply denote as Σ_n the collection of all $n-$tuples $(\sigma_1, \sigma_2, \dots, \sigma_n)$ of permutations on n letters satisfying

$$\{\sigma_1(j), \sigma_2(j), \dots, \sigma_n(j)\} = \{1, 2, \dots, n\} \tag{4.3}$$

for each $j = 1, 2, \dots, n$. We now show that the Hadamard products of weighted geometric means of $\mathbb{A}_{\sigma_1}, \mathbb{A}_{\sigma_2}, \dots, \mathbb{A}_{\sigma_n}$ for any n-tuple of permutations $(\sigma_1, \dots, \sigma_n) \in \Sigma_n$ is bounded by the Hadamard products of $A_1, \dots, A_n$. One can see that it is an extension of Ando's result in the equation (1.1).

Theorem 4.2. *Let* $\mathbb{A} = (A_1, \dots, A_n) \in \mathbb{P}^n$, $\mathbf{t} = (t_1, \dots, t_n) \in [0, 1]^n$ *and let* $\omega = (w_1, \dots, w_n) \in \Delta_n$. *For any n-tuple of permutations* $(\sigma_1, \dots, \sigma_n) \in \Sigma_n$,

$$(A_1^{-1} \circ \cdots \circ A_n^{-1})^{-1} \leq \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_1}) \circ \cdots \circ \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_n}) \leq A_1 \circ \cdots \circ A_n.$$

Proof. There is a positive linear map Φ from $M_{m^n} := M_{m^n, m^n}$ to M_m satisfying the equation (2.1). Indeed, for any n-tuple of permutations $(\sigma_1, \dots, \sigma_n) \in \Gamma_n$

$$\Phi(\mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_1}) \otimes \cdots \otimes \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_n})) = \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_1}) \circ \cdots \circ \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_n}).$$

By Proposition 4.1, the arithmetic-geometric mean inequality (P10) in Theorem 3.7, and the linearity of Φ, we obtain

$$\begin{aligned}
\mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_1}) \circ \cdots \circ \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_n}) &= \Phi(\mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_1}) \otimes \cdots \otimes \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_n})) \\
&= \Phi(\mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_1} \otimes \cdots \otimes \mathbb{A}_{\sigma_n})) \\
&\leq \Phi\left(\sum_{j=1}^{n} \Gamma(\mathbf{t}; \omega)_j \left(A_{\sigma_1(j)} \otimes \cdots \otimes A_{\sigma_n(j)}\right)\right) \\
&= \sum_{j=1}^{n} \Gamma(\mathbf{t}; \omega)_j \Phi(A_{\sigma_1(j)} \otimes \cdots \otimes A_{\sigma_n(j)}) \\
&= \sum_{j=1}^{n} \Gamma(\mathbf{t}; \omega)_j \left(A_{\sigma_1(j)} \circ \cdots \circ A_{\sigma_n(j)}\right) \\
&= \sum_{j=1}^{n} \Gamma(\mathbf{t}; \omega)_j (A_1 \circ \cdots \circ A_n) \\
&= A_1 \circ \cdots \circ A_n.
\end{aligned}$$

The sixth equality follows from the condition (4.3).

Applying the above argument to $\mathbb{A}_{\sigma_1}^{-1}, \mathbb{A}_{\sigma_2}^{-1}, \dots, \mathbb{A}_{\sigma_n}^{-1}$, we have

$$A_1^{-1} \circ \cdots \circ A_n^{-1} \geq \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_1}^{-1}) \circ \cdots \circ \mathfrak{G}_n(\mathbf{t}; \omega; \mathbb{A}_{\sigma_n}^{-1}).$$

Taking the inverse on both sides, using Choi's inequality in Theorem 2.2, Lemma 2.1 (2), and the self-duality of weighted Karcher means (P8) in Theorem 3.7 yield

$$\begin{aligned}\left[A_1^{-1}\circ\cdots\circ A_n^{-1}\right]^{-1} &\le \left[\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_1}^{-1})\circ\cdots\circ\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_n}^{-1})\right]^{-1}\\ &= \left[\Phi\left(\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_1}^{-1})\otimes\cdots\otimes\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_n}^{-1})\right)\right]^{-1}\\ &\le \Phi\left(\left[\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_1}^{-1})\otimes\cdots\otimes\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_n}^{-1})\right]^{-1}\right)\\ &= \Phi\left(\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_1}^{-1})^{-1}\otimes\cdots\otimes\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_n}^{-1})^{-1}\right)\\ &= \Phi\left(\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_1})\otimes\cdots\otimes\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_n})\right)\\ &= \mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_1})\circ\cdots\circ\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma_n}).\end{aligned}$$

□

Corollary 4.3. *Let* $\mathbb{A}=(A_1,\dots,A_n)\in\mathbb{P}^n$, $\mathbf{t}=(t_1,\dots,t_n)\in[0,1]^n$ *and let* $\omega=(w_1,\dots,w_n)\in\Delta_n$. *Then*

$$(A_1^{-1}\circ\cdots\circ A_n^{-1})^{-1}\le\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A})\circ\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_\sigma)\circ\cdots\circ\mathfrak{G}_n(\mathbf{t};\omega;\mathbb{A}_{\sigma^{n-1}})\le A_1\circ\cdots\circ A_n$$

for any n*-cyclic permutation* σ *on* $\{1,2,\dots,n\}$.

Proof. Since σ is an n-cyclic permutation, the family of permutations $\{\sigma^0,\sigma^1,\sigma^2,\dots,\sigma^{n-1}\}$ satisfies the condition (4.3) for each j, where σ^0 is the identity map. By Theorem 4.2 the desired inequality is proved. □

Corollary 4.4. *Let* $\mathbb{A}=(A_1,\dots,A_n)\in\mathbb{P}^n$. *Then*

$$(A_1^{-1}\circ\cdots\circ A_n^{-1})^{-1}\le\mathfrak{G}_n\left(\mathbf{t};\frac{1}{n}\mathbf{1}_n;\mathbb{A}\right)\circ\cdots\circ\mathfrak{G}_n\left(\mathbf{t};\frac{1}{n}\mathbf{1}_n;\mathbb{A}\right)\le A_1\circ\cdots\circ A_n.$$

Proof. For a uniform probability vector $\frac{1}{n}\mathbf{1}_n=(1/n,\dots,1/n)$ in $\mathbb{R}^n$ and any permutation σ on $\{1,\dots,n\}$, we have

$$\mathfrak{G}_n\left(\mathbf{t};\left(\frac{1}{n}\mathbf{1}_n\right)_\sigma;\mathbb{A}_\sigma\right)=\mathfrak{G}_n\left(\mathbf{t};\left(\frac{1}{n}\mathbf{1}_n\right);\mathbb{A}\right)$$

because of permutation invariance (P3) in Theorem 3.7. By Theorem 4.2 the desired inequality is obtained.

□

Corollary 4.5. *Let* $\mathbb{A}=(A_1,\dots,A_n)\in\mathbb{P}^n$. *Then*

$$(A_1^{-1}\circ\cdots\circ A_n^{-1})^{-1}\le ALM_n(\omega;\mathbb{A})\circ\cdots\circ ALM_n(\omega;\mathbb{A})\le A_1\circ\cdots\circ A_n,$$

$$(A_1^{-1}\circ\cdots\circ A_n^{-1})^{-1}\le BMP_n(\omega;\mathbb{A})\circ\cdots\circ BMP_n(\omega;\mathbb{A})\le A_1\circ\cdots\circ A_n$$

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