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ORDERED VARIATIONAL INCLUSION PROBLEM

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Abstract. In this paper, we consider an ordered variational inclusion problem involving bi-mappings. A resolvent operator is designed for bi-mappings with $\oplus$ operation. We have shown that the resolvent operator is single-valued, compression as well as Lipschitz-type-continuous. An existence as well as a convergence results are proved. The results of this paper are new and refinement of previous results.

1. Introduction

The techniques based on variational inequalities methods are very useful to solve many problems occurring in pure, applied and basic sciences. Hassouni and Moudafi [17] studied a generalization of variational inequality problem, called variational inclusion problem, which is application oriented and used as a powerful tool to solve many problems related to optimization and control, non-linear programming, engineering, elasticity theory, economics and game theory, etc..

Li et al.[18, 19, 20, 21]. and Ahmad et al.[3, 6] considered and solved some problems related to ordered variational inclusions and ordered equations in Hilbert spaces as well as in Banach spaces. Motivated by the above mentioned work due to their applications, in this paper, we consider an ordered variational inclusion problem involving bi-mappings with $\oplus$ operation. Some properties of resolvent operator under consideration are proved and applied to solve ordered variational inclusion problem in Hilbert spaces. Many previous known results related to ordered variational inclusion problems can be obtained from our results easily.

2. Preliminaries

Throughout this paper, we suppose that $\mathcal{H}_p$ is a real ordered positive Hilbert space with its norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, d is the metric induced by the norm $\|\cdot\|$ and $2^{\mathcal{H}_p}$ is the family of all nonempty subsets of $\mathcal{H}_p$.

For the presentation of the results of this paper, let us recall some known definitions and results.

Definition 2.1. A nonempty closed convex subset C of $\mathcal{H}_p$ is said to be:

(i) a cone, if for any $x \in C$ and any $\lambda > 0$, $\lambda x \in C$, and for $x \in C$ and $-x \in C$, then $x = 0$.

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(*ii*) a normal cone, if there exists a constant $N > 0$ such that for $0 \leq x \leq y$, we have $||x|| \leq N||y||$.

Definition 2.2. Let C be a cone, then

(*i*) for any $x, y \in \mathcal{H}_p$, $x \leq y$ if and only if $y - x \in C$.
(*ii*) x and y are said to be comparative to each other if and only if, we have either $x \leq y$ or $y \leq x$ and is denoted by $x \propto y$.

Definition 2.3. For arbitrary elements $x, y \in \mathcal{H}_p$, $lub\{x, y\}$ and $glb\{x, y\}$, we mean least upper bound and greatest upper bound of the set $\{x, y\}$. Suppose $lub\{x, y\}$ and $glb\{x, y\}$ exist, some binary operations are defined as follows:

(*i*) $x \vee y = lub\{x, y\}$;
(*ii*) $x \wedge y = glb\{x, y\}$;
(*iii*) $x \oplus y = (x - y) \vee (y - x)$;
(*iv*) $x \odot y = (x - y) \wedge (y - x)$.

The operations $\vee, \wedge$, $\oplus$ and $\odot$ are called OR, AND, XOR and XNOR operations, respectively.

Lemma 2.4. *If $x \propto y$, then $lub\{x, y\}$ and $glb\{x, y\}$ exist, $x - y \propto y - x$ and $0 \leq (x - y) \vee (y - x)$.*

Lemma 2.5. *For any natural number n, $x \propto y_n$ and $y_n \to y^*$ as $n \to \infty$, then $x \propto y^*$.*

Proposition 2.6. *Let $\oplus$ be an XOR operation and $\odot$ be an XNOR operation. Then the following relations hold:*

(*i*) $x \odot x = 0$, $x \odot y = y \odot x = -(x \oplus y) = -(y \oplus x)$;
(*ii*) *if* $x \propto 0$, *then* $-x \oplus 0 \leq x \leq x \oplus 0$;
(*iii*) $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;
(*iv*) $0 \leq x \oplus y$, *if* $x \propto y$;
(*v*) *if* $x \propto y$, *then* $x \oplus y = 0$ *if and only if* $x = y$;
(*vi*) $(x + y) \odot (u + v) \geq (x \odot u) + (y \odot v)$;
(*vii*) $(x + y) \odot (u + v) \geq (x \odot v) + (y \odot u)$;
(*viii*) *if* x, y *and* w *are comparative to each other, then* $(x \oplus y) \leq x \oplus w + w \oplus y$;
(*ix*) $\alpha x \oplus \beta x = |\alpha - \beta|x = (\alpha \oplus \beta)x$, *if* $x \propto 0$, $\forall\ x, y, u, v \in \mathcal{H}_p$ *and* $\alpha, \beta, \lambda \in \mathbb{R}$.

