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ULAM-HYERS STABILITY PROBLEMS AND FIXED POINTS FOR CONTRACTIVE TYPE OPERATORS ON KST-SPACES

ABDUL LATIF*, LILIANA GURAN, AND MONICA-FELICIA BOTA

Abstract. Using the concept of w-distance we prove some existence, uniqueness and Ulam-Hyers stability results for fixed point problems concerning α-ψ-contractive operators. Consequently, our results improve and generalize several known results.

1. Introduction and preliminaries

The first problem concerning the stability of group homomorphisms was raised by Ulam [28] during his talk at the University of Wisconsin in 1940. The first affirmative partial answer of his problem in the frame work of Banach spaces, was given by Hyers [9] in 1941. Thereafter, this type of stability is called the Ulam-Hyers stability. Ulam-Hyers stability results in fixed point theory have been investigated by many authors, see; [4, 5, 9, 10, 17, 19, 21, 23] and references therein.

In 2012, Samet et. al [25] introduced a notion of α-ψ-contractive type operator and proved some fixed point results for such operators. Recent work on the existence of fixed points for contractive operators can be found in [8, 12, 16, 20, 24] and references therein.

In [11], Kada et. al. introduced a notion of w-distance on metric spaces and then improved several classical results in metric fixed point theory including the Caristi fixed point theorem. While, Suzuki and Takahashi [26] introduced concepts of single and multivalued contractions with respect to w-distance and established fixed point results for such mappings, generalizing Banach contraction principal and the classical Nadler fixed point theorem. Further work in this direction can be found in [2, 15, 18, 27]. Recently, Guran [6] obtained some results concerning the Ulam-Hyers stability for KST-spaces.

In this paper we study existence, uniqueness and generalized Ulam-Hyers stability of fixed point for α-ψ-contractive type operators in the frame work of KST-spaces.

First we recall some essential definitions and fundamental results.

Let (X, d) be a metric space and $f : X \to X$ be a singlevalued operator. We will use the following notations:

$P(X)$ - *the set of all nonempty subsets of* X;

$P_{cl}(X)$ - *the set of all nonempty closed subsets of* X;

$P_{cp}(X)$ - *the set of all nonempty compact subsets of* X;

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*Corresponding author.

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$Fix(f) := \{x \in X \mid x = f(x)\}$ - *the set of all fixed points of* f.

The concept of w-distance was introduced by Kada et. al. [11]) as follows.

Definition 1.1. Let (X, d) be a metric space. Then $w : X \times X \to [0, \infty)$ is called a weak distance (briefly w-distance) on X if the following axioms are satisfied :

(1) $w(x, z) \leq w(x, y) + w(y, z)$, for any $x, y, z \in X$;
(2) for any $x \in X$, $w(x, \cdot) : X \to [0, \infty)$ is lower semicontinuous;
(3) for any $\varepsilon > 0$, exists $\delta > 0$ such that $w(z, x) \leq \delta$ and $w(z, y) \leq \delta$ implies $d(x, y) \leq \varepsilon$.

In this case, the triple (X, d, w) is called KST-space. We say, the space (X, d, w) is complete KST-space if the metric space (X, d) is complete.

Some examples of w-distance can be find in [11].

For our main results we need the following crucial result for w-distance. (see; [11, 26]).

Lemma 1.2. *Let* (X, d) *be a metric space and let* w *be a* w*-distance on* X*. Let* (x_n) *and* (y_n) *be two sequences in* X*, let* (α_n)*,* (β_n) *be sequences in* $[0, +\infty[$ *converging to zero and let* $x, y, z \in X$*. Then the following hold:*

(1) *If* $w(x_n, y) \leq \alpha_n$ *and* $w(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$*, then* $y = z$*.*
(2) *If* $w(x_n, y_n) \leq \alpha_n$ *and* $w(x_n, z) \leq \beta_n$ *for any* $n \in \mathbb{N}$*, then* (y_n) *converges to* z*.*
(3) *If* $w(x_n, x_m) \leq \alpha_n$ *for any* $n, m \in \mathbb{N}$ *with* $m > n$*, then* (x_n) *is a Cauchy sequence.*
(4) *If* $w(y, x_n) \leq \alpha_n$ *for any* $n \in \mathbb{N}$*, then* (x_n) *is a Cauchy sequence.*

Much work has been done on the existence of fixed points for contraction with respect to w- distance. For example, see [1, 7, 13, 14, 27] and references therein.

