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A GENERAL IMPLICIT ITERATIVE METHOD FOR A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce a general implicit iterative method for a countable family of nonexpansive mappings in a reflexive Banach space having a weakly continuous duality mapping. A strong convergence theorem for the sequence generated by the proposed method is established.

1. Introduction

Let E be a real Banach space with the norm $\|\cdot\|$, and let E^* be the dual space of E. Let J denote the normalized duality mapping from E into 2^{X^*} defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|,\ \|f\| = \|x\|\}, \quad \forall x \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pair between E and E^*. Let C be a nonempty closed convex subset of E. For the mapping $T : C \to C$, we denote the fixed point set of T by $Fix(T)$, that is, $Fix(T) = \{x \in C : Tx = x\}$.

Recall that the mapping $T : C \to C$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x,\ y \in C.$$

T is said to be *pseudocontractive* if there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le \|x - y\|^2, \quad \forall x,\ y \in C.$$

T is said to be *strongly pseudocontractive* it there exists a constant $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \le k\|x - y\|^2, \quad \forall x, y \in C.$$

In a Banach space E having a single-valued normalized duality mapping J, we say that an operator A is strongly positive on E if there exists a $\overline{\gamma} > 0$ with the property

$$\langle Ax, J(x)\rangle \ge \overline{\gamma}\|x\|^2, \tag{1.1}$$

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and

$$\|aI - bA\| = \sup_{\|x\| \leq 1} |\langle (aI - bA)x, J(x) \rangle|, \quad a \in [0,1], \quad b \in [-1.1],$$

for all $x \in E$, where I is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (1.1) reduce to

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.2}$$

Within the past 40 years or so, many authors have been devoting their study to the existence of fixed points and iterative construction of fixed points of nonexpansive mappings and pseudocontractive mappings. Also several iterative methods for finding a common fixed point of a family of nonexpansive mappings and pseudocontractive mappings in Hilbert spaces and Banach spaces have been introduced and studied by many authors (see [5, 13–16, 19–21, 23, 24] and references in therein). The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings (see [4, 7]). The problem of finding an optimal point which minimizes a given cost function over the common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance (see [3, 17, 25]). An algorithm solution to the problem of minimizing a quadratic function over the common set of fixed points of a family of nonexpansive mappings is of value in many applications including set theoretic signal estimation (see [10, 25]).

In 2006, Marino and Xu [13] introduced the following general iterative method for nonexpansive mapping T in a Hilbert space in an implicit way:

$$x_t = t\gamma f(x_t) + (I - tA)Tx_t, \quad \forall t \in (0, \min\{1, \|A\|^{-1}\}),$$

where $A : H \to H$ is a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$, and $f : H \to H$ is a contractive mapping (that is, there exists $k \in (0,1)$ such that $\|f(x) - f(y)\| \leq k\|x - y\|, \ \forall x, y \in H$). In 2011, Wangkeeree et al. [21] extended the result of Marino and Xu [13] to a reflexive Banach space having a weakly continuous duality mapping.

In 2007, Rafiq [18] introduced the following Mann type implicit iterative method for a hemicontractive mapping T in a Hilbert space,

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)Tx_n, \quad n \geq 1,$$

where $\{\alpha_n\} \subset [\delta, 1-\delta]$ for some $\delta \in (0,1)$, and proved a strong convergence theorem under compactness assumption on domain of T.

In 2007, Yao et al. [24] introduced the following Halpern type implicit iterative method for a continuous pseudocontractive mapping T in a uniformly smooth Banach space,

$$x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n Tx_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three sequences in $(0,1)$, and established a strong convergence theorem under appropriate control conditions.

In 2016, Jung [11] considered the following viscosity implicit iterative method for a continuous pseudocontractive mapping T in a reflexive Banach space having a weakly continuous duality mapping,

$$x_n = \alpha_n f x_n + \beta_n x_{n-1} + (1 - \alpha_n - \beta_n) T x_n, \quad \forall n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $f : C \to C$ is a continuous bounded strongly pseudocontractive mapping, and obtained a strong convergence theorem under suitable control conditions.

In this paper, as a continuation of study in this direction, we introduce a new general implicit iterative method for a countable family of nonexpansive mappings in a reflexive Banach space having a weakly continuous duality mapping. Then we establish the strong convergence of the sequence generated by proposed iterative method to a common fixed point of the mappings, which solves a ceratin variational inequality. The main results extend, improve and develop some corresponding results in [13, 21] and the references therein.

2. Preliminaries

Throughout this paper, when $\{x_n\}$ is a sequence in E, then $x_n \to x$ (respectively $x_n \rightharpoonup x$, $x_n \rightharpoonup^* x$) will denote strong (respectively weak, weak*) convergence of the sequence $\{x_n\}$ to x.

This section collects some definitions and lemmas which will be used in the proofs for the main results in the next section.

Recall that the norm of E is said to be *Gâteaux differentiable* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for each x, y in its unit sphere $U = \{x \in E : \|x\| = 1\}$. Such an E is called a *smooth* Banach space. It is known that E is smooth if and only if the normalized duality mapping J is single-valued.

By a gauge function we mean a continuous strictly increasing function φ defined on $\mathbb{R}^+ := [0, \infty)$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. The mapping $J_\varphi : E \to 2^{E^*}$ defined by

$$J_\varphi(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \varphi(\|x\|)\}, \quad \forall x \in E$$

is called the *duality mapping* with gauge function φ. In particular, the duality mapping with gauge function $\varphi(t) = t$ denoted by J, is referred to as the *normalized duality mapping*. It is known that a Banach space E is smooth if and only if the normalized duality mapping J is single-valued. The following property of duality mapping is also well-known:

$$J_\varphi(\lambda x) = \operatorname{sign} \lambda \left(\frac{\varphi(|\lambda| \cdot \|x\|)}{\|x\|} \right) J(x) \quad \text{for all } x \in E \setminus \{0\}, \quad \lambda \in \mathbb{R}, \tag{2.2}$$

where $\mathbb{R}$ is the set of all real numbers; in particular, $J(-x) = -J(x)$ for all $x \in E$.

