Linear and Nonlinear Analysis
Volume 5, Number 2, 2019, 355–369
Yokohama Publishers
ISSN 2188-8167

A CYCLIC VISCOSITY APPROXIMATION METHOD FOR THE MULTIPLE-SET SPLIT EQUALITY COMMON FIXED-POINT PROBLEM

YAQIN WANG*, XIAOLI FANG†, AND TAE-HWA KIM‡

Abstract. In this paper, we consider a newly cyclic viscosity approximation method to approximate the multiple-set split common fixed point problem governed by demicontractive mappings which are generalization of quasi-nonexpansive mappings in Hilbert spaces, and we prove that the generated sequence converges strongly to a solution of this problem. The results obtained in this paper generalize and improve the recent ones announced by many others.

1. Introduction

The split feasibility problem (SFP) in finite dimensional Hilbert spaces was introduced by Censor and Elfving [2] in 1994 for modeling inverse problems which arise from phase retrievals and in medical imagine reconstruction. Recently, it has been found that the SFP can be used in many areas such as image restoration, computer tomograph, and radiation therapy treatment planning. Some methods have been proposed to solve split feasibility problems; see, for instance, [1, 14, 15, 16].

In 2013, Moudafi and Al-Shemas [9] introduced the following new split feasibility problem, which is called the split equality fixed point problem (SEFP). Let H_1, H_2, H_3 be real Hilbert spaces, let $A : H_1 \to H_3, B : H_2 \to H_3$ be two bounded linear operators, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be two firmly quasi-nonexpansive mappings. The SEFP in [9] is to

$$\text{find } \ x^* \in F(U),\ y^* \in F(T) \ \text{ such that } \ Ax^* = By^*. \tag{1.1}$$

The interest is to cover many situations, for instance, in decomposition methods for PDF's, applications in game theory and in intensity-modulated radiation therapy (IMRT).

For solving the SEFP (1.1), Moudafi and Al-Shemas [9] introduced the following simultaneous iterative method:

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases}$$

2010 *Mathematics Subject Classification.* 47H09, 47H10, 47J05, 54H25.

Key words and phrases. Multiple-set split equality common fixed-point problems; demicontractive mappings; cyclic viscosity approximation methods.

*Supported by the grant No. 11401388, 11671365, LQ13A010007.

†Supported by the grant No. LY14A010006.

‡Corresponding author and supported by a Research Grant of Pukyong National University (2019 year).

for firmly quasi-nonexpansive mappings U and T, where $\gamma_k \in (\epsilon, \frac{2}{\lambda_A+\lambda_B} - \epsilon)$, λ_A, λ_B stand for the spectral radiuses of A^*A and B^*B, respectively.

Recently, Zhao and Wang [17] proposed the following viscosity iterative algorithm for solving the SEFP (1.1):

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k f_1(x_k) + (1-\alpha_k)((1-w_k)u_k + w_k U u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k f_2(y_k) + (1-\alpha_k)((1-w_k)v_k + w_k T v_k), \end{cases} \tag{1.2}$$

where $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ are two contractions, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are quasi-nonexpansive. They proved a strong convergence result of the algorithm (1.2) in Hilbert spaces.

On the other hand, the multiple-set split equality common fixed-point problem (MSECFP) of quasi-nonexpansive mappings studied by Zhao and Wang [18] is to

$$\text{find } \ x^* \in \cap_{i=1}^p F(U_i), \ y^* \in \cap_{j=1}^q F(T_j) \quad \text{such} \quad \text{that} \quad Ax^* = By^*, \tag{1.3}$$

where $p, q \geq 1$ are integers. They introduced two mixed cyclic and parallel iterative algorithms for solving the MSECFP (1.3) of quasi-nonexpansive mappings and proved the weak convergence of these two algorithms.

Inspired and motivated by the works mentioned above, we consider a newly cyclic viscosity approximation method for the MSECFP (1.3) of demicontractive mappings in Hilbert spaces. Under some mild assumptions we establish some strong convergence theorems.

2. Preliminaries

Throughout this paper, we always assume that H_1, H_2, H_3 are real Hilbert spaces and let $\mathbb{N}$ and $\mathbb{R}$ be the set of positive integers and real numbers, respectively. We use $\to$ and $\rightharpoonup$ to denote strong and weak convergence, respectively, and $F(T)$ denotes the set of the fixed points of a mapping T. We use $\omega_w x_k = \{x : \exists\ x_{k_j} \rightharpoonup x\}$ to stand for the weak ω-limit set of $\{x_k\}$ and use Γ to stand for the solution set of the MSECFP (1.3), i.e.,

$$\Gamma := \{(x, y) |\ x \in \cap_{i=1}^p F(U_i),\ y \in \cap_{j=1}^q F(T_j) \text{ such that } Ax = By\}.$$

Let C be a nonempty closed convex subset of a Hilbert space H. The metric (or nearest point) projection P_C from H onto C is defined as follows: Given $x \in H$, the unique point $P_C x \in C$ satisfies the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\|.$$

It is well known [10] that P_C is a nonexpansive mapping and is characterized by the inequality

$$P_C x \in C,\ \langle x - P_C x, y - P_C x\rangle \leq 0,\ \forall\ y \in C. \tag{2.1}$$

Definition 2.1. Let H be a real Hilbert space. A mapping $T : H \to H$ is said to be

(i) Lipschitzian if there exists a constant $\rho > 0$ such that

$$\|Tx - Ty\| \leq \rho\|x - y\|, \quad \forall\, x, y \in H,$$

especially, if $\rho \in (0, 1)$, T is said to be a contraction with constant ρ;

(ii) nonexpansive if $\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in H$;

(iii) quasi-nonexpansive if $F(T) \neq \emptyset$ and if $\|Tx - q\| \leq \|x - q\|, \quad \forall\, x \in H, q \in F(T)$;

(iv) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall\, x, y \in H;$$

or equivalently,

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall\, x, y \in H;$$

(v) μ-demicontractive if $F(T) \neq \emptyset$ and the exists a constant $\mu \in (-\infty, 1)$ such that

$$\|Tx - q\|^2 \leq \|x - q\|^2 + \mu\|x - Tx\|^2, \quad \forall\, x \in H, q \in F(T).$$

Remark 2.2. Notice that a 0-demicontractive mapping is exactly quasi-nonexpansive. In particular, we say that it is quasi-strictly pseudo-contractive [7] if $0 \leq \mu < 1$. Moreover, if $\mu \leq 0$, every μ-demicontractive mapping becomes quasi-nonexpansive. Therefore, it is sufficient to only take $\mu \in (0, 1)$ in (v) of Definition 2.1 in Hilbert spaces. However, as seen in (iv) of Definition 2.1, every firmly quasi-nonexpansive mapping (often called to be a directed operator [3]) is obvious (-1)-demicontractive.

It is worth noting that the class of demicontractive mappings is more general than the class of quasi-nonexpansive mappings and the class of firmly quasi-nonexpansive mappings.