A mapping $\varphi : [0, \infty) \to [0, \infty)$ is called a *comparison function* if it is increasing and $\varphi^n(t) \to 0$, $n \to \infty$, for any $t \in [0, \infty)$. We denote by Φ, the class of the corporation function $\varphi : [0, \infty) \to [0, \infty)$. For more details and examples, see [3, 22].

We recall the following essential result.

Lemma 1.3. [3, 22] *If* $\varphi : [0, \infty) \to [0, \infty)$ *is a comparison function, then:*

(1) *each iterate* φ^k *of* φ*,* $k \geq 1$*, is also a comparison function;*
(2) φ *is continuous at* 0*;*
(3) $\varphi(t) < t$*, for any* $t > 0$*.*

Next, we present the definition of α-ψ-contractive and α-admissible mappings introduced by Samet et al. [25].

We denote with Ψ the family of nondecreasing functions $\psi : [0, \infty) \to [0, \infty)$ such that
$sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n-th iterate of ψ. It is clear that if $\Psi \subset \Phi$ and hence, by Lemma 1.3 (3), for $\psi \in \Psi$ we have $\psi(t) < t$, for any $t > 0$.

Definition 1.4 ([25])**.** Let (X,d) be a metric space and $f:X\to X$ be a given mapping. We say that f is an α-ψ-contractive mapping if there exist two functions $\alpha: X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)d(f(x),f(y))\le\psi(d(x,y)),\ \text{for all } x,y\in X. \tag{1.1}$$

Remark 1.5. If $f:X\to X$ satisfies the Banach contraction principle, then f is an α-ψ-contractive mapping. In particular, if $\alpha(x,y)=1$ for all $x,y\in X$ and $\psi(t)=kt$ for all $t\ge 0$ and some $k\in[0,1)$, then α-ψ-contractive mapping reduces to classical Banach contraction mapping..

Definition 1.6 ([25])**.** Let $f:X\to X$ and $\alpha:X\times X\to[0,\infty)$. We say that f is α-admissible if

$$x,y\in X,\quad \alpha(x,y)\ge 1\Longrightarrow \alpha(f(x),f(y))\ge 1.$$

Next let us recall some important results concerning α-ψ-contractive mappings.

Theorem 1.7 ([25])**.** *Let (X,d) be a complete metric space and $f:X\to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *f is α-admissible;*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,f(x_0))\ge 1$;*
(iii) *f is continuous.*

Then, f has a fixed point.

Theorem 1.8 ([25])**.** *Let (X,d) be a complete metric space and $f:X\to X$ be an α-ψ-contractive mapping satisfying the following conditions:*

(i) *f is α-admissible;*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,f(x_0))\ge 1$;*
(iii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1})\ge 1$ for all n and $x_n\to x\in X$ as $n\to\infty$, then $\alpha(x_n,x)\ge 1$ for all n.*

Then, f has a fixed point.

2. Fixed points for α-ψ-weakly contractive operators

First, let us give the following definition as a generalization of Definition 2.1.

Definition 2.1. Let (X,d,w) be a KST-space and $f:X\to X$ be a given operator. We say that f is an α-ψ-weakly contractive operator if there exist two functions $\alpha:X\times X\to[0,\infty)$ and $\psi\in\Psi$ such that

$$\alpha(x,y)w(f(x),f(y))\le\psi(w(x,y)),\ \text{for all } x,y\in X. \tag{2.1}$$

The first our main result is the following.

