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STRONG CONVERGENCE THEOREMS BY A MODIFIED FORWARD-BACKWARD-FORWARD SPLITTING METHOD IN BANACH SPACES

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Abstract. In this paper, we consider strong convergence for a sum of a maximal monotone operator and a monotone and Lipschitz continuous mapping in a real Banach space. And we propose a modified forward-backward-forward splitting method and prove a new strong convergence theorem in a 2-uniformly convex and uniformly smooth Banach space. Further, we get new results for variational inequality problems, too.

1. Introduction

Throughout this paper, we denote by $\mathbb{N}$ the set of all positive integers. Let E be a real Banach space with norm $\|\cdot\|$ and dual space E^* and for $x \in E$ and $x^* \in E^*$, $\langle x, x^* \rangle$ be the value of x^* at x. Let $A \subset E \times E^*$ and $B \subset E \times E^*$ be maximal monotone operators such that $A + B$ is a maximal monotone operator with $(A+B)^{-1}0 \neq \emptyset$. Finding an element of $(A+B)^{-1}0$ contains many important problems such as convex minimization problems, variational inequality problems, complementary problems, and others.

In a real Hilbert space H, Lions and Mercier [16] and Passty [25] proposed the following forward-backward (F-B for short) splitting method as one of the methods of finding an element of $(A+B)^{-1}0$:

$$x_1 = x \in D(B), \quad x_{n+1} = J^A_{\lambda_n}(x_n - \lambda_n w_n) \tag{1.1}$$

for every $n \in \mathbb{N}$, where $D(B) \subset H$ is the domain of B, $D(A) \subset D(B)$, $w_n \in Bx_n$, $\{\lambda_n\} \subset (0, \infty)$ and $J^A_{\lambda_n}$ is the resolvent of A. Later, the splitting method was widely studied by Gabay [11], Chen and Rockafellar [8], Moudafi and Théra [21] and Tseng [33] and others.

Let $\alpha > 0$ and B be a single valued mapping of H into itself. B is called α-inverse-strongly-monotone if $(x - y, Bx - By) \geq \alpha \|Bx - By\|^2$ for all $x, y \in H$; see [5, 10, 17, 38]. When $\alpha = 1$, B is said to be firmly nonexpansive. Gabay [11] and many researchers [4, 9, 22, 23, 24, 26, 30, 36] studied weak and strong convergence for a F-B splitting method and modified F-B splitting methods by a maximal monotone operator A and an inverse-strongly-monotone mapping B in a real Hilbert space. In this way, F-B splitting method is considered as an algorithm of convergence to an element of $(A+B)^{-1}0$ for an inverse-strongly-monotone mapping B. And recently, Kimura and author [14] proved strong convergence for a

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modified F-B splitting method by the same A and B in a 2-uniformly convex and uniformly smooth Banach space.

On the other hand, Tseng [33] considered a monotone and Lipschitz continuous mapping which is more general than an inverse-strongly-monotone mapping and proposed the following forward-backward-forward (F-B-F for short) splitting method by a maximal monotone operator $A \subset H \times H$ and a single valued monotone operator $B : H \longrightarrow H$:

$$\begin{cases} y_1 = x \in C, \\ z_n = J^A_{\lambda_n}(y_n - \lambda_n B y_n), \\ y_{n+1} = P_C(z_n - \lambda_n(Bz_n - By_n)), \end{cases} \tag{1.2}$$

for all $n \in \mathbb{N}$, where C is nonempty closed convex subset of H, P_C is the metric projection of H onto C, $A+B$ is maximal monotone and $F = C \cap (A+B)^{-1}0 \neq \emptyset$. When B is Lipschitz continuous on $C \cup D(A)$, he proved $\{y_n\}$ generated by (1.2) converges weakly to an element of F under some conditions.

In this paper, motivated by [14, 33], we consider strong convergence for a sum of a maximal monotone operator and a monotone and Lipschitz continuous mapping in a real 2-uniformly convex and uniformly smooth Banach space E. And we introduce a modified F-B-F splitting method by a maximal monotone operator $A \subset E \times E^*$ and a single valued monotone and Lipschitz continuous mapping B of $C \cup D(A)$ into E^* as follows:

$$\begin{cases} x_1 = x \in C \cup D(A), \\ y_n = J^A_{\lambda_n} J^{-1}(Jx_n - \lambda_n Bx_n - \alpha_n(Jx_n - Ju)), \\ Jz_n = Jy_n - \lambda_n(By_n - Bx_n) \\ x_{n+1} = \Pi_C z_n \end{cases}$$

for every $n \in \mathbb{N}$, where $u \in E$, C is a nonempty closed convex subset of E, J is the duality mapping of E, $F = C \cap (A+B)^{-1}0 \neq \emptyset$, $\mathit{\Pi}_C$ is the generalized projection of E onto C, $\{\lambda_n\} \subset (0,\infty)$ and $\{\alpha_n\} \subset (0,1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then we prove $\{x_n\}$ converges strongly to $\mathit{\Pi}_F u$ under some assumptions. From this result, we get new strong convergence for a maximal monotone operator and a monotone and Lipschitz continuous mapping and for variational inequality problems in a 2-uniformly convex and uniformly smooth Banach space and a real Hilbert space.