Definition 2.3. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $F : C \to H$ is said to be

(i) monotone if $\langle Fx - Fy, x - y\rangle \geq 0, \quad \forall\, x, y \in C$;

(ii) strictly monotone if $\langle Fx - Fy, x - y\rangle > 0, \quad \forall\, x, y \in C, x \neq y$;

(iii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall x, y \in C.$$

Definition 2.4. Let H be a real Hilbert space. An operator $T : H \to H$ is called demiclosed at origin if, for any sequence $\{x_k\}$ which converges weakly to x, and if the sequence $\{Tx_k\}$ converges strongly to 0, then $Tx = 0$.

As a special case of the demicloseness principle on uniformly convex Banach spaces given by [4], we know that if C is a nonempty closed convex subset of a Hilbert space H, and $T : C \to H$ is a nonexpansive mapping. Then the mapping $I - T$ is demiclosed on C. Now the following question is naturally raised: If $T : C \to H$ is quasi-nonexpansive, is $I - T$ still demiclosed on C? The answer is negative even at 0 as follows.

Example 2.5. ([11]; Example 2.11). The mapping $T : [0,1] \to [0,1]$ is defined by

$$Tx = \begin{cases} \frac{x}{5}, \quad x \in [0, \frac{1}{2}], \\ x \sin \pi x, \quad x \in (\frac{1}{2}, 1]. \end{cases}$$

Then T is a quasi-nonexpansive mapping, but $I - T$ is not demiclosed at 0.

Lemma 2.6. ([8]). *Let T be a μ-demicontractive self mapping on H with $F(T) \neq \emptyset$ and set $T_\alpha = (1-\alpha)I + \alpha T$ for $\alpha \in [0,1]$. Then, T_α is quasi-nonexpansive provided that $\alpha \in [0, 1-\mu]$ and*

$$\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \alpha(1 - \mu - \alpha)\|x - Tx\|^2, \quad \forall\, (x, q) \in H \times F(T).$$

Lemma 2.7. ([7]; Proposition 2.1). *Assume C is a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a self-mapping of C. If T is a μ-demicontractive mapping (which is also called μ-quasi-strict pseudo-contraction in* [7]*), then the fixed point set $F(T)$ is closed and convex.*

Lemma 2.8. ([5]). *Assume $\{s_k\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{k+1} \leq (1 - \lambda_k)s_k + \lambda_k \delta_k, \\ s_{k+1} \leq s_k - \eta_k + \mu_k, \end{cases}$$

where $\{\lambda_k\}$ is a sequence in (0,1), $\{\eta_k\}$ is a sequence of nonnegative real numbers and $\{\delta_k\}$ and $\{\mu_k\}$ are two sequences in $\mathbb{R}$ such that

(i) $\sum_{k=1}^{\infty} \lambda_k = \infty$;

(ii) $\lim_{k\to\infty} \mu_k = 0$;

(iii) $\lim_{l\to\infty} \eta_{k_l} = 0$ *implies* $\limsup_{l\to\infty} \delta_{k_l} \leq 0$ *for any subsequence* $\{k_l\} \subset \{k\}$.

Then $\lim_{k\to\infty} s_k = 0$.

Lemma 2.9. ([6]). *Let X and Y be Banach spaces, A be a continuous linear operator from X to Y. Then A is weakly continuous.*

Lemma 2.10. ([13]; Proposition 2.7). *Let H be a real Hilbert space. Suppose that $F : H \to H$ is κ-Lipschitzian and η-strongly monotone over a closed convex set $C \subset H$. Then, the following $VIP(F, C)$*

$$\langle v - u^*, F(u^*)\rangle \geq 0, \quad \forall\, v \in C$$

has its unique solution $u^* \in C$.

3. Main results

In this section, we introduce a new cyclic viscosity approximation method for the MSECFP (1.3) of demicontractive mappings and prove the strong convergence of this algorithm.

Given a positive integer p, the p-mod function i takes values in the set $\{1, 2, \cdots, p\}$ as

$$i(k) = \begin{cases} p, \ r = 0; \\ r, \ 0 < r < p, \end{cases}$$

for $k = np + r$ for some integers $n \geq 0$ and $0 \leq r < p$. Given a positive integer q, the q-mod function j can be also defined in a similar way.

Put $H^* = H_1 \times H_2$. Define the inner product of H^* as follows:

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle, \ \forall \ (x_1, y_1), (x_2, y_2) \in H^*.$$

It is easy to see that H^* is also a real Hilbert space and

$$\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}, \ \forall \ (x, y) \in H^*.$$

Algorithm 3.1. Let $x_0 \in H_1, \ y_0 \in H_2$ be arbitrary given and $p, q \geq 1$ be integers. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be two contractions with constants $\rho_1, \ \rho_2 \in [0, 1)$. Let the sequences $\{\alpha_k\}, \{\beta_k\}, \{t_k\} \subset [0, 1]$. Assume that the kth iterate $x_k \in H_1, \ y_k \in H_2$ has been constructed and $Ax_k - By_k \neq 0$, then we calculate $(k+1)$th iterate $(x_{k+1}, \ y_{k+1})$ via the formula

$$\left\{ \begin{array}{l} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = t_k f_1(x_k) + (1 - t_k)(\alpha_k u_k + (1 - \alpha_k) U_{i(k)} u_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = t_k f_2(y_k) + (1 - t_k)(\beta_k v_k + (1 - \beta_k) T_{j(k)} v_k), \quad \forall \, k \geq 0. \end{array} \right. \tag{3.1}$$

Assume the stepsize γ_k is chosen in such a way that

$$\gamma_k \in (\varepsilon, \min\{\eta, \frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2}\} - \varepsilon) \tag{3.2}$$

for all $k \in \Omega$ and small enough $\varepsilon > 0$, where the index set $\Omega = \{k : Ax_k - By_k \neq 0\}$,

$$\eta = \frac{2\|Ax_l - By_l\|^2}{\|A^*(Ax_l - By_l)\|^2 + \|B^*(Ax_l - By_l)\|^2}, \ l = \min_{k \in \Omega}\{k\},$$

otherwise, $\gamma_k = \gamma$ (γ being any nonnegative value). If $Ax_k = By_k = 0$, then $u_k = x_k, \ v_k = y_k$ and

$$\left\{ \begin{array}{l} x_{k+1} = t_k f_1(x_k) + (1 - t_k)(\alpha_k x_k + (1 - \alpha_k) U_{i(k)} x_k), \\ y_{k+1} = t_k f_2(y_k) + (1 - t_k)(\beta_k y_k + (1 - \beta_k) T_{j(k)} y_k). \end{array} \right.$$

Lemma 3.2. *Assume the solution set Γ of (1.3) is nonempty. Then $\{\gamma_k\}$ defined by (3.2) is well defined and bounded.*

Proof. Take $(x, y) \in \Gamma$, i.e., $x \in \cap_{i=1}^p F(U_i), \ y \in \cap_{j=1}^q F(T_j)$ and $Ax = By$. We have

$$\langle A^*(Ax_k - By_k), x_k - x \rangle = \langle Ax_k - By_k, Ax_k - Ax \rangle$$

and

$$\langle B^*(Ax_k - By_k), y - y_k \rangle = \langle Ax_k - By_k, By - By_k \rangle.$$