Theorem 2.2. *Let (X,d,w) be a complete KST-space. Let $f:X\to X$ be an α-ψ-weakly contractive operator satisfying the following conditions:*

(i) *f is α-admissible;*
(ii) *there exists $x_0\in X$ such that $\alpha(x_0,f(x_0))\ge 1$;*
(iii) *f is continuous.*

Then, f has a fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, f(x_0)) \geq 1$ (such a point exist from condition (ii)). We define the sequence $(x_n)_{n\in\mathbb{N}}$ in X by

$$x_{n+1} = f(x_n), \text{ for all } n \in \mathbb{N}.$$

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x^* = x_n$ is a fixed point for f and thus the proof is done.

Now, we assume that:

$$x_n \neq x_{n+1} \text{for all } n \in \mathbb{N}. \tag{2.2}$$

Since f is α-admissible, we have:

$$\alpha(x_0, x_1) = \alpha(x_0, f(x_0)) \geq 1 \Longrightarrow \alpha(f(x_0), f(x_1)) = \alpha(x_1, x_2) \geq 1.$$

By induction, we get:

$$\alpha(x_n, x_{n+1}) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{2.3}$$

Applying the inequality (2.1) with $x = x_{n-1}$ and $y = x_n$, and using (2.3), we obtain:

$$w(x_n, x_{n+1}) = w(f(x_{n-1}), f(x_n)) \leq \alpha(x_{n-1}, x_n) w(f(x_{n-1}), f(x_n)) \leq \psi(w(x_{n-1}, x_n)).$$

Then $w(x_n, x_{n+1}) \leq \psi(w(x_{n-1}, x_n))$.

By induction, we obtain a sequence $(x_n)_{n\in\mathbb{N}} \in X$ such that:

(i) $x_{n+1} = f(x_n)$, for any $n \in \mathbb{N}$;

(ii) $w(x_n, x_{n+1}) \leq \psi^n(w(x_0, x_1))$, for all $n \in \mathbb{N}$.

For $n, p \in \mathbb{N}$, using (2) and the triangular inequality, we have:

$$\begin{aligned} w(x_n, x_{n+p}) &\leq w(x_n, x_{n+1}) + w(x_{n+1}, x_{n+2}) + \cdots + w(x_{n+p-1}, x_{n+p}) \\ &\leq \psi^n(w(x_0, x_1)) + \psi^{n+1}(w(x_0, x_1)) + \cdots + \psi^{n+p-1}(w(x_0, x_1)) \\ &\leq \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)). \end{aligned}$$

Since $\psi \subset \Phi$ we have that $\psi^n(t) \to 0$ as $n \to \infty$. Thus, using Lemma 1.3 we obtain:

$$\lim_{n\to\infty} w(x_n, x_{n+p}) \leq \lim_{n\to\infty} \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)) \overset{d}{\to} 0. \tag{2.4}$$

By Lemma 1.2(3), the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.

Since (X, d, w) is complete, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n \overset{d}{\to} x^*$ as $n \to \infty$. From the continuity of f, it follows that $x_{n+1} = f(x_n) \overset{d}{\to} f(x^*)$ as $n \to \infty$. By the uniqueness of the limit, we get $x^* = f(x^*)$, that is, x^* is a fixed point of f. □

Replacing the continuity condition on f of Theorem 2.2 with an other suitable condition, we obtained the following result.

Theorem 2.3. *Let (X, d, w) be a complete KST-space. Let $f : X \to X$ be an α-ψ-weakly contractive operator satisfying the following conditions:*

(i) *f is α-admissible;*

(ii) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, f(x_0)) \geq 1$;

(iii) *if* $(x_n)_{n\in\mathbb{N}}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \overset{d}{\to} x \in X$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 1$ *for all* n.

Then the fixed point equation (3.3) *has a solution.*

Proof. Following the proof of Theorem 2.2, we know that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the complete KST-space (X, d, w). Then, there exists $x^* \in X$ such that $x_n \overset{d}{\to} x^*$ as $n \to \infty$. On the other hand we have the inequality $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N}$ and by (2.3) and the hypothesis (iii), we have:

$$\alpha(x_n, x^*) \geq 1, \text{ for all } n \in \mathbb{N}. \tag{2.5}$$

For $m, n \in \mathbb{N}$ with $m > n$, from the proof of Theorem 2.2 and using the triangular inequality, we have:

$$w(x_n, x_m) \leq \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)).$$

Since $(x_n)_{n\in\mathbb{N}}$ converge to x^* and $w(x_n, \cdot)$ is lower semicontinuous we have:

$$w(x_n, x^*) \leq \lim_{m\to\infty} \inf w(x_n, x_m) \leq \lim_{m\to\infty} \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)) \leq \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)).$$