2. Preliminaries

We use $x_n \to x$ to indicate that a sequence $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ will symbolize weak convergence. We define the modulus of convexity δ_E of E as follows: δ_E is a function of $[0,2]$ into $[0,1]$ such that

$$\delta_E(\varepsilon) = \inf\{1 - \|x+y\|/2 : x, y \in E,\ \|x\| = 1,\ \|y\| = 1,\ \|x-y\| \geq \varepsilon\}$$

for every $\varepsilon \in [0,2]$. For $p > 1$, E is said to be p-uniformly convex if there exists a constant $c > 0$ with $\delta_E(\varepsilon) \geq c\varepsilon^p$ for every $\varepsilon \in [0,2]$ and we know that L_p space is p-uniformly convex if $p > 2$ and 2-uniformy convex if $1 < p \leq 2$, see [35]. We say E is uniformly convex if $\delta_E(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. It is obvious that a p-uniformly convex Banach space is uniformly convex. E is said to be strictly convex if $\|x + y\|/2 < 1$ for each $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is known that a uniformly convex Banach space is strictly convex and reflexive. The duality mapping $J : E \to 2^{E^*}$ of E is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$$

for every $x \in E$. We know that if E is strictly convex and reflexive, then, the duality mapping J of E is bijective and $J^{-1} : E^* \to 2^E$ is the duality mapping of E^*. E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for every $x, y \in S(E)$, where $S(E) = \{x \in E : \|x\| = 1\}$. E is said to be uniformly smooth if the limit (2.1) is attained uniformly for (x, y) in $S(E) \times S(E)$. It is known that the duality mapping J of E is single valued if and only if E is smooth and if J is single valued, J is norm to weak* continuous. We also know that if E is uniformly smooth, then the duality mapping J of E is uniformly continuous on bounded subsets of E, that is, for any bounded subset B of E and $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in B$, $\|x - y\| < \delta$ implies $\|Jx - Jy\| < \varepsilon$; see [31, 32] for more details.

Let $f : E \to (-\infty, \infty]$ be a proper and convex function. Then, the subdifferential ∂f of f is defined by

$$\partial f(x) = \{x^* \in E^* : f(y) \geq f(x) + \langle y - x, x^*\rangle,\ \forall y \in E\}$$

for every $x \in E$. It is known that if $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, $\partial f(x) = J(x)$ for every $x \in E$. And let $f : E \to (-\infty, \infty]$ be a proper function. Then, the conjugate function f^* of f and the biconjugate function f^{**} of f are defined by

$$f^*(x^*) = \sup_{x\in E}\{\langle x, x^*\rangle - f(x)\}\ (\forall x^* \in E^*)$$
$$f^{**}(x) = \sup_{x^*\in E^*}\{\langle x, x^*\rangle - f^*(x^*)\}\ (\forall x \in E),$$

respectively. We know that if $f : E \to (-\infty, \infty]$ is a proper, lower semicontinuous and convex function, $f^{**} = f$. The following was proved by Zălinescu [37]; see also [35].

Lemma 2.1. *Let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function and $\Psi : [0, \infty) \to [0, \infty]$ a proper, lower semicontinuous, nondecreasing and convex function. Then, the following are equivalent;*

(i) *$f(y) \geq f(x) + \langle y - x, x^*\rangle + \Psi(\|y - x\|)$ holds for every $(x, x^*) \in \partial f$ and $y \in E$;*

(ii) *$f^*(y^*) \leq f^*(x^*) + \langle x, y^* - x^*\rangle + \Psi^*(\|y^* - x^*\|)$ holds for all $(x, x^*) \in \partial f$ and $y^* \in E^*$.*

The following was proved by Xu [35]; see also [37].

Theorem 2.2. *Let E be a smooth Banach space. Then, E is 2-uniformly convex if and only if there exists a constant $c > 0$ such that for each $x, y \in E$, $\|x + y\|^2 \geq \|x\|^2 + 2\langle y, Jx\rangle + c\|y\|^2$ holds.*

Remark 2.3. Let E be a reflexive Banach space and $k > 0$. The conjugate function of $\frac{k}{2}\|x\|^2$ $(x \in E)$ is $\frac{1}{2k}\|x^*\|^2$ $(x^* \in E^*)$. Let $c > 0$. From Lemma 2.1, we have the following are equivalent;

(i) $\frac{1}{2}\|y\|^2 \geq \frac{1}{2}\|x\|^2 + \langle y - x, x^*\rangle + \frac{c}{2}\|y - x\|^2$ holds for every $x, y \in E$, $x^* \in Jx$;
(ii) $\frac{1}{2}\|y^*\|^2 \leq \frac{1}{2}\|x^*\|^2 + \langle x, y^* - x^*\rangle + \frac{1}{2c}\|y^* - x^*\|^2$ holds for all $x \in E$, $x^* \in Jx$ and $y^* \in E^*$.

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2$$

for every $x, y \in E$. It is obvious that $(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2$ for each $x, y \in E$ and $\phi(z, x) + \phi(y, u) = \phi(z, u) + \phi(y, x) - 2\langle y - z, Ju - Jx\rangle$ for all $x, y, z, u \in E$. We have the following result by Theorem 2.2 and Remark 2.3; see also [13].