By adding the two above equalities and by taking $Ax = By$ into account, we obtain

$$\begin{aligned} \|Ax_k - By_k\|^2 &= \langle A^*(Ax_k - By_k), x_k - x \rangle + \langle B^*(Ax_k - By_k), y - y_k \rangle \\ &\leq \|A^*(Ax_k - By_k)\| \|x_k - x\| + \|B^*(Ax_k - By_k)\| \|y - y_k\|. \end{aligned}$$

Consequently, for $k \in \Omega$, that is, $\|Ax_k - By_k\| > 0$, we have $\|A^*(Ax_k - By_k)\| \neq 0$ or $\|B^*(Ax_k - By_k)\| \neq 0$. And since

$$\|A^*(Ax_k - By_k)\|^2 \leq \|A^*\|^2 \|Ax_k - By_k\|^2$$

and

$$\|B^*(Ax_k - By_k)\|^2 \le \|B^*\|^2\|Ax_k - By_k\|^2,$$

we have

$$\frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2} \ge \frac{2}{\|A^*\|^2 + \|B^*\|^2} = \frac{2}{\|A\|^2 + \|B\|^2}.$$

Thus we can choose small enough $\varepsilon \in (0, \frac{1}{\|A\|^2+\|B\|^2})$. This leads that $\{\gamma_k\}$ is well defined. From (3.2) we obtain $\gamma_k \in (\varepsilon, \eta - \varepsilon)$ and η is a fixed positive number, so $\{\gamma_k\}$ is bounded. □

Lemma 3.3. *Given two bounded linear operators $A : H_1 \to H_3, B : H_2 \to H_3$, let $U_i : H_1 \to H_1$ $(1 \le i \le p)$ and $T_j : H_2 \to H_2$ $(1 \le j \le q)$ be τ_i-demicontractive and ν_j-demicontractive, respectively. Assume that the solution set Γ of (1.3) is nonempty. Then Γ is a nonempty closed convex set.*

Proof. By Lemma 2.7 we have $F(T_i)$ $(1 \le i \le p)$ and $F(U_j)$ $(1 \le j \le q)$ are both closed convex subsets, and since A and B are both linear, it is easy to see that Γ is a closed convex subset in H^*. □

Lemma 3.4. ([12]; Lemma 3.1). *Let $\{u_k\}$ be a bounded sequence of a Hilbert space H. Let p be a positive integer and $I = \{1, 2, \cdots, p\}$. If $\lim_{k\to\infty} \|u_{k+1} - u_k\| = 0$ and $x^* \in \omega_w u_k$, then for any $i \in I$, there exists a subsequence $\{u_{k_m}\}$ of $\{u_k\}$, depending on i, such that $i(k_m) = i$ for all m and $u_{k_m} \rightharpoonup x^*$, where i denotes the p-mod function.*

Theorem 3.5. *Let H_1, H_2, H_3 be real Hilbert spaces. Given two bounded linear operators $A : H_1 \to H_3, B : H_2 \to H_3$, let $U_i : H_1 \to H_1$ $(1 \le i \le p)$ and $T_j : H_2 \to H_2$ $(1 \le j \le q)$ be τ_i-demicontractive and ν_j-demicontractive, respectively. Suppose that $I - U_i$ $(1 \le i \le p)$, $I - T_j$ $(1 \le j \le q)$ are demiclosed at origin and the solution set Γ of the MSECFP (1.3) is nonempty. Assume that the following conditions are satisfied:*

(i) $\rho_1,\ \rho_2 \in [0, \frac{1}{\sqrt{2}})$;
(ii) $\lim_{k\to\infty} t_k = 0$ *and* $\sum_{k=0}^{\infty} t_k = \infty$;
(iii) $\tau < \liminf_{k\to\infty} \alpha_k \le \limsup_{k\to\infty} \alpha_k < 1$;
(iv) $\nu < \liminf_{k\to\infty} \beta_k \le \limsup_{k\to\infty} \beta_k < 1$,

where $\tau = \max_{1\le i\le p} \tau_i$, $\nu = \max_{1\le j\le q} \nu_j$.

Then the sequence $\{(x_k, y_k)\} \subset H^*$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \Gamma$ which is the unique solution of the following variational inequality problem (VIP)

$$\langle ((I - f_1)x^*, (I - f_2)y^*), (x, y) - (x^*, y^*)\rangle \ge 0, \quad \forall\, (x, y) \in \Gamma. \tag{3.3}$$

Proof. We divide the proof into several steps.

Step 1. The VIP (3.3) has a unique solution $(x^*, y^*) \in \Gamma$.

By Lemma 3.3, we know that Γ is a nonempty closed convex subset in H^*. Let $F : \Gamma \subset H^* \to H^*$ be defined by

$$F(x, y) = ((I - f_1)x, (I - f_2)y), \quad \forall\, (x, y) \in \Gamma.$$

Putting $\rho = \max\{\rho_1, \rho_2\}$, then from the condition (i) we have $\rho \in [0, \frac{1}{\sqrt{2}})$. For any $(x_1, y_1), (x_2, y_2) \in \Gamma$, since f_1 and f_2 are two contractions, we have

$$\begin{aligned}
& \langle F(x_1, y_1) - F(x_2, y_2), (x_1, y_1) - (x_2, y_2)\rangle \\
= & \langle((I - f_1)x_1 - (I - f_1)x_2, (I - f_2)y_1 - (I - f_2)y_2), (x_1 - x_2, y_1 - y_2)\rangle \\
= & \langle(I - f_1)x_1 - (I - f_1)x_2, x_1 - x_2\rangle + \langle(I - f_2)y_1 - (I - f_2)y_2, y_1 - y_2\rangle \\
\geq & \|x_1 - x_2\|^2 - \|f_1(x_1) - f_1(x_2)\|\|x_1 - x_2\| \\
& +\|y_1 - y_2\|^2 - \|f_2(y_1) - f_2(y_2)\|\|y_1 - y_2\| \\
\geq & (1 - \rho)(\|x_1 - x_2\|^2 + \|y_1 - y_2\|^2) \\
= & (1 - \rho)\|(x_1, y_1) - (x_2, y_2)\|^2,
\end{aligned}$$

which implies that F is $(1 - \rho)$-strongly monotone. And

$$\begin{aligned}
& \|F(x_1, y_1) - F(x_2, y_2)\|^2 \\
= & \|((I - f_1)x_1 - (I - f_1)x_2, (I - f_2)y_1 - (I - f_2)y_2)\|^2 \\
= & \|(I - f_1)x_1 - (I - f_1)x_2\|^2 + \|(I - f_2)y_1 - (I - f_2)y_2\|^2 \\
\leq & (1 + \rho_1)^2\|x_1 - x_2\|^2 + (1 + \rho_2)^2\|y_1 - y_2\|^2 \\
\leq & (1 + \rho)^2\|(x_1, y_1) - (x_2, y_2)\|^2,
\end{aligned}$$

which implies that F is $(1 + \rho)^2$-Lipschitzian. Therefore, it follows from Lemma 2.10 that the VIP (3.3) has a unique solution $(x^*, y^*) \in \Gamma$.