Proposition 2.7. *Let C be a normal cone in $\mathcal{H}_p$ with normal constant N, then for each $x, y \in \mathcal{H}_p$, the following relations hold:*

(*i*) $\|0 \oplus 0\| = \|0\| = 0$;
(*ii*) $\|x \vee y\| \leq \|x\| \vee \|y\| \leq \|x\| + \|y\|$;
(*iii*) $\|x \oplus y\| \leq \|x - y\| \leq N|x \oplus y\|$;
(*iv*) *if* $x \propto y$, *then* $\|x \oplus y\| = \|x - y\|$.

Definition 2.8. Let $A : \mathcal{H}_p \to \mathcal{H}_p$ be a single-valued mapping, then

(*i*) A is said to be comparison mapping, if for each $x, y \in \mathcal{H}_p$, $x \propto y$ then $A(x) \propto A(y)$, $x \propto A(x)$ and $y \propto A(y)$.

(ii) A is said to be strongly comparison mapping, if A is a comparison mapping and $A(x) \propto A(y)$ if and only if $x \propto y$, for any $x, y \in \mathcal{H}_p$.

Definition 2.9. A mapping $A : \mathcal{H}_p \to \mathcal{H}_p$ is said to be β-ordered compression mapping, if A is a comparison mapping and

$$A(x) \oplus A(y) \leq \beta(x \oplus y), \text{ for } 0 < \beta < 1.$$

Definition 2.10. Let $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Then

(i) M is said to be a comparison mapping, if for any $v_x \in M(x)$, $x \propto v_x$, and if $x \propto y$, then for any $v_x \in M(x)$ and any $v_y \in M(y)$, $v_x \propto v_y$, $\forall\ x, y \in \mathcal{H}_p$;

(ii) a comparison mapping M is said to be α-non-ordinary difference mapping, if for each $x, y \in \mathcal{H}_p$, $v_x \in M(x)$ and $v_y \in M(y)$ such that

$$(v_x \oplus v_y) \oplus \alpha(x \oplus y) = 0;$$

(iii) a comparison mapping M is said to be θ-ordered rectangular, if there exists a constant $\theta > 0$, for any $x, y \in \mathcal{H}_p$, there exist $v_x \in M(x)$ and $v_y \in M(y)$ such that

$$\langle v_x \odot v_y, -(x \oplus y)\rangle \geq \theta \|x \oplus y\|^2,$$

holds.

Definition 2.11. A mapping $M : \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be λ-XOR-ordered strongly monotone compression mapping, if $x \propto y$, then there exists a constant $\lambda > 0$ such that

$$\lambda(v_x \oplus v_y) \geq x \oplus y, \forall x, y \in \mathcal{H}_p, v_x \in M(x), v_y \in M(y).$$

Note that a non-ordinary difference mapping which is also XOR-ordered strongly monotone is called XOR-NODSM mapping.

Lemma 2.12. *Let $\{\chi_n\}$ be a nonnegative real sequence and $\{\zeta_n\}$ be a real sequence in $[0, 1]$ such that $\sum\limits_{n=0}^{\infty} \zeta_n = \infty$. If there exists a positive integer m such that*

$$\chi_n \leq (1 - \zeta_n)\chi_n + \zeta_n \eta_n,\ \forall n \geq m, \tag{2.1}$$

where $\eta_n \geq 0$, for all $n \geq 0$ and $\eta_n \to 0$ $(n \to 0)$, then $\lim\limits_{n\to\infty} \chi_n = 0$.

Now, we extend the definitions of comparison and strongly comparison mapping, XOR-ordered strongly compression mapping, XOR-NODSM mapping for bi-mappings and define a new resolvent operator associated with XOR-NODSM mapping.

Definition 2.13. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be a bi-mapping.

(i) A is said to be comparison mapping, if for each $(x, .), (y, .) \in \mathcal{H}_p \times \mathcal{H}_p$, $(x, .) \propto (y, .)$ then $A(x, .) \propto A(y, .)$, $(x, .) \propto A(x, .)$ and $(y, .) \propto A(y, .)$.

(ii) A is said to be strongly comparison mapping, if A is a comparison mapping and $A(x, .) \propto A(y, .)$ if and only if $(x, .) \propto (y, .)$, for any $(x, .), (y, .) \in \mathcal{H}_p \times \mathcal{H}_p$.