Theorem 2.4. *Let E be a 2-uniformly convex and smooth Banach space. Then, for each $x, y \in E$, $c\|x - y\|^2 \leq \phi(x, y) \leq (1/c)\|Jx - Jy\|^2$ and $c\|x - y\|^2 \leq \langle x - y, Jx - Jy\rangle \leq (1/c)\|Jx - Jy\|^2$ hold, where c is the constant in Theorem 2.2.*

Proof. Let $x, y \in E$. By Theorem 2.2, we have

$$\|x\|^2 \geq \|y\|^2 + 2\langle x - y, Jy\rangle + c\|x - y\|^2,$$

where c is the constant in Theorem 2.2. So, we get $\phi(x, y) \geq c\|x-y\|^2$. By Theorem 2.2 and Remark 2.3, we obtain

$$(1/2)\|Jy\|^2 \leq (1/2)\|Jx\|^2 + \langle x, Jy - Jx\rangle + (1/(2c))\|Jy - Jx\|^2,$$

that is,

$$\|y\|^2 - 2\langle x, Jy\rangle + \|x\|^2 \leq (1/c)\|Jx - Jy\|^2.$$

So, we have $\phi(x, y) \leq (1/c)\|Jx-Jy\|^2$. From $\langle x-y, Jx-Jy\rangle = \frac{1}{2}(\phi(x, y)+\phi(y, x))$, we get $c\|x - y\|^2 \leq \langle x - y, Jx - Jy\rangle \leq (1/c)\|Jx - Jy\|^2$. □

Let C be a nonempty closed convex subset of a strictly convex, reflexive and smooth Banach space E and let $x \in E$. Then, there exists a unique element $x_0 \in C$ such that

$$\phi(x_0, x) = \inf_{y \in C} \phi(y, x).$$

We denote x_0 by $\Pi_C x$ and call Π_C the generalized projection of E onto C; see [1, 2, 12]. We have the following result [1, 2, 12] for the generalized projection.

Lemma 2.5. *Let C be a nonempty convex subset of a smooth Banach space E, $x \in E$ and $x_0 \in C$. Then, $\phi(x_0, x) = \inf_{y \in C} \phi(y, x)$ if and only if $\langle x_0 - z, Jx - Jx_0\rangle \geq 0$ for every $z \in C$, or equivalently, $\phi(z, x) \geq \phi(z, x_0) + \phi(x_0, x)$ for all $z \in C$.*

An operator $A \subset E \times E^*$ is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for every $(x, x^*), (y, y^*) \in A$. A monotone operator A is said to be maximal if the graph of A is not properly contained in the graph of any other monotone operator. We know that a monotone operator A is maximal if and only if for $(u, u^*) \in E \times E^*$, $\langle x - u, x^* - u^* \rangle \geq 0$ for every $(x, x^*) \in A$ implies $(u, u^*) \in A$. Let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Then, it is known that the subdifferential ∂f of f is a maximal monotone operator; see [27, 28]. Let A be a single valued mapping of a nonempty convex subset C of E into E^*. A is called hemicontinuous if the real valued function $t \mapsto \langle w, A(tv + (1-t)u) \rangle$ is continuous on $[0, 1]$ for all $u, v \in C$ and $w \in E$. We know that a monotone and hemicontinuous mapping A of a reflexive Banach space E into E^* is maximal monotone. Rockafellar [29] proved the following result; see also [7].

Theorem 2.6. *Let E be a strictly convex, reflexive and smooth Banach space and let $A \subset E \times E^*$ be a monotone operator. Then, A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$, where $R(J + rA)$ is the range of $J + rA$.*

Let E be a strictly convex, reflexive and smooth Banach space and let $A \subset E \times E^*$ be a maximal monotone operator. By Theorem 2.6 and strict convexity of E, for any $x \in E$ and $r > 0$, there exists a unique element $x_r \in D(A)$ such that

$$J(x) \in J(x_r) + rAx_r.$$

We define J_r^A by $J_r^A x = x_r$ for every $x \in E$ and $r > 0$ and such J_r^A is called the resolvent of A; see [6, 32] for more details.

A function $i : \mathbb{N} \to \mathbb{N}$ is said to be eventually increasing if $\lim_{n\to\infty} i(n) = \infty$ and $i(n) \leq i(n+1)$ for all $n \in \mathbb{N}$. The following was proved by Mainge [18, Lemma 3.1], see also [3].

Lemma 2.7. *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exist $n_0 \in \mathbb{N}$ and an eventually increasing function i such that $\Gamma_{i(n)} \leq \Gamma_{i(n)+1}$ and $\Gamma_n \leq \Gamma_{i(n)+1}$ for every $n \geq n_0$.*

3. Main result

At first, we show the following result.

Lemma 3.1. *Let C be a nonempty closed convex subset of a 2-uniformly convex and smooth Banach space E, A a maximal monotone operator in $E \times E^*$, B a monotone and Lipschitz continuous mapping of $C \cup D(A)$ into E^* with a Lipschitz constant $L > 0$ such that $F = C \cap (A + B)^{-1}0 \neq \emptyset$. For every $x \in C \cup D(A)$ and $\lambda > 0$, let $S_\lambda x = J_\lambda^A J^{-1}(Jx - \lambda Bx)$ and $T_\lambda x = \Pi_C J^{-1}(JS_\lambda x - \lambda(BS_\lambda x - Bx))$. Then, the following hold;*

(i) *$\phi(z, T_\lambda x) \leq \phi(z, x) - (c - (\lambda L)^2/c)\|S_\lambda x - x\|^2$ holds for every $x \in C \cup D(A)$, $\lambda > 0$ and $z \in F$, where c is the constant in Theorem 2.2;*

(ii) *if $(c/L) > \lambda$, $F(T_\lambda) = F$, where $F(T_\lambda)$ is the set of all fixed points of T_λ;*

(iii) *F is closed convex.*

Proof. (i) Let $z \in F$, $x \in C \cup D(A)$, $\lambda > 0$, $y = S_\lambda x$ and $u = J^{-1}(Jy - \lambda(By - Bx))$. We have

$$\phi(z, x) = \phi(z, u) - \phi(y, u) + \phi(y, x) - 2\langle y - z, Ju - Jx\rangle. \tag{3.1}$$