Step 2. The sequences $\{x_k\}$ and $\{y_k\}$ are bounded.

Since $(x^*, y^*) \in \Gamma$, then $x^* \in \cap_{i=1}^p F(U_i)$, $y^* \in \cap_{j=1}^q F(T_j)$ and $Ax^* = By^*$. By (3.1) we have

$$\begin{aligned}
& \|u_k - x^*\|^2 \\
= & \|x_k - \gamma_k A^*(Ax_k - By_k) - x^*\|^2 \\
= & \|x_k - x^*\|^2 - 2\gamma_k\langle x_k - x^*, A^*(Ax_k - By_k)\rangle + \gamma_k^2\|A^*(Ax_k - By_k)\|^2 \\
= & \|x_k - x^*\|^2 - 2\gamma_k\langle Ax_k - Ax^*, Ax_k - By_k\rangle + \gamma_k^2\|A^*(Ax_k - By_k)\|^2,
\end{aligned}$$

and

$$\begin{aligned}
& \|v_k - y^*\|^2 \\
= & \|y_k + \gamma_k B^*(Ax_k - By_k) - y^*\|^2 \\
= & \|y_k - y^*\|^2 + 2\gamma_k\langle y_k - y^*, B^*(Ax_k - By_k)\rangle + \gamma_k^2\|B^*(Ax_k - By_k)\|^2 \\
= & \|y_k - y^*\|^2 + 2\gamma_k\langle By_k - By^*, Ax_k - By_k\rangle + \gamma_k^2\|B^*(Ax_k - By_k)\|^2.
\end{aligned}$$

By adding the above equalities and $Ax^* = By^*$, we have

$$\begin{aligned}
& \|u_k - x^*\|^2 + \|v_k - y^*\|^2 \\
= & \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - \gamma_k[2\|Ax_k - By_k\|^2 \\
& -\gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)]. \qquad (3.4)
\end{aligned}$$

Put $\widetilde{u}_k = \alpha_k u_k + (1-\alpha_k)U_{i(k)}u_k$ and $\widetilde{v}_k = \beta_k v_k + (1-\beta_k)T_{j(k)}v_k$. Using (ii) and Lemma 2.6 we have

$$\begin{aligned}\|\widetilde{u}_k - x^*\|^2 &= \|\alpha_k u_k + (1-\alpha_k)U_{i(k)}u_k - x^*\|^2 \\ &\leq \|u_k - x^*\|^2 - (1-\alpha_k)(\alpha_k - \tau_i)\|U_{i(k)}u_k - u_k\|^2 \\ &\leq \|u_k - x^*\|^2 - (1-\alpha_k)(\alpha_k - \tau)\|U_{i(k)}u_k - u_k\|^2 \qquad (3.5) \\ &\leq \|u_k - x^*\|^2 \qquad (3.6)\end{aligned}$$

for all sufficiently large k. Similarly, we obtain

$$\begin{aligned}&\|\widetilde{v}_k - y^*\|^2 \\ \leq\ &\|v_k - y^*\|^2 - (1-\beta_k)(\beta_k - \nu)\|T_{j(k)}v_k - v_k\|^2 \qquad (3.7) \\ \leq\ &\|v_k - y^*\|^2 \qquad (3.8)\end{aligned}$$

for all sufficiently large k. It follows from (3.1), (3.5) and the convexity of $\|\cdot\|^2$ that

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &\leq t_k\|f_1(x_k) - x^*\|^2 + (1-t_k)\|\widetilde{u}_k - x^*\|^2 \\ &\leq t_k[\rho_1\|x_k - x^*\| + \|f_1(x^*) - x^*\|]^2 + (1-t_k)(\|u_k - x^*\|^2 \\ &\quad -(1-\alpha_k)(\alpha_k - \tau)\|U_{i(k)}u_k - u_k\|^2) \\ &\leq 2t_k\rho^2\|x_k - x^*\|^2 + 2t_k\|f_1(x^*) - x^*\|^2 + (1-t_k)\|u_k - x^*\|^2 \\ &\quad -(1-t_k)(1-\alpha_k)(\alpha_k - \tau)\|U_{i(k)}u_k - u_k\|^2. \qquad (3.9)\end{aligned}$$

Replacing the role of (3.5) with (3.7), we similarly obtain

$$\begin{aligned}\|y_{k+1} - y^*\|^2 &\leq 2t_k\rho^2\|y_k - y^*\|^2 + 2t_k\|f_2(y^*) - y^*\|^2 + (1-t_k)\|v_k - y^*\|^2 \\ &\quad -(1-t_k)(1-\beta_k)(\beta_k - \nu)\|T_{j(k)}v_k - v_k\|^2. \qquad (3.10)\end{aligned}$$

It follows from (3.4), (3.9) and (3.10) that

$$\begin{aligned}&\|x_{k+1} - x^*\|^2 + \|y_{k+1} - y^*\|^2 \\ \leq\ &2t_k\rho^2(\|x_k - x^*\|^2 + \|y_k - y^*\|^2) + 2t_k(\|f_1(x^*) - x^*\|^2 \\ &+\|f_2(y^*) - y^*\|^2) + (1-t_k)(\|u_k - x^*\|^2 + \|v_k - y^*\|^2) \\ &-(1-t_k)(1-\alpha_k)(\alpha_k - \tau)\|U_{i(k)}u_k - u_k\|^2 \\ &-(1-t_k)(1-\beta_k)(\beta_k - \nu)\|T_{j(k)}v_k - v_k\|^2 \\ \leq\ &[1 - t_k(1-2\rho^2)](\|x_k - x^*\|^2 + \|y_k - y^*\|^2) \\ &+2t_k(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2) \\ &-(1-t_k)\gamma_k[2\|Ax_k - By_k\|^2 \\ &-\gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)] \\ &-(1-t_k)(1-\alpha_k)(\alpha_k - \tau)\|U_{i(k)}u_k - u_k\|^2 \\ &-(1-t_k)(1-\beta_k)(\beta_k - \nu)\|T_{j(k)}v_k - v_k\|^2.\end{aligned}$$

Then, setting $s_k = \|x_k - x^*\|^2 + \|y_k - y^*\|^2$, we get

$$s_{k+1} \leq [1 - t_k(1-2\rho^2)]s_k + 2t_k(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2)$$

$$
\begin{aligned}
&-(1-t_k)\gamma_k[2\|Ax_k - By_k\|^2 \\
&-\gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)] \\
&-(1-t_k)(1-\alpha_k)(\alpha_k - \tau)\|U_{i(k)}u_k - u_k\|^2 \\
&-(1-t_k)(1-\beta_k)(\beta_k - \nu)\|T_{j(k)}v_k - v_k\|^2 \qquad (3.11)\\
\leq\ & [1 - t_k(1-2\rho^2)]s_k + t_k(1-2\rho^2)\frac{2(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2)}{1-2\rho^2}.
\end{aligned}
$$

It follows from induction that

$$
s_{k+1} \leq \max\{s_0, \frac{2(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2)}{1-2\rho^2}\}, \quad \forall\, k \geq 0,
$$

i.e., $\{s_k\}$ is bounded. So $\{x_k\}$ and $\{y_k\}$ are also bounded.