Since $\psi \subset \Phi$ we have that $\psi^n(t) \to 0$ as $n \to \infty$. Now, using Lemma 1.3 for every $n \in \mathbb{N}$ we have that:

$$w(x_n, x^*) \leq \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)) \overset{d}{\to} 0. \tag{2.6}$$

Let $f(x^*) \in X$ and $x_n = f(x_{n-1})$. Then, by the definition of α-ψ-weakly contractive operator and letting $n \to \infty$ we obtain the following result:

$$\begin{aligned} w(x_n, f(x^*)) &= w(f(x_{n-1}), f(x^*)) \\ &\leq \alpha(x_{n-1}, x^*) w(f(x_{n-1}), f(x^*)) \\ &\leq \psi(\sum_{n=k}^{\infty} \psi^k(w(x_0, x_1))) \\ &< \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)) \overset{d}{\to} 0. \end{aligned} \tag{2.7}$$

Then, by (2.6) and (2.7), we have that $w(x_n, x^*) \overset{d}{\to} 0$ and $w(x_n, f(x^*)) \overset{d}{\to} 0$. Thus, using Lemma 1.2(1) we obtain that $x^* = f(x^*)$.

□

The following result assure the uniqueness of the fixed point on KST-spaces.

Theorem 2.4. *Adding to the hypothesis of Theorem 2.2 (resp. Theorem 2.3) the following condition:*

(H) : *for all* $x, y \in X$, *there exists* $z \in X$ *such that* $\alpha(x, z) \geq 1$ *and* $\alpha(y, z) \geq 1$.

we obtain uniqueness of the fixed point of f.

Proof. Suppose that x^* and y^* are two fixed point of f. From the new condition (H), there exists $z \in X$ such that

$$\alpha(x^*, z) \geq 1 \quad \text{and} \quad \alpha(y^*, z) \geq 1. \tag{2.8}$$

Since f is α-admissible, from (2.8), we get:

$$\alpha(x^*, f^n(z)) \geq 1 \quad \text{and} \quad \alpha(y^*, f^n(z)) \geq 1. \tag{2.9}$$

By the definition of α-ψ-weakly contractive operator and using (2.9) and (2.1), we get:

$$\begin{aligned} w(x^*, f^n(z)) &= w(f(x^*), f(f^{n-1}(z))) \\ &\leq \alpha(x^*, f^{n-1}(z))w(f(x^*), f(f^{n-1}(z))) \\ &\leq \psi(w(x^*, f^{n-1}(z))). \end{aligned}$$

This imply that:

$$w(x^*, f^n(z)) \leq \psi^{n-1}(w(x^*, z)), \text{ for all } n \in \mathbb{N}.$$

Then, letting $n \to \infty$, we have:

$$w(x^*, f^n(z)) \xrightarrow{d} 0. \tag{2.10}$$

For $x^* = f(x^*)$ we suppose that $w(x^*, x^*) \neq 0$. Then we have:
$w(x^*, x^*) = w(f(x^*), f(x^*)) \leq \alpha(x^*, x^*)w(f(x^*), f(x^*)) \leq \psi(w(x^*, x^*)) < w(x^*, x^*)$. Contradiction.