We say that a Banach space E has a *weakly continuous duality mapping* if there exists a gauge function φ such that the duality mapping J_φ is single-valued and continuous from the weak topology to the weak* topology, that is, for any $\{x_n\} \in E$

with $x_n \rightharpoonup x$, $J_\varphi(x_n) \stackrel{*}{\rightharpoonup} J_\varphi(x)$. For example, every l^p space $(1 < p < \infty)$ has a weakly continuous duality mapping with gauge function $\varphi(t) = t^{p-1}$. Set

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau, \quad \forall t \in \mathbb{R}^+.$$

Then it is known that $J_\varphi(x)$ is the subdifferential of the convex functional $\Phi(\|\cdot\|)$ at x ([1,6]).

A Banach space E is said to satisfy *Opial's condition* if, for any sequence $\{x_n\}$ in E $x_n \rightharpoonup x$ implies that $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$. It is well-known that, if E admits a weakly continuous duality mapping J_φ with gauge function φ, then E satisfies Opial's condition ([9]).

The following Lemma is a variant of Lemma 2.1 of Jung [11].

Lemma 2.1 ([11]). *Let E be a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ. Let $\{x_n\}$ be a bounded sequence of E and let $f : E \to E$ be a continuous mapping. Let $\phi : E \to \mathbb{R}$ be defined by*

$$\phi(z) = \limsup_{n\to\infty} \langle f(z), J_\varphi(z - x_n)\rangle$$

for $z \in E$. Then ϕ is a real valued continuous function on E.

We need the following well-known lemmas for the proof of our main results.

Lemma 2.2 ([1,6]). *Let E be a real Banach space and let φ be a continuous strictly increasing function on $\mathbb{R}^+$ such that $\varphi(0) = 0$ and $\lim_{r\to\infty} \varphi(r) = \infty$. Define*

$$\Phi(t) = \int_0^t \varphi(\tau)d\tau \text{ for all } t \in \mathbb{R}^+.$$

Then (i) *The following inequalities hold:*

$$\Phi(kt) \leq k\Phi(t), \quad 0 < k < 1,$$

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\varphi(x + y)\rangle \text{ for all } x, y \in E,$$

where $j_\varphi(x + y) \in J_\varphi(x + y)$.

(ii) *Assume that a sequence $\{x_n\}$ in E is weakly convergent to a point x. Then there holds the identity*

$$\limsup_{n\to\infty} \Phi(\|x_n - y\|) = \limsup_{n\to\infty} \Phi(\|x_n - x\|) + \Phi(\|y - x\|), \quad x, \ y \in E.$$

Lemma 2.3 ([8]). *Let C be a nonempty closed subset of a Banach space E, and let $T : C \to E$ be a continuous strongly pseudocontractive mappings with a pseudocontractive coefficient $k \in (0, 1)$ satisfying*

$$\lim_{\lambda\to 0^+} \frac{d((1-\lambda)x + \lambda Tx, C)}{\lambda} = 0, \quad \forall x \in C,$$

where d denotes the distance to C (equivalently, the weakly inward condition under additional assumption that C is convex). Then T has a unique fixed point.

Lemma 2.4 ([22]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\lambda_n)s_n + \lambda_n\delta_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$ *or, equivalently,* $\prod_{n=0}^{\infty}(1-\lambda_n) = 0$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=0}^{\infty} \lambda_n|\delta_n| < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 ([6]). *If E is a Banach space such that E^* is strictly convex, then E is smooth and any duality mapping is norm-to-weak* continuous.*

Lemma 2.6 ([9] Demiclosedness principle). *Let E be a reflexive Banach space with Opial's condition, let C be a nonempty closed convex subset of E, and let $T : C \to E$ be a nonexapnsive mapping. Then the mapping $I - T$ is demiclosed on C, where I is the identity mapping; that is, $x_n \rightharpoonup x$ in E and $(I-T)x_n \to y$ imply that $x \in C$ and $(I-T)x = y$.*

Lemma 2.7 ([2]). *Let C be a nonempty closed convex subset of a Banach space E. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of mappings of C into itself. Suppose that*

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}x - T_n\| : x \in C\} < \infty.$$

Then, for each $y \in C$, $\{T_n y\}$ converges strongly to some point of C. Moreover, let T be a mapping of C into itself defined by $Ty = \lim_{n\to\infty} y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Tx - T_n x\| : x \in C\} = 0$.

3. General implicit iterative algorithms

In a Banach space E having a weakly continuous duality mapping J_φ with gauge function φ, we say that an operator A is *strongly positive* if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, J_\varphi(x)\rangle \geq \overline{\gamma}\|x\|\varphi(\|x\|) \tag{3.1}$$

and

$$\|aI - bA\| = \sup_{\|x\|\leq 1} |\langle (aI - bA)x, J_\varphi(x)\rangle|, \quad a \in [0,1], \quad b \in [-1,1],$$

where I is the identity mapping. If $E := H$ is a real Hilbert space, then the inequality (3.1) reduce to (1.2)

The following result is Lemma 3.1 of [21].

Lemma 3.1 ([21]). *Let E be a Banach space having a weakly continuous duality mapping J_φ with gauge function φ such that φ is invariant on $[0,1]$, i.e., $\varphi([0,1]) \subset [0,1]$. Assume that A is a strongly positive linear bounded operator on E with a coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \varphi(1)\|A\|^{-1}$. Then $\|I - \rho A\| \leq \varphi(1)(1 - \rho\overline{\gamma})$.*

We prepare the following result.