Step 3. The sequence $\{(x_k, y_k)\}$ converges strongly to (x^*, y^*).

It follows from (3.1) and (3.6) that

$$
\begin{aligned}
&\|x_{k+1} - x^*\|^2 \\
\leq\ & t_k^2\|f_1(x_k) - x^*\|^2 + 2t_k(1-t_k)\langle f_1(x_k) - x^*, \widetilde{u}_k - x^*\rangle \\
&+(1-t_k)^2\|\widetilde{u}_k - x^*\|^2 \\
\leq\ & t_k^2\|f_1(x_k) - x^*\|^2 + t_k(1-t_k)(\|f_1(x_k) - f_1(x^*)\|^2 + \|\widetilde{u}_k - x^*\|^2) \\
&+(1-t_k)^2\|\widetilde{u}_k - x^*\|^2 + 2t_k(1-t_k)\langle f_1(x^*) - x^*, \widetilde{u}_k - x^*\rangle \\
\leq\ & t_k^2\|f_1(x_k) - x^*\|^2 + t_k(1-t_k)(\rho_1^2\|x_k - x^*\|^2 + \|u_k - x^*\|^2) \\
&+(1-t_k)^2\|u_k - x^*\|^2 + 2t_k(1-t_k)\langle f_1(x^*) - x^*, \widetilde{u}_k - x^*\rangle \\
\leq\ & t_k(1-t_k)\rho^2\|x_k - x^*\|^2 + (1-t_k)\|u_k - x^*\|^2 \\
&+t_k^2\|f_1(x_k) - x^*\|^2 + 2t_k(1-t_k)\langle f_1(x^*) - x^*, \widetilde{u}_k - x^*\rangle. \qquad (3.12)
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&\|y_{k+1} - y^*\|^2 \\
\leq\ & t_k(1-t_k)\rho^2\|y_k - y^*\|^2 + (1-t_k)\|v_k - y^*\|^2 + t_k^2\|f_2(y_k) - y^*\|^2 \\
&+2t_k(1-t_k)\langle f_2(y^*) - y^*, \widetilde{v}_k - y^*\rangle. \qquad (3.13)
\end{aligned}
$$

By (3.4), (3.12) and (3.13) we get

$$
\begin{aligned}
s_{k+1} \leq\ & t_k(1-t_k)\rho^2(\|x_k - x^*\|^2 + \|y_k - y^*\|^2) \\
&+(1-t_k)(\|u_k - x^*\|^2 + \|v_k - y^*\|^2) \\
&+t_k^2(\|f_1(x_k) - x^*\|^2 + \|f_2(y_k) - y^*\|^2) \\
&+2t_k(1-t_k)(\langle f_1(x^*) - x^*, \widetilde{u}_k - x^*\rangle + \langle f_2(y^*) - y^*, \widetilde{v}_k - y^*\rangle) \\
\leq\ & [1 - t_k(1 - (1-t_k)\rho^2)]s_k + t_k^2(\|f_1(x_k) - x^*\|^2 + \|f_2(y_k) - y^*\|^2) \\
&+2t_k(1-t_k)(\langle f_1(x^*) - x^*, \widetilde{u}_k - x^*\rangle + \langle f_2(y^*) - y^*, \widetilde{v}_k - y^*\rangle) \\
=\ & (1-\lambda_k)s_k + \lambda_k\delta_k, \qquad (3.14)
\end{aligned}
$$

where $\lambda_k = t_k(1-(1-t_k)\rho^2)$,

$$\begin{aligned}\delta_k &= \frac{t_k(\|f_1(x_k)-x^*\|^2+\|f_2(y_k)-y^*\|^2)}{1-(1-t_k)\rho^2}\\ &\quad+\frac{2(1-t_k)(\langle f_1(x^*)-x^*,\widetilde{u}_k-x^*\rangle+\langle f_2(y^*)-y^*,\widetilde{v}_k-y^*\rangle)}{1-(1-t_k)\rho^2}.\end{aligned}$$

From (3.1), (3.5) and (3.7) we have

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &\le t_k\|f_1(x_k)-x^*\|^2+(1-t_k)\|\widetilde{u}_k-x^*\|^2\\ &\le t_k\|f_1(x_k)-x^*\|^2+(1-t_k)(\|u_k-x^*\|^2\\ &\quad-(1-\alpha_k)(\alpha_k-\tau)\|U_{i(k)}u_k-u_k\|^2),\\ \|y_{k+1}-x^*\|^2 &\le t_k\|f_2(y_k)-y^*\|^2+(1-t_k)\|\widetilde{v}_k-y^*\|^2\\ &\le t_k\|f_2(y_k)-y^*\|^2+(1-t_k)(\|v_k-y^*\|^2\\ &\quad-(1-\beta_k)(\beta_k-\nu)\|T_{j(k)}v_k-v_k\|^2).\end{aligned}$$

From the above two inequalities and (3.4) we obtain

$$\begin{aligned}s_{k+1} &\le \|u_k-x^*\|^2+\|v_k-y^*\|^2+t_k(\|f_1(x_k)-x^*\|^2+\|f_2(y_k)-y^*\|^2)\\ &\quad-(1-t_k)((1-\alpha_k)(\alpha_k-\tau)\|U_{i(k)}u_k-u_k\|^2\\ &\quad+(1-\beta_k)(\beta_k-\nu)\|T_{j(k)}v_k-v_k\|)\\ &\le s_k+t_k(\|f_1(x_k)-x^*\|^2+\|f_2(y_k)-y^*\|^2)-\gamma_k[2\|Ax_k-By_k\|^2\\ &\quad-\gamma_k(\|A^*(Ax_k-By_k)\|^2+\|B^*(Ax_k-By_k)\|^2)]\\ &\quad-(1-t_k)[(1-\alpha_k)(\alpha_k-\tau)\|U_{i(k)}u_k-u_k\|^2\\ &\quad+(1-\beta_k)(\beta_k-\nu)\|T_{j(k)}v_k-v_k\|]\\ &\le s_k-\eta_k+\mu_k, \qquad (3.15)\end{aligned}$$

where $\mu_k = t_k(\|f_1(x_k)-x^*\|^2+\|f_2(y_k)-y^*\|^2)$,

$$\begin{aligned}\eta_k &= \gamma_k[2\|Ax_k-By_k\|^2-\gamma_k(\|A^*(Ax_k-By_k)\|^2+\|B^*(Ax_k-By_k)\|^2)]\\ &\quad+(1-t_k)[(1-\alpha_k)(\alpha_k-\tau)\|U_{i(k)}u_k-u_k\|^2\\ &\quad+(1-\beta_k)(\beta_k-\nu)\|T_{j(k)}v_k-v_k\|].\end{aligned}$$

It follows from the condition (ii) that $\Sigma\lambda_k=\infty$ and $\lim_{k\to\infty}\mu_k=0$ due to the boundedness of $\{x_k\}$ and $\{y_k\}$.