Definition 2.14. A mapping $A : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ is said to be β-ordered compression mapping, if A is a comparison mapping and

$$A(x,.) \oplus A(y,.) \leq \beta((x,.) \oplus (y,.)), \text{ for } 0 < \beta < 1.$$

Definition 2.15. Let $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Then

(i) M is said to be a comparison mapping, if for any $v_x \in M(x,.)$, $(x,.) \propto v_x$, and if $(x,.) \propto (y,.)$, then for any $v_x \in M(x,.)$ and any $v_y \in M(y,.)$, $v_x \propto v_y$, $\forall\ (x,.),(y,.) \in \mathcal{H}_p \times \mathcal{H}_p$;

(ii) a comparison mapping M is said to be α-non-ordinary difference mapping, if for each $(x,.),(y,.) \in \mathcal{H}_p \times \mathcal{H}_p$, $v_x \in M(x,.)$ and $v_y \in M(y,.)$ such that
$$(v_x \oplus v_y) \oplus \alpha((x,.) \oplus (y,.)) = 0;$$

(iii) a comparison mapping M is said to be θ-ordered rectangular, if there exists a constant $\theta > 0$, for any $(x,.),(y,.) \in \mathcal{H}_p \times \mathcal{H}_p$, there exist $v_x \in M(x,.)$ and $v_y \in M(y,.)$ such that
$$\langle v_x \odot v_y, -((x,.) \oplus (y,.))\rangle \geq \theta \|(x,.) \oplus (y,.)\|^2,$$
holds.

Definition 2.16. A mapping $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be λ-XOR-ordered strongly monotone compression mapping, if $(x,.) \propto (y,.)$, then there exists a constant $\lambda > 0$ such that

$$\lambda(v_x \oplus v_y) \geq ((x,.) \oplus (y,.)), \forall (x,.),(y,.) \in \mathcal{H}_p \times \mathcal{H}_p, v_x \in M(x,.), v_y \in M(y,.).$$

Definition 2.17. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be a strongly comparison and β-ordered compression mapping. Then, a comparison set-valued mapping $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is said to be (α,λ)-XOR-NODSM, if M is a α-non-ordinary difference mapping and λ-XOR-ordered strongly monotone mapping and $[A \oplus \lambda M](\mathcal{H}_p \times \mathcal{H}_p) = \mathcal{H}_p \times \mathcal{H}_p$, for $\lambda, \beta, \alpha > 0$.

Definition 2.18. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be a strongly comparison and β-ordered compression mapping. Suppose that $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ is a set-valued, (α,λ)-XOR-NODSM mapping. The resolvent operator $\mathcal{J}^A_{\lambda,M} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ associated with A and M is defined by

$$\mathcal{J}^A_{\lambda,M}(x,.) = [A(.,y) \oplus \lambda M(.,y)]^{-1}(x,.), \forall (x,.) \in \mathcal{H}_p \times \mathcal{H}_p, \tag{2.2}$$

where $\lambda > 0$ is a constant.

Proposition 2.19. *Let $A : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be a strongly comparison, β-ordered compression mapping and $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be the set-valued θ-ordered rectangular mapping with $\theta\lambda > \beta$. Then the resolvent operator $\mathcal{J}^A_{\lambda,M} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ is a single-valued, for all $\lambda > 0$.*

Proof. For any given $(u,.) \in \mathcal{H}_p \times \mathcal{H}_p$ and a constant $\lambda > 0$, let $(x,.),(y,.) \in [A \oplus \lambda M]^{-1}(u,.)$. Then, let

$$v_x = \frac{1}{\lambda}((u,.) \oplus A(x,.)) \in M(x,.),$$

and

$$v_y = \frac{1}{\lambda}((u,.) \oplus A(y,.)) \in M(y,.).$$

Using (i) and (ii) of Proposition 2.6, we have

$$\begin{aligned} v_x \odot v_y &= \frac{1}{\lambda}((u,.) \oplus A(x,.)) \odot \frac{1}{\lambda}((u,.) \oplus A(y,.)) \\ &= \frac{1}{\lambda}[((u,.) \oplus A(x,.)) \odot ((x,.) \oplus A(y,.))] \\ &= -\frac{1}{\lambda}[((u,.) \oplus A(x,.)) \oplus ((u,.) \oplus A(y,.))] \\ &= -\frac{1}{\lambda}[((u,.) \oplus (u,.) \oplus (A(x,.) \oplus A(y,.))] \\ &= -\frac{1}{\lambda}[0 \oplus (A(x,.) \oplus A(y,.))] \\ &\leq -\frac{1}{\lambda}[A(x,.) \oplus A(y,.)]. \end{aligned} \tag{2.3}$$