Then we have:

$$w(x^*, x^*) = 0. \tag{2.11}$$

By (2.10) and (2.11) and using Lemma 1.2(1) we have that:

$$f^n(z) \xrightarrow{d} x^*. \tag{2.12}$$

Similarly, for $y^* = f(y^*)$ using (2.9) and (2.1), we get:

$$f^n(z) \xrightarrow{d} y^* \text{ as } n \to \infty. \tag{2.13}$$

Using (2.12) and (2.13), the uniqueness of the limit gives us $x^* = y^*$. □

Now, we present some examples in support of our new results,

Example 2.5. Let $(X, \|\cdot\|, w)$ be a KST-space, where $X = \mathbb{R}^+ \cup \{0\}$ is a normed linear space. Let $f : X \to X$ be a mapping given by $f(x) = \frac{1}{4}x$. Define a w-distance on X by $w(x, y) = \|y\|$. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function such that $\psi(t) = \frac{1}{2}t$. We define the mapping $\alpha : X \times X \to \mathbb{R}^+$ by $\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in [0, 1], \\ 0, & \text{otherwise.} \end{cases}$

Then all the hypotheses of Theorem 2.2 (respectively Theorem 2.3) are satisfied and consequently, f has a fixed point.

Proof. Clearly, $(X, \|\cdot\|, w)$ is a complete KST space and, obviously, f is a continuous mapping. We show that f is an α-admissible mapping. Let $x, y \in X$, if $\alpha(x, y) \geq 1$, then $x, y \in [0, 1]$. On the other hand, for all $x \in [0, 1]$ we have $f(x) = \frac{1}{4}x < x \leq 1$. It follows that $\alpha(f(x), f(y)) \geq 1$. Hence, the assertion holds. Note that $\alpha(0, f(0)) \geq 1$. Now, we check the validity of contractive condition (2.1). Let $x, y \in X$,then we have

$$\begin{aligned}\alpha(x,y)w(f(x),f(y)) &= w(f(x),f(y)) = \|f(y)\| \\ &= \left\|\frac{1}{4}y\right\| \leq \frac{1}{2}\|y\| = \frac{1}{2}w(x,y) = \psi(w(x,y)).\end{aligned}$$

That is; the contractive condition satisfied. Now, if $\{x_n\}$ is a sequence on X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$ then $\{x_n\} \subset [0, 1]$ and hence $x \in [0, 1]$. This implies $\alpha(x_n, x) \leq 1$ for all $n \in \mathbb{N}$. Note that Theorem (2.2) (also Theorem (2.3)) guarantees only the existence of a fixed point but not the uniqueness. For $x, y, z \in X$ such that $\alpha(x, z) \geq 1$, $\alpha(y, z) \geq 1$, result that $x, y, z \in [0, 1]$. Then, on this example, is true the Theorem (2.4) and 0 is the only fixed point of f. □

Example 2.6. Let (X, d, w) be a KST-space, where $X = [0, 1]$ and $d(x, y) = |x - y|$ is the usual metric. Let $w : X \times X \to \mathbb{R}^+$ be a w-distance such that $w(x, y) = \max\{d(f(x), y), d(f(x), f(y))\}$. We define the mapping $f : X \to X$ by $f(x) = \begin{cases} \frac{1}{3}, & \text{for } x \in [0,1). \\ 0, & \text{for } x = 1. \end{cases}$

Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function such that $\psi(t) = \frac{1}{3}t$. We define the mapping $\alpha : X \times X \to \mathbb{R}^+$ by

$$\alpha(x,y) = \begin{cases} 1, & \text{if } x, y \in ([0,\frac{1}{3}] \times [\frac{1}{3},1]) \cup [\frac{1}{3},1] \times [0,\frac{1}{3}]) \\ 0, & \text{otherwise.} \end{cases}$$

Then all the hypotheses of Theorem 2.2 (respectively Theorem 2.3) are satisfied and consequently, f has a fixed point.

Proof. We observe that (X, d, w) is a complete KST space but f is not a continuous mapping. Now we show that f is an admissible mapping. Let $(x, y) \in X \times X$ such that $\alpha(x, y) \geq 1$. From the definition of α there are two possibilities. First, if $(x, y) \in [0, \frac{1}{3}] \times [\frac{1}{3}, 1]$ then we have $(f(x), f(y)) \in [\frac{1}{3}, 1] \times [0, \frac{1}{3}]$ which implies that $\alpha(f(x), f(y)) = 1$. And second, if $(x, y) \in [\frac{1}{3}, 1] \times [0, \frac{1}{3}]$, then we get $(f(x), f(y)) \in [0, \frac{1}{3}] \times [\frac{1}{3}, 1]$ which implies $\alpha(f(x), f(y)) = 1$. Thus the mapping f is α-admissible. In the view of the previous arguments taking $x_0 = 0$, we have $\alpha(x_0, f(x_0)) = \alpha(0, \frac{1}{3}) = 1$. Now, we check the validity of the contraction condition (2.1).