Proposition 3.2. *Let E be a Banach space having a weakly continuous duality mapping J_φ with gauge function φ such that φ is invariant on $[0,1]$. Let $T : E \to E$ be a nonexpansive mapping, and let $h : E \to E$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient $k \in (0,1)$. Let $A : E \to E$ be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\varphi(1)\overline{\gamma}}{k}$. Then the following hold:*

(a) *There exists a unique path $t \mapsto x_t \in E$, $t \in (0, \min\{1, \|A\|^{-1}\})$, satisfying*

$$x_t = t\gamma h(x_t) + (I - tA)Tx_t.$$

(b) *If v is a fixed point of T, then for each $t \in (0, \min\{1, \|A\|^{-1}\})$*

$$\langle (A - \gamma h)x_t, J_\varphi(x_t - v)\rangle \le \langle A(I - T)x_t, J_\varphi(x_t - v)\rangle.$$

(c) *In particular, if T has a fixed point in E, then the path $\{x_t\}$ is bounded and $\|x_t - Tx_t\| \to 0$ as $t \to 0$.*

Proof. First, we observe that J_φ is single-valued and so E is smooth.

(a) For each $t \in (0, \min\{1, \|A\|^{-1}\})$, the mapping $G_t : E \to E$ defined by

$$G_t(x) := t\gamma h(x) + (I - tA)Tx, \quad x \in H$$

is continuous strongly pseudocontractive with a pseudocontractive coefficient $1 - t(\varphi(1)\overline{\gamma} - \gamma k) \in (0,1)$. Indeed, from (2.2) and Lemma 3.1, for each $x,\ y \in E$, we derive

$$\begin{aligned}
\langle G_t x - G_t y, J_\varphi(x-y)\rangle &= t\gamma\langle h(x) - h(y), J_\varphi(x-y)\rangle \\
&\quad + \langle (I - tA)(Tx - Ty), J_\varphi(x-y)\rangle \\
&= t\gamma \frac{\varphi(\|x-y\|)}{\|x-y\|}\langle h(x) - h(y), J(x-y)\rangle \\
&\quad + \frac{\varphi(\|x-y\|)}{\|x-y\|}\langle (I - tA)(Tx - Ty), J(x-y)\rangle \\
&\le t\gamma k\|x-y\|\varphi(\|x-y\|) + \|I - tA\|\|Tx - Ty\|\varphi(\|x-y\|) \\
&\le t\gamma k\|x-y\|\varphi(\|x-y\|) + \varphi(1)(1 - t\overline{\gamma})\|x-y\|\varphi(\|x-y\|) \\
&\le (1 - t(\varphi(1)\overline{\gamma} - \gamma k))\|x-y\|\varphi(\|x-y\|),
\end{aligned}$$

and so,

$$\langle G_t x - G_t y, J(x-y)\rangle \le (1 - t(\varphi(1)\overline{\gamma} - \gamma k))\|x-y\|^2.$$

Thus, by Lemma 2.3, there exists a unique fixed point $x_t \in E$ of G_t such that

$$x_t = t\gamma h(x_t) + (I - tA)Tx_t. \tag{3.2}$$

To see the continuity, let $t,\ t_0 \in (0, \min\{1, \|A\|^{-1}\})$. Then we get

$$\begin{aligned}
&\|x_t - x_{t_0}\|\varphi(\|x_t - x_0\|)\\
&= \langle t\gamma h(x_t) + (I - tA)Tx_t - (t_0\gamma h(x_{t_0}) + (I - t_0A)Tx_{t_0}), J_\varphi(x_t - x_{t_0})\rangle\\
&= \langle (t - t_0)\gamma h(x_t) + t_0\gamma(h(x_t) - h(x_{t_0})) - (t - t_0)ATx_t, J_\varphi(x_t - x_{t_0})\rangle\\
&\quad + \langle (I - t_0A)(Tx_t - Tx_{t_0}), J_\varphi(x_t - x_{t_0})\rangle\\
&\le (\gamma\|h(x_t)\| + \|ATx_t\|)(t - t_0)\varphi(\|x_t - x_{t_0}\|) + t_0\gamma k\|x_t - x_{t_0}\|\varphi(\|x_t - x_0\|)\\
&\quad + \varphi(1)(1 - t_0\overline{\gamma})\|x_t - x_{t_0}\|\|\varphi(\|x_t - x_0\|)\\
&\le (\gamma\|h(x_t)\| + \|ATx_t\|)(t - t_0)\varphi(\|x_t - x_{t_0}\|) + t_0\gamma k\|x_t - x_{t_0}\|\varphi(\|x_t - x_0\|)\\
&\quad + (1 - t_0\varphi(1)\overline{\gamma})\|x_t - x_{t_0}\|\|\varphi(\|x_t - x_0\|).
\end{aligned}$$

It follows that

$$\|x_t - x_{t_0}\| \le \frac{\gamma\|h(x_t)\| + \|ATx_t\|}{t_0(\varphi(1)\overline{\gamma} - \gamma k)}|t - t_0|.$$

This shows that x_t is locally Lipschitzian and hence continuous.

(b) Suppose that v is a fixed point of T. Since T is nonexpansive, we have for all $x,\ y \in E$

$$\begin{aligned}
\langle (I - T)x - (I - T)y, J_\varphi(x - y)\rangle &= \|x - y\|\varphi(\|x - y\|) - \langle Tx - Ty, J_\varphi(x - y)\rangle\\
&\ge \|x - y\|\varphi(\|x - y\|) - \|x - y\|\varphi(\|x - y\|) = 0.
\end{aligned}$$

Thus, from (3.2) we obtain

$$\begin{aligned}
\langle (A - \gamma h)x_t, J_\varphi(x_t - v)\rangle &= -\frac{1}{t}\langle (I - tA)(I - T)x_t, J_\varphi(x_t - v)\rangle\\
&= -\frac{1}{t}\langle (I - T)x_t - (I - T)v, J_\varphi(x_t - v)\rangle\\
&\quad + \langle A(I - T)x_t, J_\varphi(x_t - v)\rangle\\
&\le \langle A(I - T)x_t, J_\varphi(x_t - v)\rangle.
\end{aligned}$$

(c) Let $v \in Fix(T)$. From strong pseudocontractivity of h, it follows that

$$\langle h(x_t) - h(v), J_\varphi(x_t - v)\rangle \le k\|x_t - v\|\varphi(\|x_t - v\|).$$