Thus, we have

$$v_x \odot v_y \leq -\frac{1}{\lambda}[A(x,.) \oplus A(y,.)]. \tag{2.4}$$

Since M is θ-ordered rectangular mapping, A is β-ordered compression mapping and using (2.4), we have

$$\begin{aligned} \theta\|(x,.) \oplus (y,.)\|^2 &\leq \langle v_x \odot v_y, -((x,.) \oplus (y,.))\rangle \\ &\leq \langle -\frac{1}{\lambda}[A(x,.) \oplus A(y,.)], -((x,.) \oplus (y,.))\rangle \\ &\leq \frac{1}{\lambda}\langle A(x,.) \oplus A(y,.), (x,.) \oplus (y,.)\rangle \\ &\leq \frac{1}{\lambda}\langle \beta((x,.) \oplus (y,.)), (x,.) \oplus (y,.)\rangle \\ &\leq \frac{\beta}{\lambda}\langle (x,.) \oplus (y,.), (x,.) \oplus (y,.)\rangle \\ &= \frac{\beta}{\lambda}\|(x,.) \oplus (y,.)\|^2, \end{aligned}$$

i.e.,

$$\begin{aligned} \theta\|(x,.) \oplus (y,.)\|^2 &\leq \frac{\beta}{\lambda}\|x \oplus y\|^2, \\ \Big(\theta - \frac{\beta}{\lambda}\Big)\|(x,.) \oplus (y,.)\|^2 &\leq 0, \text{ for } \theta\lambda > \beta, \end{aligned}$$

which implies that

$$\|(x,.) \oplus (y,.)\| = 0 \Rightarrow (x,.) \oplus (y,.) = 0. \tag{2.5}$$

Therefore, $(x,.) = (y,.)$. Hence the resolvent operator $\mathcal{J}^A_{\lambda,M}$ is single-valued, for $\theta\lambda > \beta$.

□

Proposition 2.20. *Let $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a (α, λ)-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^A_{\lambda,M}$. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be a strongly comparison mapping with respect to $\mathcal{J}^A_{\lambda,M}$. Then the resolvent operator $\mathcal{J}^A_{\lambda,M} : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ is a comparison mapping.*

Proof. Let M be a (α, λ)-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^A_{\lambda,M}$. That is, M is α-non-ordinary difference and λ-XOR-ordered strongly monotone comparison mapping with respect to $\mathcal{J}^A_{\lambda,M}$ so that $(x, .) \propto \mathcal{J}^A_{\lambda,M}(x, .)$. For any $(x, .), (y, .) \in \mathcal{H}_p \times \mathcal{H}_p$, let $(x, .) \propto (y, .)$ and

$$v^*_x = \frac{1}{\lambda}((x, .) \oplus A(\mathcal{J}^A_{\lambda,M}(x, .))) \in M(\mathcal{J}^A_{\lambda,M}(x, .)) \tag{2.6}$$

and

$$v^*_y = \frac{1}{\lambda}((y, .) \oplus A(\mathcal{J}^A_{\lambda,M}(y, .))) \in M(\mathcal{J}^A_{\lambda,M}(y, .)). \tag{2.7}$$

Since M is λ-XOR-ordered strongly monotone mapping, using (2.6) and (2.7), we have

$$(x, .) \oplus (y, .) \leq \lambda[(v^*_x \oplus v^*_y]$$
$$(x, .) \oplus (y, .) \leq \Big((x, .) \oplus A(\mathcal{J}^A_{\lambda,M}(x, .))\Big) \oplus \Big((y, .) \oplus A(\mathcal{J}^A_{\lambda,M}(y, .))\Big)$$
$$(x, .) \oplus (y, .) \leq \Big((x, .) \oplus (y, .))\Big) \oplus \Big(A(\mathcal{J}^A_{\lambda,M}(x, .)) \oplus A(\mathcal{J}^A_{\lambda,M}(y, .))\Big)$$
$$0 \leq A(\mathcal{J}^A_{\lambda,M}(x, .)) \oplus A(\mathcal{J}^A_{\lambda,M}(y, .))$$
$$0 \leq \Big[A(\mathcal{J}^A_{\lambda,M}(x, .)) - A(\mathcal{J}^A_{\lambda,M}(y, .))\Big] \vee \Big[A(\mathcal{J}^A_{\lambda,M}(y, .)) - A(\mathcal{J}^A_{\lambda,M}(x, .))\Big]$$
$$0 \leq \Big[A(\mathcal{J}^A_{\lambda,M}(x, .)) - A(\mathcal{J}^A_{\lambda,M}(y, .))\Big] \text{ or } 0 \leq \Big[A(\mathcal{J}^A_{\lambda,M}(y, .)) - A(\mathcal{J}^A_{\lambda,M}(x, .))\Big].$$

Thus, we have

$$A(\mathcal{J}^A_{\lambda,M}(x, .)) \geq A(\mathcal{J}^A_{\lambda,M}(y, .)) \text{ or } A(\mathcal{J}^A_{\lambda,M}(y, .)) \geq A(\mathcal{J}^A_{\lambda,M}(x, .)),$$

which implies that

$$A(\mathcal{J}^A_{\lambda,M}(x, .)) \propto A(\mathcal{J}^A_{\lambda,M}(y, .)).$$