From $\frac{1}{\lambda}(Jx - Jy) - Bx \in Ay$ and $\frac{1}{\lambda}(Jy - Ju) + Bx = By$, $\frac{1}{\lambda}(Jx - Ju) \in (A + B)y$. Since $0 \in (A + B)z$ and $(A + B)$ is monotone, we get

$$\langle y - z, Jx - Ju\rangle \geq 0. \tag{3.2}$$

By (3.1) and (3.2),

$$\phi(z, x) \geq \phi(z, u) - \phi(y, u) + \phi(y, x). \tag{3.3}$$

From Lemma 2.5,

$$]\phi(z, T_\lambda x) \leq \phi(z, u) - \phi(T_\lambda x, u). \tag{3.4}$$

By (3.3) and (3.4) and Theorem 2.4, we get

$$\begin{aligned} \phi(z, T_\lambda x) &\leq \phi(z, x) + \phi(y, u) - \phi(y, x) - \phi(T_\lambda x, u) \\ &\leq \phi(z, x) + \phi(y, u) - \phi(y, x) \\ &\leq \phi(z, x) + (1/c)\|Jy - Ju\|^2 - c\|x - y\|^2 \\ &= \phi(z, x) + (\lambda^2/c)\|By - Bx\|^2 - c\|x - y\|^2 \\ &\leq \phi(z, x) + ((\lambda^2 L^2)/c)\|x - y\|^2 - c\|x - y\|^2 \\ &= \phi(z, x) - (c - (\lambda^2 L^2)/c)\|x - y\|^2. \end{aligned}$$

(ii) Let $z \in F$. We have $z \in C$ and $Jz - \lambda Bz \in Jz + \lambda Az$, that is, $z = J_\lambda^A J^{-1}(Jz - \lambda Bz) = S_\lambda z$. Hence, we get

$$T_\lambda z = \Pi_C J^{-1}(Jz - \lambda(Bz - Bz)) = \Pi_C z = z.$$

So, $F \subset F(T_\lambda)$. Let $z \in F$ and $u \in F(T_\lambda)$. From (i), we obtain

$$\phi(z, u) = \phi(z, T_\lambda u) \leq \phi(z, u) - (c - (\lambda^2 L^2)/c)\|u - S_\lambda u\|^2.$$

By $(c - (\lambda^2 L^2)/c) > 0$, we get

$$u = S_\lambda u = J_\lambda^A J^{-1}(Ju - \lambda Bu)$$

which implies $u \in F$. So, we get $F(T_\lambda) \subset F$. Therefore, $F = F(T_\lambda)$ holds.
(iii) If $(c/L) > \lambda$, $\phi(z, T_\lambda x) \leq \phi(z, x)$ for every $x \in C \cup D(A)$ and $z \in F = F(T_\lambda)$ from (i) and (ii). So, we have F is closed and convex from the result in [19, 20]. □

Now, we get the following strong convergence theorem.

Theorem 3.2. *Let C be a nonempty closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E and assume that A, B, L and F are the same as Lemma 3.1 such that for a bounded sequence $\{u_n\} \subset C$ and $\{\lambda_n\} \subset (0, \infty)$ with $\inf_{n\in\mathbb{N}} \lambda_n > 0$, $\|u_n - J_{\lambda_n}^A J^{-1}(Ju_n - \lambda_n Bu_n)\| \to 0$ implies $\omega_w(u_n) \subset F$, where*

$\omega_w(u_n)$ is the set of all weak cluster points of $\{u_n\}$. Let $u \in E$ and $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in C \cup D(A), \\ y_n = J_{\lambda_n}^A J^{-1}(Jx_n - \lambda_n B x_n - \alpha_n(Jx_n - Ju)), \\ Jz_n = Jy_n - \lambda_n(By_n - Bx_n) \\ x_{n+1} = \Pi_C z_n \end{cases}$$

for every $n \in \mathbb{N}$, where $0 < \inf_{n\in\mathbb{N}} \lambda_n \le \sup_{n\in\mathbb{N}} \lambda_n < c/L$ where c is the constant in Theorem 2.2 and $0 < \alpha_n \le 1$ for all $n \in \mathbb{N}$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $\Pi_F u$.

Proof. From Lemma 3.1 (iii), F is closed and convex and hence, Π_F is well defined. Let $z \in F$. We have

$$\phi(z, x_n) = \phi(z, z_n) - \phi(y_n, z_n) + \phi(y_n, x_n) - 2\langle y_n - z, Jz_n - Jx_n\rangle.$$

By $(1/\lambda_n)(Jx_n - Jy_n) - Bx_n - (\alpha_n/\lambda_n)(Jx_n - Ju) \in Ay_n$ and $(1/\lambda_n)(Jy_n - Jz_n) + Bx_n = By_n$, $(1/\lambda_n)(Jx_n - Jz_n) - (\alpha_n/\lambda_n)(Jx_n - Ju) \in (A+B)y_n$. Since $0 \in (A+B)z$ and A+B is monotone, we get

$$\langle y_n - z, Jx_n - Jz_n - \alpha_n(Jx_n - Ju)\rangle \ge 0$$

for every $n \in \mathbb{N}$. So we get,

$$\phi(z, z_n) \le \phi(z, x_n) + \phi(y_n, z_n) - \phi(y_n, x_n) - 2\alpha_n\langle y_n - z, Jx_n - Ju\rangle \tag{3.5}$$

for all $n \in \mathbb{N}$. And

$$2\langle y_n - z, Jx_n - Ju\rangle = -\phi(y_n, x_n) + \phi(y_n, u) + \phi(z, x_n) - \phi(z, u)$$

holds. So, from (3.5),

$$\begin{aligned} \phi(z, z_n) \le\ & \phi(z, x_n) + \phi(y_n, z_n) - (1-\alpha_n)\phi(y_n, x_n) \\ & -\alpha_n(\phi(y_n, u) + \phi(z, x_n) - \phi(z, u)) \end{aligned} \tag{3.6}$$

for each $n \in \mathbb{N}$. By Lemma 2.5, we have

$$\phi(z, x_{n+1}) \le \phi(z, z_n) - \phi(x_{n+1}, z_n). \tag{3.7}$$

And from Theorem 2.4,

$$\begin{aligned} \phi(y_n, x_n) &\ge c\|y_n - x_n\|^2 \\ \phi(y_n, z_n) &\le (1/c)\|Jy_n - Jz_n\|^2 = (\lambda_n^2/c)\|By_n - Bx_n\|^2 \\ &\le ((\lambda_n^2 L^2)/c)\|y_n - x_n\|^2. \end{aligned} \tag{3.8}$$