Thus we have

$$\begin{aligned}
&\|x_t - v\|\varphi(\|x_t - v\|)\\
&= \langle (I - tA)(Tx_t - v) + t(\gamma h(x_t) - Av), J_\varphi(x_t - v)\rangle\\
&\le \varphi(1)(1 - t\overline{\gamma})\|x_t - v\|\varphi(\|x_t - v\|) + t\langle \gamma h(x_t) - Av, J_\varphi(x_t - v)\rangle\\
&\le (1 - t\varphi(1)\overline{\gamma})\|x_t - v\|\varphi(\|x_t - v\|) + t\gamma\langle h(x_t) - h(v), J_\varphi(x_t - v)\rangle\\
&\quad + \langle \gamma h(v) - Av, J_\varphi(x_t - v)\rangle\\
&\le (1 - t\varphi(1)\overline{\gamma})\|x_t - v\|\varphi(\|x_t - v\|) + t\gamma k\|x_t - v\|\varphi(\|x_t - v\|)\\
&\quad + t\|\gamma h(v) - Av\|\varphi(\|x_t - v\|).
\end{aligned}$$

It follows that

$$\|x_t - v\| \le \frac{\|\gamma h(v) - Av\|}{\varphi(1)\overline{\gamma} - \gamma k}.$$

Hence $\{x_t\}$ is bounded for $t \in (0, \min\{1, \|A\|^{-1}\})$. Since $\|Tx_t - v\| \le \|x_t - v\|$, $\{Tx_t\}$ is bounded and so are $\{ATx_t\}$ and $\{Ax_t\}$. Moreover, since h is a bounded mapping, $\{h(x_t)\}$ is bounded. This implies that

$$\|x_t - Tx_t\| = t\|\gamma h(x_t) - ATx_t\| \to 0 \quad \text{as } t \to 0.$$

□

We prove the following result for the existence of a solution of a certain variational inequality related to A.

Theorem 3.3. *Let E be a a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ such that φ is invariant on $[0,1]$. Let $T : E \to E$ be a nonexpansive mapping with $Fix(T) \ne \emptyset$, and let $h : E \to E$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient $k \in (0,1)$. Let $A : E \to E$ be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\overline{\gamma}\varphi(1)}{k}$. Let $\{x_t\}$ be defined by*

$$x_t = t\gamma h(x_t) + (I - tA)Tx_t. \tag{3.3}$$

Then, as $t \to 0$, $\{x_t\}$ converges strongly to a fixed point p of T, where p is the unique solution in $Fix(T)$ to the variational inequality

$$\langle (A - \gamma h)p, J_\varphi(p - q)\rangle \le 0, \quad \forall q \in Fix(T). \tag{3.4}$$

Proof. First, we notice that the definition of weak continuity of duality mapping J_φ implies that E is smooth. Since E is reflexive, E^* is strictly convex. By Lemma 2.5, J_φ is norm-to-weak* continuous.

Also, we note that by Proposition 3.2 (c), $\{x_t\}$, $\{h(x_t)\}$, $\{Tx_t\}$, $\{Ax_t\}$ and $\{ATx_t\}$ are bounded for $t \in (0, \min\{1, \|A\|^{-1}\})$. As a consequence, we have

$$\|x_t - Tx_t\| = t\|\gamma h(x_t) - ATx_t\| \to 0 \quad \text{as } t \to 0. \tag{3.5}$$

Now, let $t_m \in (0, \min\{1, \|A\|^{-1}\})$ be such that $t_m \to 0$ and let $\{x_m\} := \{x_{t_m}\}$ be a subsequence of $\{x_t\}$. It follows from (3.3) that

$$x_m = t_m\gamma h(x_m) + (I - t_mA)Tx_m.$$

Let $p \in Fix(T)$. Then we deduce

$$x_m - p = (I - t_mA)(Tx_m - Tp) + t_m(\gamma h(x_m) - Ap)$$

and

$$\begin{aligned}
\|x_m - p\|\varphi(\|x_m - p\|) &= \langle x_m - p, J_\varphi(x_m - p)\rangle \\
&\le t_m\langle \gamma h(x_m) - Ap, J_\varphi(x_m - p)\rangle \\
&\quad + \langle (I - t_mA)(Tx_m - Tp), J_\varphi(x_m - p)\rangle \\
&\le t_m\langle \gamma h(x_m) - Ap, J_\varphi(x_m - p)\rangle \\
&\quad + \varphi(1)(1 - t_m\overline{\gamma})\|x_m - p\|\varphi(\|x_m - p\|) \\
&\le t_m\langle \gamma h(x_m) - Ap, J_\varphi(x_m - p)\rangle \\
&\quad + (1 - t_m\varphi(1)\overline{\gamma})\|x_m - p\|\varphi(\|x_m - p\|).
\end{aligned}$$

Thus it follows that

$$\varphi(1)\overline{\gamma}\|x_m - p\|\varphi(\|x_m - p\|) \le \langle \gamma h(x_m) - Ap, J_\varphi(x_m - p)\rangle. \tag{3.6}$$

On the other hand, since $\{x_m\}$ is bounded and E is reflexive, $\{x_m\}$ has a weakly convergent subsequence $\{x_{m_k}\}$, say $x_{m_k} \rightharpoonup q \in E$. From (3.5), it follows that

$$\|x_m - Tx_m\| = t_m\|\gamma h(x_m) - ATx_m\| \to 0.$$

Thus, by Lemma 2.6, $q \in Fix(T)$. Therefore, by (3.6) and the assumption that J_φ is weakly continuous, we get

$$\|x_m - q\|\varphi(\|x_m - q\|) \le \frac{1}{\varphi(1)\overline{\gamma}}\langle \gamma h(x_m) - Aq, J_\varphi(x_m - q)\rangle \to 0.$$

Since φ is continuous and strictly increasing, we must have $x_{m_k} \to q$.