Case I. Let $x \in [0, \frac{1}{3}]$ and $y = 1$. Then we have:

$$\begin{aligned}\alpha(x,y)w(f(x),f(y)) \leq w(f(x),f(y)) &= \max\{d(f(f(x)),f(y)),d(f(f(x)),f(f(y)))\} \\ &= \max\{|f(f(x))-f(y)|,|f(f(x))-f(f(y))|\} \\ &= \max\left\{\left|f\left(\frac{1}{3}\right)-f(1)|,\left|f\left(\frac{1}{3}\right)-f(0)\right|\right\} \\ &= \max\left\{\left|\frac{1}{3}-0\right|,\left|\frac{1}{3}-\frac{1}{3}\right|\right\} \qquad (2.14)\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{3}\max\left\{\left|\frac{1}{3}-1\right|, \left|\frac{1}{3}-0\right|\right\} \\
&= \frac{1}{3}\max\{|f(x)-y|, |f(x)-f(y)|\} \\
&= \frac{1}{3}\max\{d(f(x),y), d(f(x),f(y))\} \\
&= \frac{1}{3}w(x,y) = \psi(w(x,y)).
\end{aligned}$$

Case II. Let $x = 1$ and $y \in [0, \frac{1}{3}]$. Then we have:

$$\begin{aligned}
\alpha(x,y)w(f(x),f(y)) &\leq w(f(x),f(y)) \\
&= \max\{d(f(f(x)),f(y)), d(f(f(x)),f(f(y)))\} \\
&= \max\{|f(f(x))-f(y)|, |f(f(x))-f(f(y))|\} \\
&= \max\left\{\left|f(0)-f\left(\frac{1}{3}\right)\right|, \left|f(0)-f\left(\frac{1}{3}\right)\right|\right\} \\
&= \max\left\{\left|\frac{1}{3}-\frac{1}{3}\right|, \left|\frac{1}{3}-\frac{1}{3}\right|\right\} \\
&\leq \frac{1}{3}\max\left\{\left|0-\frac{1}{3}\right|, \left|\frac{1}{3}-\frac{1}{3}\right|\right\} \\
&= \frac{1}{3}\max\{|f(x)-y|, |f(x)-f(y)|\} \\
&= \frac{1}{3}\max\{d(f(x),y), d(f(x),f(y))\} \\
&= \frac{1}{3}w(x,y) \\
&= \psi(w(x,y)).
\end{aligned} \tag{2.15}$$

Thus f is α-ψ-weakly contractive operator on X. Now, let $\{x_n\}$ be a sequence on X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to +\infty$, for some $x \in X$. Then, clearly $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}$. Further, note that for any $(x, y) \in X \times X$ there is and $z = \frac{1}{3} \in X$ such that $\alpha(x, z) = \alpha(y, z) = 1$. Thus all the hypotheses of Theorem 2.4 are satisfied and hence f has unique fixed point in X. Note that the unique fixed point of f is $x = \frac{1}{3}$. □

3. Ulam-Hyers w-stability for fixed point problems

Definition 3.1. Let (X, d) be a metric space and $f : X \to X$ be an operator. By definition, the fixed point equation

$$x = f(x) \tag{3.1}$$

is called generalized Ulam-Hyers stable if and only if there exists increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous at 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each $u^* \in X$ an ε-solution of the fixed point equation (3.3), i.e. u^* satisfies the inequality

$$d(u^*, f(u^*)) \leq \varepsilon \tag{3.2}$$

there exists a solution $x^* \in X$ of the equation (3.3) such that

$$d(u^*, x^*) \leq \psi(\varepsilon).$$

If there exists $c > 0$ such that $\psi(t) = c \cdot t$, for each $t \in \mathbb{R}_+$, then the fixed point equation (3.3) is said to be Ulam-Hyers stable.