Hence, by (3.5), (3.7) and (3.8),

$$\begin{aligned} \phi(z, x_{n+1}) &\le \phi(z, z_n) - \phi(x_{n+1}, z_n) \\ &\le \phi(z, x_n) - \phi(x_{n+1}, z_n) - (c - (\lambda_n^2 L^2)/c)\|y_n - x_n\|^2 \\ &\quad - 2\alpha_n\langle y_n - z, Jx_n - Ju\rangle \end{aligned} \tag{3.9}$$

for every $n \in \mathbb{N}$ and by (3.6)-(3.8), we get

$$\phi(z, x_{n+1}) \le \phi(z, x_n) - \phi(x_{n+1}, z_n) \tag{3.10}$$

$$- ((1-\alpha_n)c - (\lambda_n^2 L^2)/c)\|y_n - x_n\|^2$$
$$- \alpha_n(\phi(y_n,u) + \phi(z,x_n) - \phi(z,u))$$

for all $n \in \mathbb{N}$. From $0 < \inf_{n\in\mathbb{N}} \lambda_n \le \sup_{n\in\mathbb{N}} \lambda_n < c/L$ and $\{\alpha_n\} \subset (0,1]$ with $\alpha_n \to 0$, there exist $k_1, k_2 > 0$ and $n_0 \in \mathbb{N}$ such that $c - (\lambda_n^2 L^2)/c > k_1$ for each $n \in \mathbb{N}$ and $(1-\alpha_n)c - (\lambda_n^2 L^2)/c > k_2$ for every $n \ge n_0$. Hence, by (3.9) and (3.10), we obtain

$$\begin{aligned} \phi(z,x_{n+1}) &\le \phi(z,x_n) - \phi(x_{n+1},z_n) - k_1\|y_n - x_n\|^2 \\ &- 2\alpha_n\langle y_n - z, Jx_n - Ju\rangle \end{aligned} \tag{3.11}$$

for all $n \in \mathbb{N}$ and

$$\begin{aligned} \phi(z,x_{n+1}) &\le \phi(z,x_n) - \phi(x_{n+1},z_n) - k_2\|y_n - x_n\|^2 \\ &- \alpha_n(\phi(y_n,u) + \phi(z,x_n) - \phi(z,u)) \end{aligned} \tag{3.12}$$

for each $n \ge n_0$. At first, we prove $\{x_n\}$ is bounded. If $\phi(z,x_n)$ decreases, it is trivial. If not so, by Lemma 2.7, there exist $n_1 \in \mathbb{N}$ and an eventually increasing function j such that $\phi(z,x_{j(n)}) \le \phi(z,x_{j(n)+1})$ and $\phi(z,x_n) \le \phi(z,x_{j(n)+1})$ for each $n \ge n_1$. There exists $N \in \mathbb{N}$ with $j(n) \ge n_0$ $(\forall n \ge N)$. From (3.12), we obtain

$$\phi(y_{j(n)},u) + \phi(z,x_{j(n)}) - \phi(z,u) \le 0$$

for all $n \ge \max\{n_1, N\}$ which implies that $\{x_{j(n)}\}$ is bounded. By $\phi(z,x_n) \le \phi(z,x_{j(n)+1})$ and (3.12), we have

$$\phi(z,x_n) \le \phi(z,x_{j(n)+1}) \le \phi(z,x_{j(n)}) + \alpha_{j(n)}\phi(z,u) \le \phi(z,x_{j(n)}) + \phi(z,u)$$

for each $n \ge \max\{n_1, N\}$. So, we have $\{x_n\}$ is bounded. Next, we show $\{y_n\}$ is bounded. From (3.12),

$$k_2\|y_n - x_n\|^2 \le \phi(z,x_n) + \alpha_n\phi(z,u) \le \phi(z,x_n) + \phi(z,u)$$

for every $n \ge n_0$. Hence, $\{y_n\}$ is bounded.

(i). If $\{\phi(\Pi_F u, x_n)\}$ is not decreasing, by Lemma 2.7, there exist $n_2 \in \mathbb{N}$ and an eventually increasing function i such that $\phi(\Pi_F u, x_{i(n)}) \le \phi(\Pi_F u, x_{i(n)+1})$ and $\phi(\Pi_F u, x_n) \le \phi(\Pi_F u, x_{i(n)+1})$ for every $n \ge n_2$. There exists $m \in \mathbb{N}$ such that $i(n) \ge n_0$ $(\forall n \ge m)$ holds. From (3.12), we have

$$\begin{aligned} &\phi(\Pi_F u, x_{i(n)+1}) \\ &\quad\le \phi(\Pi_F u, x_{i(n)}) - k_2\|y_{i(n)} - x_{i(n)}\|^2 + \alpha_{i(n)}\phi(\Pi_F u, u) \\ &\quad\le \phi(\Pi_F u, x_{i(n)+1}) - k_2\|y_{i(n)} - x_{i(n)}\|^2 + \alpha_{i(n)}\phi(\Pi_F u, u) \end{aligned} \tag{3.13}$$