Now, we will show that every weakly convergent subsequence of $\{x_m\}$ has the same limit. Suppose that $x_{m_k} \rightharpoonup q$ and $x_{m_j} \rightharpoonup p$. Then, by the above argument, we have $q,\ p \in F(T)$, and $x_{m_k} \to q$ and $x_{m_j} \to p$. From (3.6), we derive

$$\|x_{m_k} - p\|\varphi(\|x_{m_k} - p\|) \le \frac{1}{\varphi(1)\overline{\gamma}}\langle \gamma h(x_{m_k} - Ap, J_\varphi(x_{m_k} - p)\rangle$$

and

$$\|x_{m_j} - q\|\varphi(\|x_{m_j} - q\|) \le \frac{1}{\varphi(1)\overline{\gamma}}\langle \gamma h(x_{m_j} - Aq, J_\varphi(x_{m_j} - q)\rangle.$$

Taking limits, we obtain

$$\Phi(\|q - p\|) = \|q - p\|\varphi(\|q - p\|) \le \frac{1}{\varphi(1)\overline{\gamma}}\langle \gamma h(q) - Ap, J_\varphi(q - p)\rangle \tag{3.7}$$

and

$$\Phi(\|p - q\|) = \|p - q\|\varphi(\|p - q\|) \le \frac{1}{\varphi(1)\overline{\gamma}}\langle \gamma h(p) - Aq, J_\varphi(p - q)\rangle. \tag{3.8}$$

Moreover, by Proposition 3.1 (b), we have

$$\begin{aligned}\langle Aq - \gamma h(q), J_\varphi(q - p)\rangle &= \lim_{k\to\infty}\langle Ax_{m_k} - \gamma h(x_{m_k}), J_\varphi(x_{m_k} - p)\rangle \\ &\le \lim_{k\to\infty}\langle A(I - T)x_{m_k}, J_\varphi(x_{m_k} - p)\rangle = 0.\end{aligned} \tag{3.9}$$

and

$$\langle Ap - \gamma h(p), J_\varphi(p - q) \le 0. \tag{3.10}$$

Adding up (3.7) and (3.8) yields

$$\begin{aligned}
&2\Phi(\|p-q\|) = 2\|p-q\|\varphi(\|p-q\|) \\
&= \frac{1}{\varphi(1)\overline{\gamma}}[\langle \gamma h(q) - Ap, J_\varphi(q-p)\rangle + \langle \gamma h(p) - Aq, J_\varphi(p-q)\rangle] \\
&= \frac{1}{\varphi(1)\overline{\gamma}}[\langle \gamma h(q) - \gamma h(p), J_\varphi(q-p)\rangle + \langle \gamma h(p) - Ap, J_\varphi(q-p)\rangle \\
&\quad + \langle \gamma h(p) - \gamma h(q), J_\varphi(p-q)\rangle + \langle \gamma h(q) - Aq, J_\varphi(p-q)\rangle] \\
&= \frac{1}{\varphi(1)\overline{\gamma}}[2\langle \gamma h(p) - \gamma h(q), J_\gamma(p-q)\rangle + \langle Ap - \gamma h(p), J_\varphi(p-q)\rangle \\
&\qquad + \langle Aq - \gamma h(q), J_\varphi(q-p)\rangle].
\end{aligned}$$

From (3.9) and (3.10), we obtain

$$\begin{aligned}
2\Phi(\|p-q\|) &\le \frac{2\gamma}{\varphi(1)\overline{\gamma}}\langle h(p) - h(q), J_\varphi(p-q)\rangle \\
&\le \frac{2k\gamma}{\varphi(1)\overline{\gamma}}\|p-q\|\varphi(\|p-q\|) = \frac{2k\gamma}{\varphi(1)\overline{\gamma}}\Phi(\|p-q\|).
\end{aligned}$$

That is,

$$(\varphi(1)\overline{\gamma} - k\gamma)\Phi(\|p-q\|) \le 0.$$

Since $\varphi(1)\overline{\gamma} - k\gamma > 0$, this implies that $\Phi(\|p-q\|) \le 0$, that is, $p = q$. Hence $\{x_m\}$ is strongly convergent to a point in $Fix(T)$ as $t_m \to 0$.

The same argument shows that if $t_l \to 0$, then the other subsequence $\{x_l\} := \{x_{t_l}\}$ of $\{x_t\}$ is strongly convergent to the same limit. Thus, as $t \to 0$, $\{x_t\}$ converges strongly to a point in $Fix(T)$. Denote $p := \lim_{t\to 0} x_t$. By Proposition 3.1 (b), we have for $q \in Fix(T)$

$$\langle (A - \gamma h)x_t, J_\varphi(x_t - q)\rangle \le \langle A(I-T)x_t, J_\varphi(x_t - q)\rangle. \tag{3.11}$$

Since $(I-T)x_t \to 0$ by Proposition 3.1 (c), noting that J_φ is norm-to-weak* and taking the limit as $t \to 0$ in (3.11), we obtain

$$\langle (A - \gamma h)p, J_\varphi(p-q)\rangle \le 0, \quad \forall q \in Fix(T). \tag{3.12}$$

The above same argument may be used to conclude that p is the unique solution of the variational inequality (3.12). This completes the proof. □

By using Lemma 2.7 and Theorem 3.3, we have the following main result for a countable family of nonexpansive mappings.

Theorem 3.4. *Let E be a a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ such that φ is invariant on $[0,1]$. Let $\{T_i\}_{i=1}^\infty$ be a countable family of nonexpansive mappings from E into itself such that $\bigcap_{i=1}^\infty Fix(T_i) \neq \emptyset$. Let $h : E \to E$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient $k \in (0,1)$. Let $A : E \to E$ be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$. Assume that*

$0 < \gamma < \frac{\varphi(1)\overline{\gamma}}{k} 0 < \varphi(1)\overline{\gamma} - \gamma k < 1$. *For arbitrary initial value* $x_0 \in E$, *let* $\{x_n\}$ *be a sequence generated by the following iterative algorithm:*

$$x_n = \alpha_n \gamma h(x_n) + \beta_n x_{n-1} + ((1-\beta_n)I - \alpha_n A)T_n x_n, \quad n \geq 1, \tag{3.13}$$

where $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are two sequences in* $(0,1)$ *satisfying the conditions:*

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty} \beta_n = 0$*;*
(C2) $\sum_{n=1}^{\infty} \frac{\alpha_n}{\alpha_n+\beta_n} = \infty$.