For Ulam-Hyers stability results concerning fixed point problems, see [4, 17, 21, 23] and references therein.

Now, we recall the notion of weakly Picard operator, see [21, 23].

Definition 3.2. Let (X,d) be a metric space. An operator $f : X \to X$ is weakly Picard operator (briefly WPO) if the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations for f starting from $x \in X$ converges, for all $x \in X$ and its limit is a fixed point for f.

If f is WPO, then we consider the operator

$$f^\infty : X \to X \text{ defined by } f^\infty(x) := \lim_{n\to\infty} f^n(x).$$

Note that $f^\infty(X) = Fix(f)$. Applications for such operator are given in [21, 23].

Recently, Guran in [6] defined the Ulam-Hyers stability of fixed point equations on KST-spaces. Here, we define a general notion and called it *generalized Ulam-Hyers w-stability.*

Definition 3.3. Let (X, d, w) be a KST-space and $f : X \to X$ be an operator. The fixed point equation

$$x = f(x) \tag{3.3}$$

is called generalized Ulam-Hyers w-stable (with respect to w-distance) if and only if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for every $\varepsilon > 0$ and for each $u^* \in X$ an ε-solution of the fixed point equation (3.3), i.e. u^* satisfies the inequality

$$w(u^*, f(u^*)) \leq \varepsilon \tag{3.4}$$

there exists a solution $x^* \in X$ of the equation (3.3) such that

$$w(u^*, x^*) \leq \psi(\varepsilon).$$

Motivated by the work of Petru et. al [19] about Ulam-Hyers stability problem, we obtain the following result, which improve and generalize a number of known results.

Theorem 3.4. *Let (X, d, w) be a complete KST-space. Suppose that all the hypotheses of Theorem 2.4 hold and additionally that the function $\beta : [0, \infty) \to [0, \infty)$, $\beta(r) := r - \psi(r)$ is strictly increasing and onto. Then*

(a) *the fixed point equation* (3.3) *is generalized Ulam-Hyers w-stable.*

(b) *$Fix(f) = \{x^*\}$ and if $x_n \in X$, $n \in \mathbb{N}$ are such that $w(x_n, f(x_n)) \xrightarrow{d} 0$, as $n \to \infty$, then $x_n \xrightarrow{d} x^*$, as $n \to \infty$, i.e. the fixed point equation* (3.3) *is well posed with respect to w-distance.*

(c) *If $g : X \to X$ is such that there exists $\eta \in [0, \infty)$ with*

$$w(g(x), f(x)) \le \eta, \ \textit{for all } x \in X,$$

then

$$y^* \in Fix(g) \Longrightarrow d(y^*, x^*) \le \beta^{-1}(\eta).$$

Proof. (a) By the proof of Theorem 2.2 we get the conclusion that $f : X \to X$ is a weakly Picard operator with respect to w-distance, so $Fix(f) = \{x^*\}$. Let $\varepsilon > 0$ and $u^* \in X$ be a solution of (3.3), i.e,

$$w(u^*, f(u^*)) \le \varepsilon.$$

Since f is α-ψ-weakly contractive operator and since $x^* \in Fix(f)$, from (H) there exists $u^* \in X$ such that $\alpha(u^*, x^*) \ge 1$. We obtain

$$\begin{aligned} w(u^*, x^*) = w(u^*, f(x^*)) &\le w(u^*, f(u^*)) + w(f(u^*), f(x^*)) \\ &\le \varepsilon + \alpha(u^*, x^*) w(f(u^*), f(x^*)) \\ &\le \varepsilon + \psi(w(x^*, u^*)). \end{aligned}$$

Therefore,

$$\begin{aligned} \beta(w(u^*, x^*)) &:= w(u^*, x^*) - \psi(w(u^*, x^*)) \\ &\le \varepsilon + \psi(w(u^*, x^*)) - \psi(w(u^*, x^*)) \\ &\le \varepsilon. \end{aligned}$$

and thus $w(u^*, x^*) \le \beta^{-1}(\varepsilon)$. Consequently, the fixed point equation (3.3) is generalized Ulam-Hyers w-stable.