Since A is strongly comparison mapping with respect to $\mathcal{J}^A_{\lambda,M}$. Therefore, $\mathcal{J}^A_{\lambda,M}(x, .) \propto \mathcal{J}^A_{\lambda,M}(y, .)$. That is, the resolvent operator $\mathcal{J}^A_{\lambda,M}$ is a comparison mapping. □

Proposition 2.21. *Let $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a (α, λ)-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^A_{\lambda,M}$. Let $A : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be a comparison and β-ordered compression mapping with respect to $\mathcal{J}^A_{\lambda,M}$, for $\alpha\lambda > \beta$. Then the following condition holds:*

$$\mathcal{J}^A_{\lambda,M}(x, .) \oplus \mathcal{J}^A_{\lambda,M}(y, .) \leq \frac{1}{(\alpha\lambda \oplus \beta)}(x, .) \oplus (y, .), \forall (x, .), (y, .) \in \mathcal{H}_p \times \mathcal{H}_p. \tag{2.8}$$

Proof. Let $(x, .), (y, .) \in \mathcal{H}_p \times \mathcal{H}_p$,

$$v_{x^*} = \frac{1}{\lambda}\big((x, .) \oplus A(\mathcal{J}^A_{\lambda,M}(x, .))\big) \in M(\mathcal{J}^A_{\lambda,M}(x, .))$$

and

$$v_{y^*} = \frac{1}{\lambda}\big((y,.) \oplus A(\mathcal{J}^A_{\lambda,M}(y,.))\big) \in M(\mathcal{J}^A_{\lambda,M}(y,.)).$$

As M be an (α,λ)-XOR-NODSM set-valued mapping with respect to $\mathcal{J}^A_{\lambda,M}$ and A is β-ordered compression mapping with respect to $\mathcal{J}^A_{\lambda,M}$. It follows that M is also an α-non-ordinary difference mapping with respect to $\mathcal{J}^A_{\lambda,M}$, we have

$$(v_{x^*} \oplus v_{y^*}) \oplus \alpha(\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(y,.)) = 0, \tag{2.9}$$

and

$$\begin{aligned} v_{x^*} \oplus v_{y^*} &= \frac{1}{\lambda}[((x,.) \oplus A(\mathcal{J}^A_{\lambda,M}(x,.))) \oplus ((y,.) \oplus A(\mathcal{J}^A_{\lambda,M}(y,.)))] \\ &= \frac{1}{\lambda}[((x,.) \oplus (y,.)) \oplus (A(\mathcal{J}^A_{\lambda,M}(x,.)) \oplus A(\mathcal{J}^A_{\lambda,M}(y,.)))] \\ &\leq \frac{1}{\lambda}[((x,.) \oplus (y,.)) \oplus \beta(\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(y,.))]. \end{aligned}$$

From (2.9), we have

$$\begin{aligned} \alpha(\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(y,.)) &= v_{x^*} \oplus v_{y^*} \\ &\leq \frac{1}{\lambda}[((x,.) \oplus (y,.)) \oplus \beta(\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(y,.))], \end{aligned} \tag{2.10}$$

i.e.,

$$\alpha\lambda(\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(y,.)) \leq [((x,.) \oplus (y,.)) \oplus \beta(\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(y,.))].$$

Now,

$$\begin{aligned} &\big(\alpha\lambda(\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(y,.)\big) \oplus \big(\beta(\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(y,.)\big) \\ &\qquad \leq ((x,.) \oplus (y,.)) \oplus 0 = (x,.) \oplus (y,.) \\ &\big(\alpha\lambda \oplus \beta\big)\big(\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(y,.)\big) \leq (x,.) \oplus (y,.). \end{aligned}$$

It follows that $\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(x,.) \leq \Big(\frac{1}{(\alpha\lambda\oplus\beta)}\Big)(x,.) \oplus (y,.)$ and consequently, we have

$$\mathcal{J}^A_{\lambda,M}(x,.) \oplus \mathcal{J}^A_{\lambda,M}(y,.) \leq \frac{1}{(\alpha\lambda \oplus \beta)}(x,.) \oplus (y,.). \tag{2.11}$$

□

3. Formulation of the problem and existence results

Let $C \subseteq \mathcal{H}_p$ be a normal cone with constant N. Let $P : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be a bi-mapping and $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. We consider the following problem:

Find $(x,.) \in \mathcal{H}_p \times \mathcal{H}_p$ such that

$$0 \in P(x,y) \oplus M(x,y), \tag{3.1}$$

for some fixed $y \in \mathcal{H}_p$.