which implies

$$k_2\|y_{i(n)} - x_{i(n)}\|^2 \le \alpha_{i(n)}\phi(\Pi_F u, u)$$

for all $n \ge \max\{n_2, m\}$. From this and $\alpha_n \to 0$, we get

$$\|x_{i(n)} - y_{i(n)}\| \to 0. \tag{3.14}$$

Let $w_n = J_{\lambda_n}^A J^{-1}(Jx_n - \lambda_n Bx_n)$. Since $(1/\lambda_n)(Jx_n - Jy_n) - Bx_n - (\alpha_n/\lambda_n)(Jx_n - Ju) \in Ay_n$, $(1/\lambda_n)(Jx_n - Jw_n) - Bx_n \in Aw_n$ and A is monotone,

$$\langle y_n - w_n, Jw_n - Jy_n - \alpha_n(Jx_n - Ju)\rangle \geq 0.$$

By Theorem 2.4,

$$\begin{aligned} \alpha_n\|y_n - w_n\| \cdot \|Jx_n - Ju\| &\geq -\alpha_n\langle y_n - w_n, Jx_n - Ju\rangle \\ &\geq \langle y_n - w_n, Jy_n - Jw_n\rangle \\ &\geq c\|y_n - w_n\|^2 \end{aligned}$$

for each $n \in \mathbb{N}$. By $\alpha_n \to 0$, we obtain

$$\|y_n - w_n\| \to 0. \tag{3.15}$$

From (3.14) and (3.15), $\|x_{i(n)} - w_{i(n)}\| \to 0$. By the assumption, we have $\omega_w(x_{i(n)}) \subset F$. From (3.11),

$$\begin{aligned} \phi(\Pi_F u, x_{i(n)+1}) &\leq \phi(\Pi_F u, x_{i(n)}) - 2\alpha_{i(n)}\langle y_{i(n)} - \Pi_F u, Jx_{i(n)} - Ju\rangle \\ &\leq \phi(\Pi_F u, x_{i(n)+1}) - 2\alpha_{i(n)}\langle y_{i(n)} - \Pi_F u, Jx_{i(n)} - Ju\rangle \end{aligned}$$

which implies

$$\langle y_{i(n)} - \Pi_F u, Jx_{i(n)} - Ju\rangle \leq 0 \tag{3.16}$$

for every $n \geq n_2$. By Theorem 2.4,

$$\begin{aligned} &\langle y_{i(n)} - \Pi_F u, Jx_{i(n)} - Ju\rangle \\ &\quad = \langle y_{i(n)} - \Pi_F u, Jx_{i(n)} - Jy_{i(n)}\rangle + \langle y_{i(n)} - \Pi_F u, Jy_{i(n)} - J\Pi_F u\rangle \\ &\qquad + \langle y_{i(n)} - \Pi_F u, J\Pi_F u - Ju\rangle \\ &\quad \geq -\|y_{i(n)} - \Pi_F u\| \cdot \|Jy_{i(n)} - Jx_{i(n)}\| + c\|y_{i(n)} - \Pi_F u\|^2 \\ &\qquad + \langle y_{i(n)} - \Pi_F u, J\Pi_F u - Ju\rangle. \end{aligned}$$

So, from (3.16), we get

$$\begin{aligned} &-\|y_{i(n)} - \Pi_F u\| \cdot \|Jy_{i(n)} - Jx_{i(n)}\| + c\|y_{i(n)} - \Pi_F u\|^2 \\ &\qquad + \langle y_{i(n)} - \Pi_F u, J\Pi_F u - Ju\rangle \leq 0 \end{aligned} \tag{3.17}$$

for every $n \geq n_2$. Let $\{x_{n_j}\}$ be a subsequence of $\{x_{i(n)}\}$ such that $x_{n_j} \rightharpoonup w$. We have $w \in F$. From (3.14), $y_{n_j} \rightharpoonup w$. Since the duality mapping J of E is uniformly continuous on bounded subsets of E and (3.14), $\|Jy_{i(n)} - Jx_{i(n)}\| \to 0$. Hence, by (3.17), lower-semicontinuity of norm and Lemma 2.5, we obtain

$$\begin{aligned} 0 &\geq c\liminf_{j\to\infty}\|y_{n_j} - \Pi_F u\|^2 + \liminf_{j\to\infty}\langle y_{n_j} - \Pi_F u, J\Pi_F u - Ju\rangle \\ &\geq c\|w - \Pi_F u\|^2 + \langle w - \Pi_F u, J\Pi_F u - Ju\rangle \geq c\|w - \Pi_F u\|^2 \end{aligned}$$

which implies $w = \Pi_F u$. Therefore, $x_{i(n)} \rightharpoonup \Pi_F u$ and $y_{i(n)} \rightharpoonup \Pi_F u$. From (3.17), we have

$$\begin{aligned} 0 &\geq \limsup_{n\to\infty}(-\|y_{i(n)} - \Pi_F u\| \cdot \|Jy_{i(n)} - Jx_{i(n)}\| + c\|y_{i(n)} - \Pi_F u\|^2 \\ &\quad + \langle y_{i(n)} - \Pi_F u, J\Pi_F u - Ju\rangle) \end{aligned}$$

$$= \quad c \limsup_{n\to\infty} \|y_{i(n)} - \Pi_F u\|^2.$$

Hence, we get $y_{i(n)} \to \Pi_F u$. By (3.14),

$$x_{i(n)} \to \Pi_F u.$$

Since J is norm to weak* continuous,

$$\begin{aligned} \phi(\Pi_F u, x_{i(n)}) &= \|\Pi_F u\|^2 - 2\langle \Pi_F u, Jx_{i(n)}\rangle + \|x_{i(n)}\|^2 \\ &\to \|\Pi_F u\|^2 - 2\langle \Pi_F u, J\Pi_F u\rangle + \|\Pi_F u\|^2 = 0. \end{aligned}$$