(b) Using similarly steps as in the proof of Theorem 2.2 for $\alpha(x_{n-2}, x_{n-1}) \ge 1$ we obtain for $x_{n-1} = f(x_{n-2})$ the following inequality:

$$\begin{aligned} w(x_{n-1}, x_n) \le w(f(x_{n-2}, f(x_{n-1}))) &\le \alpha(x_{n-2}, x_{n-1}) w(f(x_{n-2}), f(x_{n-1})) \\ &\le \psi(\sum_{n=k}^{\infty} \psi^{k-2}(w(x_0, x_1))) \\ &< \sum_{n=k}^{\infty} \psi^{k-2}(w(x_0, x_1)) \xrightarrow{d} 0. \end{aligned} \tag{3.5}$$

Since f is α-ψ-contractive operator and since $x^* \in Fix(f)$, from (H) there exists $x_n \in X$ such that $\alpha(x^*, x_n) \ge 1$. By Theorem 2.3, for $m, n \in \mathbb{N}$, with $m > n$ we obtain: $w(x_n, x_m) \le \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1))$. Since $(x_m)_{m \in \mathbb{N}}$ converge to x^* and $w(x_n, \cdot)$ is lower semicontinuous we have

$$w(x_{n-1}, x^*) \le \lim_{m \to \infty} \inf(x_n, x_m) \le \lim_{m \to \infty} \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)) \le \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)).$$

By the properties of function ψ, using Lemma 1.3 and letting $n \to \infty$ we obtain:

$$w(x_{n-1}, x^*) \le \sum_{n=k}^{\infty} \psi^k(w(x_0, x_1)) \xrightarrow{d} 0. \tag{3.6}$$

Then, by (3.5) and (3.6) we have: $w(x_{n-1}, x_n) \overset{d}{\to} 0$ and $w(x_{n-1}, x^*) \overset{d}{\to} 0$. Using Lemma 1.2(2) we get that $x_n \overset{d}{\to} x^*$. So, the fixed point equation (3.3) is well posed with respect to w-distance.

(c) Since f is α-ψ-contractive operator and since $x^* \in Fix(f)$, from (H), there exists $x \in X$ such that $\alpha(x^*, x) \geq 1$. Using the triangle inequality we obtain:

$$\begin{aligned} w(x, x^*) &\leq w(x, f(x)) + w(f(x), x^*) \\ &= w(x, f(x)) + w(f(x), f(x^*)) \\ &\leq w(x, f(x)) + \alpha(x, x^*) w(f(x), f(x^*)) \\ &\leq w(x, f(x)) + \psi(w(x, x^*)). \end{aligned}$$

Therefore

$$\begin{aligned} \beta(w(x, x^*)) &:= w(x, x^*) - \psi(w(x, x^*)) \\ &\leq w(x, f(x)) - \psi(w(x, x^*)) + \psi(w(x, x^*)) \\ &\leq w(x, f(x)). \end{aligned}$$

We have the following estimation:

$$w(x, x^*) \leq \beta^{-1}(w(x, f(x))). \tag{3.7}$$

For any operator $g : X \to X$ with $y \in Fix(g)$ if we denote $x := y^*$, there exists $\eta \in [0, \infty)$ with $w(g(x), f(x)) < \eta$ such that:

$$w(y^*, x^*) \leq \beta^{-1}(w(y^*), f(y^*)) \leq \beta^{-1}(w(g(y^*), f(y^*))) \leq \beta^{-1}(\eta).$$

□

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Abdul Latif
Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah-21589, Saudi Arabia
E-mail address: alatif@kau.edu.sa

Liliana Guran
Department of Pharmaceutical Sciences, "Vasile Goldiş" Western University of Arad, Revoluţiei Avenue, no. 94-96, 310025, Arad, Romania
E-mail address: lguran@uvvg.ro

Monica-Felicia Bota
Department of Mathematics, Babeş-Bolyai University Kogălniceanu Street No.1, 400084, Cluj-Napoca, Romania
E-mail address: bmonica@math.ubbcluj.ro