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|T_{n+1}x - T_n x\| < \infty$ *for any bounded subset* D *of* E*. Let* T *be a mapping from* E *into itself defined by* $Tx = \lim_{n\to\infty} T_n x$ *for all* $x \in E$ *and suppose that* $F = Fix(T) = \bigcap_{i=1}^{\infty} Fix(T_i)$*. Then* $\{x_n\}$ *converges strongly to a point* p *in* $\bigcap_{i=1}^{\infty} Fix(T_i)$*, which is the unique solution in* $\bigcap_{i=1}^{\infty} Fix(T_i)$ *of the variational inequality*

$$\langle (A-\gamma h)p, J_\varphi(p-q)\rangle \leq 0, \quad \forall q \in \bigcap_{i=1}^{\infty} Fix(T_i). \tag{3.14}$$

Proof. Let $p \in F = Fix(T) = \bigcap_{i=1}^{\infty} Fix(T_i)$ be the unique solution of the variational inequality (3.14) (The existence of p follows from Theorem 3.3). In fact. $p := \lim_{t\to 0} x_t$ with $x_t \in E$ being defined by $x_t = t\gamma h(x_t) + (I - tA)Tx_t$.

From now, by condition (C1), we may assume, without loss of generality, that $\alpha_n \leq (1-\beta_n)\varphi(1)\|A\|^{-1}$ and $\beta_n + \alpha_n(\varphi(1)\overline{\gamma} - \gamma k) < 1$. Since A is a strongly positive linear operator on E, we have

$$\|A\| = \sup\{|\langle Au, J_\varphi(u)\rangle| : u \in E,\ \|u\| = 1\}.$$

So, it follows that for $u \in E$ and $\|u\| = 1$,

$$\begin{aligned}
\langle ((1-\beta_n)I - \alpha_n A)u, J_\varphi(u)\rangle &= (1-\beta_n)\langle u, J_\varphi(u)\rangle - \alpha_n\langle Au, J_\varphi(u)\rangle \\
&= (1-\beta_n)\|u\|\varphi(\|u\|) - \alpha_n\langle Au, J_\varphi(u)\rangle \\
&= (1-\beta_n)\varphi(1) - \alpha_n\langle Au, J_\varphi(u)\rangle \\
&\geq (1-\beta_n)\varphi(1) - \alpha_n\|A\| \\
&\geq 0,
\end{aligned}$$

that is, $(1-\beta_n)I - A$ is positive, and

$$\begin{aligned}
\|(1-\beta_n)I - \alpha_n A\| =&\ \sup\{\langle ((1-\beta_n)I - \alpha_n A)u, J_\varphi(u)\rangle : u \in E, \|u\| = 1\} \\
=&\ \sup\{\langle (1-\beta_n)\varphi(1) - \alpha_n\langle Au, J_\varphi(u)\rangle : u \in E, \|u\| = 1\} \\
\leq&\ \varphi(1)(1-\beta_n - \alpha_n\overline{\gamma}) \\
\leq&\ 1 - \beta_n - \alpha_n\varphi(1)\overline{\gamma}).
\end{aligned}$$

Next, we show that $\{x_n\}$ is well defined. For each $n \geq 1$, define a mapping $S : E \to E$ by

$$Sx = \alpha_n\gamma h(x) + \beta_n x_{n-1} + ((1-\beta_n)I - \alpha_n A)T_n x, \quad \forall x \in E.$$

Then, for every $x, y \in E$, we have

$$\begin{aligned}\langle Sx - Sy, J_\varphi(x-y)\rangle &= \alpha_n\gamma\langle h(x) - h(y), J_\varphi(x-y)\rangle \\ &\quad + \langle((1-\beta_n)I - \alpha_n A)(T_n x - T_n y), J_\varphi(x-y)\rangle \\ &\leq \alpha_n\gamma k\|x-y\|\varphi(\|x-y\|) \\ &\quad + (1-\beta_n - \alpha_n\varphi(1)\overline{\gamma})\|x-y\|\varphi(\|x-y\|) \\ &= (1-\beta_n - \alpha_n(\varphi(1)\overline{\gamma} - \gamma k))\|x-y\|\varphi(\|x-y\|).\end{aligned}$$

Therefore, S is a continuous strong pseudocontractive mapping with a pseudocontractive coefficient $0 < 1 - \beta_n - \alpha_n(\varphi(1)\overline{\gamma} - \gamma k) < 1$ for each $n \geq 1$. By Lemma 2.3, we see that there exists a unique fixed point x_n for each $n \geq 1$ such that

$$x_n = \alpha_n\gamma h(x_n) + \beta_n x_{n-1} + ((1-\beta_n)I - \alpha_n A)T_n x_n.$$

That is, the sequence $\{x_n\}$ is well defined.

Now, we divide the proof into several steps as follows.

Step 1. We show that $\{x_n\}$ is bounded. To this end, let $q \in \bigcap_{i=1}^\infty Fix(T_i)$. Then, noting that

$$x_n - q = \alpha_n(\gamma h(x_n) - Aq) + \beta_n(x_{n-1} - q) + ((1-\beta_n)I - \alpha_n A)(T_n x_n - q),$$

and

$$\langle h(x_n) - h(q), J_\varphi(x_n - q)\rangle \leq k\|x_n - q\|\varphi(\|x_n - q\|),$$