Next we show that, for any subsequence $\{k_l\}\subset\{k\}$,

$$\lim_{l\to\infty}\eta_{k_l}=0 \;\;\Rightarrow\;\; \limsup_{l\to\infty}\delta_{k_l}\le 0.$$

Indeed, for any $\{k_l\}\subset\{k\}$ and $\lim_{l\to\infty}\eta_{k_l}=0$, by the conditions (ii)-(iv), for any $i\in\{1,2,\cdots,p\}, j\in\{1,2,\cdots,q\}$ we have

$$\lim_{l\to\infty}\|u_{k_l}-U_{i(k_l)}u_{k_l}\|=0, \qquad (3.16)$$

$$\lim_{l\to\infty}\|v_{k_l}-T_{j(k_l)}v_{k_l}\|=0. \qquad (3.17)$$

And due to the assumption (3.2) on $\{\gamma_k\}$ and $\lim_{l\to\infty}\eta_{k_l}=0$, we have

$$\begin{aligned}&\varepsilon(\|A^*(Ax_{k_l}-By_{k_l})\|^2+\|B^*(Ax_{k_l}-By_{k_l})\|^2)\\ &\le 2\|Ax_{k_l}-By_{k_l}\|^2-\gamma_{k_l}(\|A^*(Ax_{k_l}-By_{k_l})\|^2\\ &\quad+\|B^*(Ax_{k_l}-By_{k_l})\|^2).\end{aligned} \tag{3.18}$$

Since $\lim_{l\to\infty}\eta_{k_l}=0$, the right side of (3.18) immediately converges to zero, i.e.,

$$\lim_{l\to\infty}(\|A^*(Ax_{k_l}-By_{k_l})\|^2+\|B^*(Ax_{k_l}-By_{k_l})\|^2)=0.$$

Therefore, in view of (3.18) again, we readily see

$$\lim_{l\to\infty}\|Ax_{k_l}-By_{k_l}\|^2=0.$$

Furthermore, we get

$$\lim_{l\to\infty}\|u_{k_l}-x_{k_l}\|=\lim_{l\to\infty}\gamma_{k_l}\|A^*(Ax_{k_l}-By_{k_l})\|=0, \tag{3.19}$$

$$\lim_{l\to\infty}\|v_{k_l}-y_{k_l}\|=\lim_{l\to\infty}\gamma_{k_l}\|B^*(Ax_{k_l}-By_{k_l})\|=0. \tag{3.20}$$

Then it follows from (3.1), (3.16), (3.19) and the condition (ii) that

$$\begin{aligned}&\|x_{k_l+1}-x_{k_l}\|\\ =\;&\|t_{k_l}f_1(x_{k_l})+(1-t_{k_l})(\alpha_{k_l}u_{k_l}+(1-\alpha_{k_l})U_{i(k_l)}u_{k_l})-x_{k_l}\|\\ \le\;&t_{k_l}\|f_1(x_{k_l})-x_{k_l}\|+(1-t_{k_l})\alpha_{k_l}\|u_{k_l}-x_{k_l}\|\\ &+(1-t_{k_l})(1-\alpha_{k_l})\|U_{i(k_l)}u_{k_l}-x_{k_l}\|\\ \le\;&t_{k_l}\|f_1(x_{k_l})-x_{k_l}\|+\|u_{k_l}-x_{k_l}\|+\|U_{i(k_l)}u_{k_l}-u_{k_l}\|\to 0.\end{aligned}$$

Similarly, by (3.1), (3.17), (3.20) and the condition (ii) we have $\|y_{k_l+1}-y_{k_l}\|\to 0$. Now we claim that $\omega_w(x_{k_l},y_{k_l})\subset\Gamma$. In fact, for any $(\widetilde{x},\widetilde{y})\in\omega_w(x_{k_l},y_{k_l})$, we get $\widetilde{x}\in\omega_w(x_{k_l})$ and $\widetilde{y}\in\omega_w(y_{k_l})$. Since $\lim_{l\to\infty}\|x_{k_l+1}-x_{k_l}\|=0$, use Lemma 3.4 to choose a subsequence $\{m_l\}\subset\{k_l\}$ (depending on i), for any (fixed) $i\in\{1,2,\cdots,p\}$, such that $x_{m_l}\rightharpoonup\widetilde{x}$ and $i(m_l)=i$ for all l. And by (3.19) we have $u_{m_l}\rightharpoonup\widetilde{x}$. It turns out that

$$\lim_{l\to\infty}\|u_{m_l}-U_iu_{m_l}\|=\lim_{l\to\infty}\|u_{m_l}-U_{i(m_l)}u_{m_l}\|=0.$$

Since $I-U_i$ is demiclosed at the origin, we have $\widetilde{x}\in F(U_i)$ for all $1\le i\le p$, that is, $\widetilde{x}\in\cap_{i=1}^{p}F(U_i)$. Similarly, we can prove that $\widetilde{y}\in\cap_{j=1}^{q}F(T_j)$. On the other hand, it follows from $(\widetilde{x},\widetilde{y})\in\omega_w(x_{k_l},y_{k_l})$ and Lemma 2.9 that $A\widetilde{x}-B\widetilde{y}\in\omega_w(A\widetilde{x}_{k_l}-B\widetilde{y}_{k_l})$, and the weakly lower semicontinuity of the norm imply

$$\|A\widetilde{x}-B\widetilde{y}\|\le\liminf_{l\to\infty}\|Ax_{k_l}-By_{k_l}\|=0.$$

Hence $(\widetilde{x},\widetilde{y})\in\Gamma$, i.e., $\omega_w(x_{k_l},y_{k_l})\subset\Gamma$. It is easy to see that $\lim_{k\to\infty}(1-(1-t_k)\rho^2)=1-\rho^2$ and $\lim_{k\to\infty}t_k(\|f_1(x_k)-x^*\|^2+\|f_2(y_k)-y^*\|^2)=0$. Finally, completing $\limsup_{l\to\infty}\delta_{k_l}\le 0$, we only need to prove

$$\limsup_{l\to\infty}(\langle f_1(x^*)-x^*,\widetilde{u}_{k_l}-x^*\rangle+\langle f_2(y^*)-y^*,\widetilde{v}_{k_l}-y^*\rangle)\le 0.$$