We call problem (3.1) as ordered variational inclusion problem involving XOR operation (in short, OVIP).

Below we list some special cases of problem (3.1).

(i) If $P = 0$, and $M(x,y) = M(x)$ then OVIP (3.1) reduces to the problem of finding $x \in \mathcal{H}_p$ such that

$$0 \in M(x). \tag{3.2}$$

Problem (3.2) is introduced and studied by Li [20].

(ii) If $P(x,y) = P(x)$, $M(x,y) = M(x)$, then problem (3.1) becomes the problem of finding $x \in \mathcal{H}_p$ such that

$$0 \in P(x) \oplus M(x) \tag{3.3}$$

Problem (3.3) is introduced and studied by Iqbal et al. [7].

Hence, we claim that our problem is much more general than many existing problems in the literature. The following lemma is a fixed point formulation of OVIP (3.1).

Lemma 3.1. *The OVIP* (3.1) *admits a solution* $(x,.) \in \mathcal{H}_p \times \mathcal{H}_p$ *if and only if it satisfies the following equation:*

$$(x,.) = \mathcal{J}^A_{\lambda,M}[\lambda P(x,y) \oplus A(x,y)], \tag{3.4}$$

where $\lambda > 0$ *is constant.*

Proof. Proof is a direct consequence of the definition of resolvent operator (2.2). □

Theorem 3.2. *Let* $P, A : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ *be the bi-mappings such that* P *is a comparison,* γ*-ordered compression mapping and* A *is a comparison and* β*-ordered compression mapping, respectively. Suppose that* $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ *is a* (α,λ)*-XOR-NODSM set-valued mapping. In addition, if for all* $\lambda, \alpha > 0$*, the following conditions are satisfied:*

$$\begin{cases} |(|\lambda|\gamma \oplus \beta)| < \frac{|\alpha\lambda \oplus \beta|}{N}; \\ \alpha\lambda > \beta, \end{cases} \tag{3.5}$$

then, OVIP (3.1) *admits a solution* $(x^*,.) \in \mathcal{H}_p \times \mathcal{H}_p$*, which is a fixed point of* $\mathcal{J}^A_{\lambda,M}[\lambda P(x^*,.) \oplus A(x^*,.)]$.

Proof. By Proposition 2.20, if $(x_1,.) \propto (x_2,.)$, then

$$\mathcal{J}^A_{\lambda,M}[\lambda P(x_1,y) \oplus A(x_1,y)] \propto \mathcal{J}^A_{\lambda,M}[\lambda P(x_2,y) \oplus A(x_2,y)].$$

Since P is γ-ordered compression, A is β-ordered compression mapping and using Proposition 2.21, we have

$$\begin{aligned} 0 &\leq \mathcal{J}^A_{\lambda,M}[\lambda P(x_1,y) \oplus A(x_1,y)] \oplus \mathcal{J}^A_{\lambda,M}[\lambda P(x_2,y) \oplus A(x_2,y)] \\ &\leq \frac{1}{(\alpha\lambda \oplus \beta)}\Big([\lambda P(x_1,y) \oplus A(x_1,y)] \oplus [\lambda P(x_2,y) \oplus A(x_2,y)]\Big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(\alpha\lambda \oplus \beta)}\Big([|\lambda|(P(x_1, y) \oplus P(x_2, y))] \oplus [A(x_1, y) \oplus A(x_2, y)]\Big) \\
&\leq \frac{1}{(\alpha\lambda \oplus \beta)}\Big([|\lambda|\gamma(x_1, y) \oplus (x_2, y))] \oplus [\beta((x_1, y)) \oplus (x_2, y))]\Big) \\
&= \frac{1}{(\alpha\lambda \oplus \beta)}\Big([|\lambda|\gamma \oplus \beta](x_1, y) \oplus (x_2, y)\Big) \\
&= \frac{(|\lambda|\gamma \oplus \beta)}{(\alpha\lambda \oplus \beta)}\Big((x_1, y) \oplus (x_2, y)\Big),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
(3.6) \qquad 0 &\leq \mathcal{J}^A_{\lambda,M}[\lambda P(x_1, y) \oplus A(x_1, y)] \oplus \mathcal{J}^A_{\lambda,M}[\lambda P(x_2, y) \oplus A(x_2, y)] \\
&\leq \Theta(x_1, y) \oplus x_2, y),
\end{aligned}
$$

$$
\text{where } \Theta = \frac{(|\lambda|\gamma \oplus \beta)}{(\alpha\lambda \oplus \beta)}.
$$

Now by using the definition of normal cone and Proposition 2.7, from (3.6), we have

$$
\Big\|\mathcal{J}^A_{\lambda,M}[\lambda P(x_1, y) \oplus A(x_1, y)] - \mathcal{J}^A_{\lambda,M}[\lambda P(x_2, y) \oplus A(x_2, y)]\Big\| \leq N|\Theta|\|(x_1, y) - (x_2, y)\|.
$$