we induce

$$\begin{aligned}\|x_n - q\|\varphi(\|x_n - q\|) &= \langle x_n - q, J_\varphi(x_n - q)\rangle \\ &= \alpha_n\langle\gamma h(x_n) - Aq, J_\varphi(x_n - q)\rangle \\ &\quad + \beta_n\langle x_{n-1} - q, J_\varphi(x_n - q)\rangle \\ &\quad + \langle((1-\beta_n)I - \alpha_n A)(Tx_n - q), J_\varphi(x_n - q)\rangle \\ &\leq \alpha_n\gamma\langle h(x_n) - h(q), J_\varphi(x_n - q)\rangle \\ &\quad + \alpha_n\langle\gamma h(q) - Aq, J_\varphi(x_n - q)\rangle \\ &\quad + \beta_n\|x_{n-1} - q\|\varphi(\|x_n - q\|) \\ &\quad + \|(1-\beta_n)I - \alpha_n A\|\|Tx_n - q\|\varphi(\|x_n - q\|) \\ &\leq \alpha_n\gamma k\|x_n - q\|\varphi(\|x_n - q\|) \\ &\quad + (1-\beta_n - \alpha_n\varphi(1)\overline{\gamma})\|x_n - q\|\varphi(\|x_n - q\|) \\ &\quad + \alpha_n\|\gamma h(q) - Aq\|\varphi(\|x_n - q\|) \\ &\quad + \beta_n\|x_{n-1} - q\|\varphi(\|x_n - q\|) \\ &= (1-\beta_n - \alpha_n(\varphi(1)\overline{\gamma} - \gamma k))\|x_n - q\|\varphi(\|x_n - q\|) \\ &\quad + \alpha_n\|\gamma h(q) - Aq\|\varphi(\|x_n - q\|) \\ &\quad + \beta_n\|x_{n-1} - q\|\varphi(\|x_n - q\|),\end{aligned}$$

which implies

$$\|x_n - q\| \leq \frac{\beta_n}{\beta_n + \alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}\|x_{n-1} - q\| + \frac{\alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}{\beta_n + \alpha_n(\varphi(1)\overline{\gamma} - \gamma k)} \cdot \frac{\|\gamma h(q) - Aq\|}{(\varphi(1)\overline{\gamma} - \gamma k)}.$$

By induction, we have

$$\|x_n - q\| \leq \max\left\{\|x_0 - q\|, \frac{\|\gamma h(q) - Aq\|}{\varphi(1)\overline{\gamma} - \gamma k}\right\} \text{ for } n \geq 1.$$

Hence $\{x_n\}$ is bounded. Moreover, since h is a bounded mapping, $\{h(x_n)\}$ is bounded. Also, since $\|Tx_n - q\| \leq \|x_n - q\|$, $\{T_n x_n\}$ and $\{AT_n x_n\}$ are bounded.

Step 2. We show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. First, since

$$\begin{aligned}\|x_n - T_n x_n\| &= \|\alpha_n(\gamma h(x_n) - AT_n x_n) + \beta_n(x_{n-1} - T_n x_n)\| \\ &\leq \alpha_n\|\gamma h(x_n) - AT_n x_n\| + \beta_n\|x_{n-1} - T_n x_n\|.\end{aligned}$$

it follows from the condition (C1) and boundedness of $\{x_n\}$, $\{h(x_n)\}$, $\{T_n x_n\}$ and $\{AT_n x_n\}$ that

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.15}$$

Now, since

$$\begin{aligned}\|x_n - Tx_n\| &\leq \|x_n - T_n x_n\| + \|T_n x_n - Tx_n\| \\ &\leq \sup\{\|Tz - T_n z\| : z \in \{x_n\}\} + \|x_n - T_n x_n\|,\end{aligned}$$

by assumption on T and (3.15), we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Step 3. We show that $\limsup_{n\to\infty}\langle(\gamma h - A)p, J_\varphi(x_n - p)\rangle \leq 0$. To show this, we first note that

$$\begin{aligned}x_t - x_n &= t\gamma h(x_t) + Tx_t - tATx_t - x_n \\ &= t(\gamma h(x_t) - Ax_t) + (Tx_t - Tx_n) + (Tx_n - x_n) + tA(Tx_t - x_t) \\ &= t(\gamma h(x_t) - Ax_t) + (Tx_t - Tx_n) + (Tx_n - x_n) + t^2 A(ATx_t - \gamma h(x_t)).\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_t - x_n\|\varphi(\|x_t - x_n\|) \\ &= t\langle\gamma h(x_t) - Ax_t, J_\varphi(x_t - x_n)\rangle + \langle Tx_t - Tx_n, J_\varphi(x_t - x_n)\rangle \\ &\quad + \langle Tx_n - x_n, J_\varphi(x_t - x_n)\rangle + t^2\langle A(ATx_t - \gamma h(x_t)), J_\varphi(x_t - x_n)\rangle \\ &\leq t\langle\gamma h(x_t) - Ax_t, J_\varphi(x_t - x_n)\rangle + \|x_t - x_n\|\varphi(\|x_t - x_n\|) \\ &\quad + \|Tx_n - x_n\|\varphi(\|x_t - x_n\|) \\ &\quad + t^2\|A(ATx_t - \gamma h(x_t))\|\varphi(\|x_t - x_n\|),\end{aligned}$$

which implies that

$$\begin{aligned}\langle\gamma h(x_t) - Ax_t, J_\varphi(x_n - x_t)\rangle \leq \frac{1}{t}&\|Tx_n - x_n\|\varphi(\|x_t - x_n\|) \\ &+ t\|A\|\|ATx_t - \gamma h(x_t)\|\varphi(\|x_t - x_n\|).\end{aligned}$$

Hence, by $\limsup_{n\to\infty}\varphi(\|x_t - x_n\|) < \infty$ and Step 2, we have

$$\limsup_{n\to\infty}\langle\gamma h(x_t) - Ax_t, J_\varphi(x_n - x_t)\rangle \leq tM,$$

where $M > 0$ is a constant such that $\|A\|\|ATx_t - \gamma h(x_t)\|\varphi(\|x_t - x_n\|) \le M$ for all $n \ge 1$ and $t \in (0, \min\{1, \|A\|^{-1}\})$. Thus, by Lemma 2.1, we conclude

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma h(p) - Ap, J_\varphi(x_n - p)\rangle &= \lim_{t\to 0}\limsup_{n\to\infty}\langle \gamma h(x_t) - Ax_t, J_\varphi(x_n - x_t)\rangle \\ &\le \lim_{t\to 0} tM = 0.\end{aligned}$$