Indeed, from (3.16)-(3.20), for any $i \in \{1, 2, \cdots, p\}, j \in \{1, 2, \cdots, q\}$, we have

$$\lim_{l\to\infty} \|U_{i(k_l)}u_{k_l} - x_{k_l}\| = \lim_{l\to\infty} \|T_{j(k_l)}v_{k_l} - y_{k_l}\| = 0,$$

furthermore, we obtain

$$\begin{aligned}
&\limsup_{l\to\infty}[\langle f_1(x^*) - x^*, \widetilde{u}_{k_l} - x^*\rangle + \langle f_2(y^*) - y^*, \widetilde{v}_{k_l} - y^*\rangle] \\
= &\limsup_{l\to\infty}[\langle f_1(x^*) - x^*, \alpha_{k_l}u_{k_l} + (1-\alpha_{k_l})U_{i(k_l)}u_{k_l} - x^*\rangle \\
&+\langle f_2(y^*) - y^*, \beta_{k_l}v_{k_l} + (1-\beta_{k_l})T_{j(k_l)}v_{k_l} - y^*\rangle] \\
\leq &\limsup_{l\to\infty}[\langle f_1(x^*) - x^*, \alpha_{k_l}x_{k_l} + (1-\alpha_{k_l})x_{k_l} - x^*\rangle \\
&+\langle f_2(y^*) - y^*, \beta_{k_l}y_{k_l} + (1-\beta_{k_l})y_{k_l} - y^*\rangle] \\
&+\limsup_{l\to\infty}\langle f_1(x^*) - x^*, \alpha_{k_l}(u_{k_l} - x_{k_l}) + (1-\alpha_{k_l})(U_{i(k_l)}u_{k_l} - x_{k_l})\rangle \\
&+\limsup_{l\to\infty}\langle f_2(y^*) - y^*, \beta_{k_l}(v_{k_l} - y_{k_l}) + (1-\beta_{k_l})(T_{j(k_l)}v_{k_l} - y_{k_l})\rangle \\
= &\limsup_{l\to\infty}(\langle f_1(x^*) - x^*, x_{k_l} - x^*\rangle + \langle f_2(y^*) - y^*, y_{k_l} - y^*\rangle). \qquad (3.21)
\end{aligned}$$

By the boundedness of $\{(x_{k_l}, y_{k_l})\}$ in H^*, there exists a point $(p, q) \in H^*$ and a subsequence $\{(x_{k'_l}, y_{k'_l})\}$ of $\{(x_{k_l}, y_{k_l})\}$ in H^* such that $(x_{k'_l}, y_{k'_l}) \rightharpoonup (p, q)$ and

$$\begin{aligned}
&\limsup_{l\to\infty}[\langle f_1(x^*) - x^*, x_{k_l} - x^*\rangle + \langle f_2(y^*) - y^*, y_{k_l} - y^*\rangle] \\
= &\lim_{l\to\infty}[\langle f_1(x^*) - x^*, x_{k'_l} - x^*\rangle + \langle f_2(y^*) - y^*, y_{k'_l} - y^*\rangle]. \qquad (3.22)
\end{aligned}$$

Since $(p, q) \in \omega_w(x_{k'_l}, y_{k'_l}) \subset \omega_w(x_{k_l}, y_{k_l}) \subset \Gamma$, it directly follows from (3.3), (3.21) and (3.22) that

$$\begin{aligned}
&\limsup_{l\to\infty}[\langle f_1(x^*) - x^*, \widetilde{u}_{k_l} - x^*\rangle + \langle f_2(y^*) - y^*, \widetilde{v}_{k_l} - y^*\rangle] \\
\leq &\lim_{l\to\infty}[\langle f_1(x^*) - x^*, x_{k'_l} - x^*\rangle + \langle f_2(y^*) - y^*, y_{k'_l} - y^*\rangle] \\
= &\langle f_1(x^*) - x^*, p - x^*\rangle + \langle f_2(y^*) - y^*, q - y^*\rangle \\
= &-\langle (I - f_1)x^* - (I - f_2)y^*, (p, q) - (x^*, y^*)\rangle \leq 0,
\end{aligned}$$

i.e., $\limsup_{l\to\infty} \delta_{k_l} \leq 0$. Therefore it follows from Lemma 2.8 that $\lim_{k\to\infty} s_k = 0$, that is

$$\lim_{k\to\infty}(\|x_k - x^*\|^2 + \|y_k - y^*\|^2) = 0,$$

which implies that $\{(x_k, y_k)\}$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \Gamma$ which is the unique solution of the VIP (3.3). □

Taking $U_1 = U_2 = \cdots = U_p = U$, $T_1 = T_2 = \cdots = T_q = T$, the Algorithm 3.1 reduces to the following algorithm:

Algorithm 3.6. Let $x_0 \in H_1$, $y_0 \in H_2$ be arbitrary given and $p, q \geq 1$ be integers. Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be two contractions with constants ρ_1, $\rho_2 \in [0, 1)$. Let the sequences $\{\alpha_k\}, \{\beta_k\}, \{t_k\} \subset [0, 1]$. Assume that the kth iterate

$(x_k, y_k) \in H^*$ has been constructed and $Ax_k - By_k \neq 0$, then we calculate $(k+1)$th iterate $(x_{k+1},\ y_{k+1})$ in H^* via the formula

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = t_k f_1(x_k) + (1-t_k)(\alpha_k u_k + (1-\alpha_k)Uu_k), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = t_k f_2(y_k) + (1-t_k)(\beta_k v_k + (1-\beta_k)Tv_k), \quad \forall\, k \geq 0. \end{cases}$$

Assume the stepsize γ_k is chosen in such a way that

$$\gamma_k \in (\varepsilon, \min\{\eta, \frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2}\} - \varepsilon)$$

for all $k \in \Omega$ and small enough $\varepsilon > 0$, where the index set $\Omega = \{k : Ax_k - By_k \neq 0\}$,

$$\eta = \frac{2\|Ax_l - By_l\|^2}{\|A^*(Ax_l - By_l)\|^2 + \|B^*(Ax_l - By_l)\|^2},\ l = \min_{k\in\Omega}\{k\},$$

otherwise, $\gamma_k = \gamma$ (γ being any nonnegative value). If $Ax_k = By_k = 0$, then $u_k = x_k,\ v_k = y_k$ and

$$\begin{cases} x_{k+1} = t_k f_1(x_k) + (1-t_k)(\alpha_k x_k + (1-\alpha_k)Ux_k), \\ y_{k+1} = t_k f_2(y_k) + (1-t_k)(\beta_k y_k + (1-\beta_k)Ty_k). \end{cases}$$

By Theorem 3.5, we have the following result.

Corollary 3.7. *Let H_1, H_2, H_3 be real Hilbert spaces. Given two bounded linear operators $A : H_1 \to H_3, B : H_2 \to H_3$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be τ-demicontractive and μ-demicontractive, respectively. Suppose that $I-U$, $I-T$ are demiclosed at origin and the solution set Γ of the SEFP (1.1) is nonempty. Assume that the following conditions are satisfied:*

(i) $\rho_1,\ \rho_2 \in [0, \frac{1}{\sqrt{2}})$;
(ii) $\lim_{k\to\infty} t_k = 0$ *and* $\sum_{k=0}^{\infty} t_k = \infty$*;*
(iii) $\tau < \liminf_{k\to\infty} \alpha_k \leq \limsup_{k\to\infty} \alpha_k < 1$;
(iv) $\nu < \liminf_{k\to\infty} \beta_k \leq \limsup_{k\to\infty} \beta_k < 1$.