It is clear from (3.5), that $|\Theta| < \frac{1}{N}$. It follows that $\mathcal{J}^A_{\lambda,M}[\lambda P(.,.) \oplus A(.,.)]$ is contraction mapping. Therefore, there exists a unique $(x^*, .) \in \mathcal{H}_p \times \mathcal{H}_p$ such that

$$
(x^*, .) = \mathcal{J}^A_{\lambda,M}[\lambda P(x^*, .) \oplus A(x^*, .)].
$$

By Lemma 3.1, $(x^*, .)$ is a unique solution of OVIP (3.1), which is a fixed point of $\mathcal{J}^A_{\lambda,M}[\lambda P(x^*, .) \oplus A(x^*, .)]$. □

4. Convergence Analysis

First we establish an Ishikawa type iterative algorithm based on Lemma 3.1 for finding the approximate solution of OVIP (3.1), and then we prove a convergence result. If $(x_n, .) \to (x^*, .)$, we mean $x_n \to x^*$ and vice versa.

Iterative Algorithm 4.1. Let $A, P : \mathcal{H}_p \times \mathcal{H}_p \to \mathcal{H}_p$ be the bi-mappings and $M : \mathcal{H}_p \times \mathcal{H}_p \to 2^{\mathcal{H}_p}$ be a set-valued mapping. Given any $(x_0, .) \in \mathcal{H}_p \times \mathcal{H}_p$, compute the sequence $\{x_n\}$ converges to x^* such that $(x_n, .)$ converges to $(x^*, .)$ and defined by the following iterative scheme:

$$
(4.1) \qquad \begin{cases} (x_{n+1}, .) = (1 - a_n)(x_n, .) + a_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(y_n, .) \oplus A(y_n, .)]\Big) + a_n\alpha_n, \\ (y_n, .) = (1 - b_n)(x_n, .) + b_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x_n, .) \oplus A(x_n, .)]\Big) + b_n\beta_n. \end{cases}
$$

where $0 \leq a_n, b_n \leq 1$, $\sum_{n=0}^{\infty} a_n = \infty, \forall\ n \geq 0$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $\mathcal{H}_p \times \mathcal{H}_p$ introduced to take into account the possible inexact computation provided that $\alpha_n \oplus 0 = \alpha_n$ and $\beta_n \oplus 0 = \beta_n$, $\forall\ n \geq 0$.

Remark 4.1. If $P(x_n,.) = P(x_n), P(y_n,.) = P(y_n), A(x_n,.) = A(x_n), A(y_n,.) = A(y_n), b_n = 0, \forall\, n \geq 0$, then Algorithm 4.1 becomes Mann type iterative algorithm. Also, we remark that for suitable choices of operators involved in Algorithm 4.1, we can easily obtain many more algorithms studied by several authors for solving ordered variational inclusion problems.

Theorem 4.2. *Let M, A and P be the same as in Theorem 3.2 such that all the conditions of Theorem 3.2 are satisfied. Additionally, if the following conditions are satisfied:*

$$\begin{cases} |(|\lambda|\gamma \oplus \beta)| < |\alpha\lambda \oplus \beta| \\ \alpha\lambda > \beta, \end{cases} \tag{4.2}$$

and $\lim_{n\to\infty} \|\alpha_n \vee (-\alpha_n)\| = \lim_{n\to\infty} \|\beta_n \vee (-\beta_n)\| = 0$, then the sequence $\{x_n\}$ generated by Algorithm 4.1 converges strongly to the unique solution x^* of OVIP (3.1).

Proof. We show that the sequence $\{x_n\}$ converges strongly to the unique solution x^* of OVIP (3.1). Theorem 3.2 implies that $(x^*,.)$ i.e x^* is a unique solution of OVIP (3.1). Then, we have

$$\begin{cases} (x^*,.) = (1-a_n)(x^*,.) + a_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*,.) \oplus A(x^*,.)]\Big) \\ \quad = (1-b_n)(x^*,.) + b_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*,.) \oplus A(x^*,.)]\Big). \end{cases} \tag{4.3}$$