Step 4. We show that $\lim_{n\to\infty}\|x_n - p\| = 0$, where $p \in F$ is the unique solution of the variational inequality (3.14). Indeed, using the equality

$$\begin{aligned}x_n - p &= \alpha_n(\gamma h(x_n) - Ap) + \beta_n(x_{n-1} - p) + ((1-\beta_n)I - \alpha_n A)(T_n x_n - p) \\ &= \alpha_n(\gamma h(x_n) - \gamma h(p)) + \alpha_n(\gamma h(p) - Ap) \\ &\quad + \beta_n(x_{n-1} - p) + ((1-\beta_n)I - \alpha_n A)(T_n x_n - p)\end{aligned}$$

and the inequality $\langle h(x_n) - h(p), J_\varphi(x_n - p)\rangle \le k\|x_n - p\|\varphi(\|x_n - p\|)$, we have

$$\begin{aligned}\Phi(\|x_n - p\|) &\le \Phi(\|\beta_n(x_{n-1} - p)\|) + \alpha_n\gamma\langle h(x_n) - h(p), J_\varphi(x_n - p)\rangle \\ &\quad + \alpha_n\langle \gamma h(p) - Ap, J_\varphi(x_n - p)\rangle \\ &\quad + \langle ((1-\beta_n)I - \alpha_n A)(Tx_n - p), J_\varphi(x_n - p)\rangle \\ &\le \beta_n\Phi(\|x_{n-1} - p\|) + \alpha_n\gamma k\|x_n - p\|\varphi(\|x_n - p\|) \\ &\quad + \alpha_n\langle \gamma h(p) - Ap, J_\varphi(x_n - p)\rangle \\ &\quad + (1 - \beta_n - \alpha_n\varphi(1)\overline{\gamma})\|x_n - p\|\varphi(\|x_n - p\|) \\ &= \beta_n\Phi(\|x_n - p\|) + (1 - \beta_n - \alpha_n(\varphi(1)\overline{\gamma} - \gamma k))\Phi(\|x_n - p\|) \\ &\quad + \alpha_n\langle \gamma h(p) - Ap, J_\varphi(x_n - p)\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\Phi(\|x_n - p\|) &\le \frac{\beta_n}{\beta_n + \alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}\Phi(\|x_{n-1} - p\|) \\ &\quad + \frac{\alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}{\beta_n + \alpha_n(\varphi(1)\overline{\gamma} - \gamma k)} \cdot \frac{\langle \gamma h(p) - Ap, J_\varphi(x_n - p)\rangle}{\varphi(1)\overline{\gamma} - \gamma k)} \\ &= \left(1 - \frac{\alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}{\beta_n + \alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}\right)\Phi(\|x_{n-1} - p\|) \\ &\quad + \frac{\alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}{\beta_n + \alpha_n(\varphi(1)\overline{\gamma} - \gamma k)} \cdot \frac{\langle \gamma h(p) - Ap, J_\varphi(x_n - p)\rangle}{\varphi(1)\overline{\gamma} - \gamma k)} \\ &= (1 - \lambda_n)\Phi(\|x_{n-1} - p\|) + \lambda_n\delta_n,\end{aligned} \tag{3.16}$$

where $\lambda_n = \frac{\alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}{\beta_n + \alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}$ and $\delta_n = \frac{1}{\varphi(1)\overline{\gamma} - \gamma k}\langle \gamma h(p) - Ap, J_\varphi(x_n - p)\rangle$. We note that

$$0 \le \frac{\alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}{\beta_n + \alpha_n(\varphi(1)\overline{\gamma} - \gamma k)} \le 1 \text{ and } \frac{(\varphi(1)\overline{\gamma} - \gamma k)\alpha_n}{\alpha_n + \beta_n} < \frac{\alpha_n(\varphi(1)\overline{\gamma} - \gamma k)}{\alpha_n(\varphi(1)\overline{\gamma} - \gamma k) + \beta_n}.$$

From the condition (C2) and Step 3, it is easily seen that $\sum_{n=1}^\infty \lambda_n = \infty$ and $\limsup_{n\to\infty}\delta_n \le 0$. Thus, applying Lemma 2.4 to (3.16), we conclude that

$$\lim_{n\to\infty} x_n = p.$$

This completes the proof. □

As an immediate result of Theorem 3.4, we have the following result.

Corollary 3.5. *Let E be a a reflexive Banach space having a weakly continuous duality mapping J_φ with gauge function φ such that φ is invariant on $[0,1]$. Let $T : E \to E$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$, and let $h : E \to E$ be a continuous bounded strongly pseudocontractive mapping with a pseudocontractive coefficient $k \in (0,1)$. Let $A : E \to E$ be a strongly positive linear bounded operator with a coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \frac{\varphi(1)\overline{\gamma}}{k}$ and $\varphi(1)\overline{\gamma} - \gamma k < 1$. For arbitrary initial value $x_0 \in E$, let $\{x_n\}$ be a sequence generated by the following iterative method:*

$$x_n = \alpha_n \gamma h(x_n) + \beta_n x_{n-1} + ((1-\beta_n)I - \alpha_n A)Tx_n, \quad n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$ satisfying the conditions (C1) *and* (C2) *in Theorem 3.4. Then $\{x_n\}$ converges strongly to a fixed point p of T, which is the unique solution in $Fix(T)$ of the variational inequality* (3.4).

Remark 3.6.
1) Theorem 3.3 extends and improves the corresponding results of Marino and Xu [13] and Wangkeeree *et al.* [21] in the following aspects:
   (a) The contractive mapping f in [13, 21] is replaced by a continuous bounded strongly pseudocontractive mapping h.
   (b) The Hilbert space H in [13] is extended to a reflexive Banach space E having a weakly continuous duality mapping J_φ with gauge function φ.
   (c) One nonexpansive mapping in [13] is replaced by a countable family of nonexpansive mappings.
2) Theorem 3.3 also says that Theorem 3.2 of Jung [12] in case of closed subspace $C = E$ holds in a reflexive Banach space E having a weakly continuous duality mapping J_φ with gauge function φ such that φ is invariant on $[0,1]$.
3) It is worth pointing out that the general implicit iterative method in Theorem 3.4 is a new ones for finding a common fixed point of a countable family of nonexpansive mappings in a reflexive Banach space E having a weakly continuous duality mapping J_φ with gauge function φ.

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