From (3.13) and $\alpha_{i(n)} \to 0$, we obtain $\phi(\Pi_F u, x_{i(n)+1}) \to 0$. By $\phi(\Pi_F u, x_n) \leq \phi(\Pi_F u, x_{i(n)+1})$ for each $n \geq n_2$, we have

$$\phi(\Pi_F u, x_n) \to 0$$

which implies $x_n \to \Pi_F u$ from Theorem 2.4.

(ii). Suppose that $\phi(\Pi_F u, x_n)$ is decreasing. There exists $\lim_{n\to\infty} \phi(\Pi_F u, x_n)$. By (3.12) and $\alpha_n \to 0$, we get $\|x_n - y_n\| \to 0$. On the other hand, we have (3.15). So, from $\|x_n - w_n\| \to 0$ and the assumption, $\omega_w(x_n) \subset F$. We show that

$$\limsup_{n\to\infty} \langle \Pi_F u - y_n, Jx_n - Ju\rangle \geq 0. \tag{3.18}$$

Suppose that $\limsup_{n\to\infty} \langle \Pi_F u - y_n, Jx_n - Ju\rangle = l < 0$. There exists $n_3 \in \mathbb{N}$ such that $\langle \Pi_F u - y_n, Jx_n - Ju\rangle \leq \frac{1}{2} l$ for every $n \geq n_3$. By (3.11),

$$-l\alpha_n \leq 2\alpha_n \langle y_n - \Pi_F u, Jx_n - Ju\rangle \leq \phi(\Pi_F u, x_n) - \phi(\Pi_F u, x_{n+1})$$

for each $n \geq n_3$ which implies

$$\sum_{n=n_3}^{\infty} (-l)\alpha_n \leq \phi(\Pi_F u, x_{n_3}) < \infty.$$

From $\sum_{n=1}^{\infty} \alpha_n = \infty$, this is a contradiction. So, we obtain (3.18). Next,

$$\begin{aligned} \langle \Pi_F u - y_n, Jx_n - Ju\rangle &= \langle \Pi_F u - y_n, Jx_n - Jy_n\rangle + \langle \Pi_F u - y_n, Jy_n - J\Pi_F u\rangle \\ &\quad + \langle \Pi_F u - y_n, J\Pi_F u - Ju\rangle \\ &\leq \|\Pi_F u - y_n\| \cdot \|Jx_n - Jy_n\| - \frac{1}{2}\phi(\Pi_F u, y_n) \\ &\quad + \langle \Pi_F u - y_n, J\Pi_F u - Ju\rangle \end{aligned}$$

for all $n \in \mathbb{N}$. Since J is uniformly continuous on bounded subsets of E and $\|x_n - y_n\| \to 0$, we have

$$\|Jx_n - Jy_n\| \to 0. \tag{3.19}$$

So, we have

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \langle \Pi_F u - y_n, Jx_n - Ju\rangle \\ &\leq -\frac{1}{2} \liminf_{n\to\infty} \phi(\Pi_F u, y_n) + \limsup_{n\to\infty} \langle \Pi_F u - y_n, J\Pi_F u - Ju\rangle. \end{aligned}$$

And there exists a subsequence $\{y_{n_j}\}$ of $\{y_n\}$ such that $y_{n_j} \rightharpoonup w \in F$ and

$$\limsup_{n\to\infty}\langle \Pi_F u - y_n, J\Pi_F u - Ju\rangle = \lim_{j\to\infty}\langle \Pi_F u - y_{n_j}, J\Pi_F u - Ju\rangle.$$

From Lemma 2.5,

$$\lim_{j\to\infty}\langle \Pi_F u - y_{n_j}, J\Pi_F u - Ju\rangle = \langle \Pi_F u - w, J\Pi_F u - Ju\rangle \le 0$$

holds. Hence, we get

$$\liminf_{n\to\infty}\phi(\Pi_F u, y_n) = 0.$$

From $\|x_n - y_n\| \to 0$, (3.19) and

$$\begin{aligned}
&|\phi(\Pi_F u, x_n) - \phi(\Pi_F u, y_n)| \\
&\quad\le 2\|\Pi_F u\|\cdot\|Jx_n - Jy_n\| + |\|x_n\|^2 - \|y_n\|^2| \\
&\quad\le 2\|\Pi_F u\|\cdot\|Jx_n - Jy_n\| + \|x_n - y_n\|\cdot(\|x_n\| + \|y_n\|),
\end{aligned}$$

we obtain $\liminf_{n\to\infty}\phi(\Pi_F u, x_n) = 0$ which implies $\lim_{n\to\infty}\phi(\Pi_F u, x_n) = 0$. Therefore, $\{x_n\}$ converges strongly to $\Pi_F u$ from Theorem 2.4. □

4. Deduced results

To begin with, we get a new strong convergence theorem for a sum of maximal monotone operators by Theorem 3.2.