Using Algorithm 4.1, (4.3), Proposition 2.6 and Proposition 2.21, it follows that

$$\begin{aligned} 0 &\leq (x_{n+1},.) \oplus (x^*,.) \\ &= \Big[(1-a_n)(x_n,.) + a_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(y_n,.) \oplus A(y_n,.)]\Big) + a_n\alpha_n\Big] \\ &\quad \oplus\Big[(1-a_n)(x^*,.) + a_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*,.) \oplus A(x^*,.)]\Big) + a_n 0\Big] \\ &\leq (1-a_n)(x_n,.) \oplus (x^*,.) + a_n(\alpha_n \oplus 0) \\ &\quad + a_n\Big[\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(y_n,.) \oplus A(y_n,.)]\Big) \oplus \Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*,.) \oplus A(x^*,.)]\Big)\Big] \\ &\leq (1-a_n)(x_n \oplus x^*) + a_n(\alpha_n \oplus 0) \\ &\quad + a_n\Big[\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(y_n,.) \oplus A(y_n,.)]\Big) \oplus \Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*,.) \oplus A(x^*,.)]\Big)\Big] \\ &\leq (1-a_n)(x_n \oplus x^*) + a_n(\alpha_n \oplus 0) + \Theta a_n((y_n,.) \oplus (x^*,.)) \\ &\leq (1-a_n)(x_n \oplus x^*) + a_n(\alpha_n \oplus 0) + \Theta a_n(y_n \oplus x^*). \end{aligned} \tag{4.4}$$

$$\text{where } \Theta = \frac{(|\lambda|\gamma \oplus \beta)}{(\alpha\lambda \oplus \beta)}.$$

Now we evaluate,

$$\begin{aligned} 0 &\leq (y_n,.) \oplus (x^*,.) \\ &= \Big[(1-b_n)(x_n,.) + b_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x_n,.) \oplus A(x_n,.)]\Big) + b_n\beta_n\Big] \\ &\quad \oplus\Big[(1-b_n)(x^*,.) + b_n\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*,.) \oplus A(x^*,.)]\Big) + b_n 0\Big] \\ &\leq (1-b_n)(x_n,.) \oplus (x^*,.) + b_n(\beta_n \oplus 0) \end{aligned}$$

$$
\begin{aligned}
&+b_n\Big[\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x_n,.)\oplus A(x_n,.)]\Big)\oplus\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*,.)\oplus A(x^*,.)]\Big)\Big]\\
\leq\ &(1-b_n)(x_n\oplus x^*)+b_n(\beta_n\oplus 0)\\
&+b_n\Big[\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x_n,.)\oplus A(x_n,.)]\Big)\oplus\Big(\mathcal{J}^A_{\lambda,M}[\lambda P(x^*,.)\oplus A(x^*,.)]\Big)\Big]\\
\leq\ &(1-b_n)(x_n\oplus x^*)+b_n(\beta_n\oplus 0)+\Theta b_n((x_n,.)\oplus(x^*,.))\\
\leq\ &(1-b_n)(x_n\oplus x^*)+b_n(\beta_n\oplus 0)+\Theta b_n(x_n\oplus x^*). \qquad (4.5)
\end{aligned}
$$

Combining (4.4) and (4.5), it follows that

$$
\begin{aligned}
0\ \leq\ &(x_{n+1},.)\oplus(x^*,.)\\
\leq\ &(1-a_n)(x_n\oplus x^*)+\Theta a_n\big[(1-b_n)(x_n\oplus x^*)+\Theta b_n(x_n\oplus x^*)\\
&+b_n(\beta_n\oplus 0)\big]+a_n(\alpha_n\oplus 0)\\
\leq\ &(1-a_n(1-2\Theta))(x_n\oplus x^*)+a_n\big[\Theta b_n(\beta_n\oplus 0)+(\alpha_n\oplus 0)\big]. \qquad (4.6)
\end{aligned}
$$

Using definition of normal cone and Proposition 2.7, we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|\ \leq\ &N(1-a_n(1-2\Theta))\|x_n-x^*\|\\
&+Na_n(1-2\Theta)\left(\frac{\Theta b_n\|\beta_n\vee(-\beta_n)\|+\|\alpha_n\vee(-\alpha_n)\|}{(1-2\Theta)}\right).
\end{aligned}
\qquad (4.7)
$$

By setting $\eta_n=\frac{\Theta b_n\|\beta_n\vee(-\beta_n)\|+\|\alpha_n\vee(-\alpha_n)\|}{(1-2\Theta)}$, $\chi_n=\|x_n-x^*\|$, $\zeta_n=Na_n(1-2\Theta)$, inequality (4.7) can be rewritten as

$$\chi_n\leq(1-\zeta_n)\chi_n+\zeta_n\eta_n. \qquad (4.8)$$

From Lemma 2.12 and using the hypothesis $\lim_{n\to\infty}\|\alpha_n\vee(-\alpha_n)\|=\lim_{n\to\infty}\|\beta_n\vee(-\beta_n)\|=0$, we deduce that $\chi_n\to 0$, as $n\to\infty$, and so $\{x_n\}$ converges strongly to a unique solution x^* of OVIP (3.1). □

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