Then the sequence $\{(x_k, y_k)\}$ generated by Algorithm 3.6 converges strongly to a solution (x^*, y^*) of the the SEFP (1.1) which is the unique solution of the VIP (3.3).

Let $\alpha_k = \beta_k = w_k$, $t_k = \alpha_k (k \geq 0)$ and $\nu = \tau = 0$. Since every 0-demicontractive mapping is quasi-nonexpansive, from Corollary 3.7, we also have the following corollary.

Corollary 3.8. *Let H_1, H_2, H_3 be real Hilbert spaces. Given two bounded linear operators $A : H_1 \to H_3, B : H_2 \to H_3$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ $(1 \leq j \leq q)$ be quasi-nonexpansive with the solution set Γ of the SEFP (1.1) is nonempty. Assume that the following conditions are satisfied:*

(i) $\rho_1,\ \rho_2 \in [0, \frac{1}{\sqrt{2}})$;
(ii) $\lim_{k\to\infty} \alpha_k = 0$ *and* $\sum_{k=0}^{\infty} \alpha_k = \infty$*;*
(iii) $I-U$, $I-T$ *are demiclosed at origin;*

(iv) $w_k \in (0,1)$ *such that* $0 < \liminf_{k\to\infty} w_k \le \limsup_{k\to\infty} w_k < 1$.

Then the sequence $\{(x_k, y_k)\}$ *generated by Algorithm 3.6 converges strongly to a solution* (x^*, y^*) *of the the SEFP (1.1) which is the unique solution of the VIP (3.3).*

Remark 3.9. Theorem 3.5 extends and develops Theorem 3.2 in [17] from the following aspects:

(a) Two quasi-nonexpansive mappings U and T are extended to two finite family of demicontractive mappings $\{U_i\}_{i=1}^p$ and $\{T_j\}_{j=1}^q$, then the split equality fixed point problem is extended to the multiple-set split equality common fixed-point problem.

(b) The parameter sequence $\{\omega_k\}$ is replaced by two different parameter sequences $\{\alpha_k\}$ and $\{\beta_k\}$.

(c) The advantage of our choice (3.2) of the stepsizes $\{\gamma_k\}$ lies in the fact that no prior information about the operator norms of A and B is required, and still convergence is guaranteed.

(d) In [17], the authors didn't give the unique solution proof of the VIP (3.3), which leads to an incomplete result. In this paper we prove it; see Step 1 in the proof of Theorem 3.5.

Firstly we shall give an example which satisfies all the conditions of the solution set Γ of the MSECFP (1.3), the mappings $\{U_i\}_{i=1}^p$, and $\{T_j\}_{j=1}^q$ in Theorem 3.5.

Example 3.10. Let $H_1 = H_2 = H_3 = \ell_2$ and let $i \in \{1, 2, \cdots, p\}$ and $j \in \{1, 2, \cdots, q\}$ be arbitrarily fixed. Let $U_i,\ T_j : \ell_2 \to \ell_2$ be defined by $U_i x = -2ix$ and $T_j x = -(2j+1)x$ for all $x \in \ell_2$. Then it is easy to see that $\cap_{i=1}^p F(U_i) = \{0\} = \cap_{j=1}^q F(T_j)$ and $A0 = 0 = B0$. Thus $\Gamma = \{(0,0)\} \neq \emptyset$. Also U_i is τ_i-demicontractive and T_j is ν_j-demicontractive by Example 2.5 in [11], where $\tau_i = \frac{2i-1}{2i+1}$ and $\nu_j = \frac{j}{j+1}$; then $I - U_i$ and $I - T_j$ are demiclosed at 0 by Remark 2.12 in [11].

References

[1] C. Byrne, *Iterative oblique projection onto convex sets and the split feasibility problem*, Inverse Problems **18** (2002), 441–453.

[2] Y. Censor and T. Elfving, *A multiprojection algorithm using Bergman projections in a product space*, Numer. Algorithms **8** (1994), 221–239.

[3] Y. Censor and A. Segal, *The split common fixed point problem for directed operators*, J. Convex Anal. **16** (2009), 587–600.

[4] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, vol. 28 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, UK, 1990.

[5] S. He and C. Yang, *Solving the variational inequality problem defined on intersection of finite level sets,* Abstr. Appl. Anal. **2013**(2013), Article ID 942315.

[6] M. C. Joshi and R. K. Bose, *Some Topics in Nonlinear Functional Analysis*, John Wiley and Sons, Inc., New York, 1985.

[7] G. Marino and H. K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), 336–346.

[8] A. Moudafi, *The split common fixed-point problem for demicontractive mappings*, Inverse Problems **26** (2010), 587–600.

[9] A. Moudafi and E. Al-Shemas, *Simultaneous iterative methods for split equality problem*, Trans. Math. Program. Appl. **1** (2013), 1–11.

[10] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, (2000).

[11] Y. Q. Wang and T. H. Kim, *Simultaneous iterative algorithm for the split equality fixed-point problem of demicontractive mappings* , J. Nonlinear Sci. Appl. **10** (2017), 154–165.

[12] Y. Q. Wang, T. H. Kim, X. L. Fang and H. M. He, *The split common fixed-point problem for demicontractive mappings and quasi-nonexpansive mappings*, J. Nonlinear Sci. Appl. **10** (2017), 2976–2985.

[13] I. Yamada, *The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings*, in: Butnariu, D., Censor Y., Reich S., eds. *Inherently Parallel Algorithm for Feasibility and Optimization and Their Applications.* Elsevier, 2001.

[14] Y.-H. Yao, R. P. Agarwal, M. Postolache and Y.-C. Liou, *Algorithms with strong convergence for the split common solution of the feasibility problem and fixed point problem*, Fixed Point Theory Appl. **2014** (2014), 14 pages.

[15] Y.-H. Yao, M. Postolache and Y.-C. Liou, *Strong convergence of a self-adaptive method for the split feasibility problem*, Fixed Point Theory Appl. **2013** (2013), 12 pages.

[16] Y.-H. Yao, J. Wu and Y.-C. Liou, *Regularized methods for the split feasibility problem*, Abstr. Appl. Anal. **2012** (2012), 13 pages.

[17] J. Zhao and S. Wang, *Viscosity approximation methods for the split equality common fixed point problem of quasi-nonexpansive operators*, Acta Math. Sci. **36B** (2016), 1474–1486.

[18] J. Zhao and S. Wang, *Mixed iterative algorithms for the multiple-set split equality common fixed-point problems without prior knowledge of operator norms*, Optimization **65** (2016), 1069–1083.

Manuscript received 4 May 2019
revised 22 August 2019

Yaqin Wang
Department of Mathematics, Shaoxing University, Shaoxing 312000, China
E-mail address: `wangyaqin0579@126.com`

Xiaoli Fang
Department of Mathematics, Shaoxing University, Shaoxing 312000, China
E-mail address: `fxl0418@126.com`

Tae-Hwa Kim
Department of Applied Mathematics, College of Natural Sciences, Pukyong National University, Busan 48513, Korea
E-mail address: `taehwa@pknu.ac.kr`