Theorem 4.1. *Let E be a 2-uniformly convex and uniformly smooth Banach space, A a maximal monotone operator in $E\times E^*$, B a monotone and Lipschitz continuous mapping of E into E^* with a Lipschitz constant $L > 0$ such that $F = (A+B)^{-1}0 \neq \emptyset$. Let $u \in E$ and $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_1 = x \in E, \\ y_n = J^A_{\lambda_n}J^{-1}(Jx_n - \lambda_n Bx_n - \alpha_n(Jx_n - Ju)), \\ x_{n+1} = J^{-1}(Jy_n - \lambda_n(By_n - Bx_n)) \end{cases}$$

for every $n \in \mathbb{N}$, where $0 < \inf_{n\in\mathbb{N}}\lambda_n \le \sup_{n\in\mathbb{N}}\lambda_n < c/L$ where c is the constant in Theorem 2.2 and $0 < \alpha_n \le 1$ for all $n \in \mathbb{N}$ with $\alpha_n \to 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $\Pi_F u$.

Proof. It is known that $A + B$ is a maximal monotone operator, see [29]. We show the assumption in Theorem 3.2. Let $\{u_n\}$ be a bounded sequence and $\{\lambda_n\} \subset (0,\infty)$ with $\inf_{n\in\mathbb{N}}\lambda_n > 0$ such that $\|u_n - J^A_{\lambda_n}J^{-1}(Ju_n - \lambda_n Bu_n)\| \to 0$ and $\{u_{n_i}\}$ a subsequence of $\{u_n\}$ with $u_{n_i} \rightharpoonup u$. Let $v_n = J^A_{\lambda_n}J^{-1}(Ju_n - \lambda_n Bu_n)$. Since $\frac{1}{\lambda_{n_i}}(Ju_{n_i} - Jv_{n_i}) - Bu_{n_i} + Bv_{n_i} \in (A+B)v_{n_i}$ and $A + B$ is monotone, we have

$$\langle v_{n_i} - w, \frac{1}{\lambda_{n_i}}(Ju_{n_i} - Jv_{n_i}) - Bu_{n_i} + Bv_{n_i} - w^*\rangle \ge 0$$

for every $\langle w, w^*\rangle \in (A+B)$. Hence, we get

$$\langle v_{n_i} - w, -w^*\rangle \ \ge\ \langle v_{n_i} - w, \frac{1}{\lambda_{n_i}}(Jv_{n_i} - Ju_{n_i}) + Bu_{n_i} - Bv_{n_i}\rangle$$

$$
\begin{aligned}
&\geq -\frac{1}{\lambda_{n_i}}\|v_{n_i}-w\|\cdot\|Jv_{n_i}-Ju_{n_i}\|-\|v_{n_i}-w\|\cdot\|Bu_{n_i}-Bv_{n_i}\| \\
&\geq -\frac{1}{\lambda_{n_i}}\|v_{n_i}-w\|\cdot\|Jv_{n_i}-Ju_{n_i}\|-\|v_{n_i}-w\|\cdot L\|u_{n_i}-v_{n_i}\|
\end{aligned}
$$

for all $i \in \mathbb{N}$. Since J is uniformly continuous on bounded subsets of E and $\|u_{n_i} - v_{n_i}\| \to 0$, $\|Ju_{n_i} - Jv_{n_i}\| \to 0$. By $v_{n_i} \rightharpoonup u$ and $\inf_{n\in\mathbb{N}} \lambda_n > 0$, we obtain

$$\langle u - w, -w^*\rangle \geq 0.$$

From maximality of $A + B$, we get $u \in (A + B)^{-1}0 = F$. So, $\omega_w(u_n) \subset F$ holds. Therefore, the proof is complete by Theorem 3.2. □

Let C be a nonempty closed convex subset of E and A a single valued mapping of C into E^*. We consider the variational inequality problem [15] for A, that is, the problem of finding an element $z \in C$ such that

$$\langle x - z, Az\rangle \geq 0 \text{ for all } x \in C.$$

The set of all solutions of the variational inequality problem for A is denoted by $VI(C, A)$. Tufa and Zegeye [34] proved the strong convergence theorem of variational inequality problems for a monotone and Lipschitz continuous mapping in a 2-uniformly convex and uniformly smooth Banach sapce. From Theorem 3.2, we have a new result which is defferent from that.

Theorem 4.2. *Let C be a nonempty closed convex subset of a 2-uniformly convex and uniformly smooth Banach space E. Let B be a monotone and Lipschitz continuous mapping of C into E^* with a Lipschitz constant $L > 0$ such that $VI(C, B) \neq \emptyset$. Let $u \in E$ and $\{x_n\}$ a sequence generated by*

$$
\begin{cases}
x_1 = x \in C, \\
y_n = \Pi_C J^{-1}(Jx_n - \lambda_n Bx_n - \alpha_n(Jx_n - Ju)) \\
x_{n+1} = \Pi_C J^{-1}(Jy_n - \lambda_n(By_n - Bx_n))
\end{cases}
$$

for every $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0,\infty)$ and $\{\alpha_n\} \subset (0, 1]$ are real sequences. Suppose that $0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < c/L$, where c is the constant in Theorem 2.2, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $\Pi_{VI(C,B)}u$.

Proof. Let i_C be the indicator function of C. We have that $i_C : E \to (-\infty, \infty]$ is a proper lower semicontinuous and convex function and hence, the subdifferential ∂i_C is maximal monotone. Let $A = \partial i_C$. Then, it is easy to see that $D(\partial i_C) = C$, $J^A_\lambda x = \Pi_C x$ for every $\lambda > 0$ and $x \in E$ and $(A + B)^{-1}0 = VI(C, B)$.

We prove the assumption in Theorem 3.2. Let $\{u_n\}$ be a bounded sequence in C and $\{\lambda_n\} \subset (0,\infty)$ with $\inf_{n\in\mathbb{N}} \lambda_n > 0$ such that $\|u_n - \Pi_C J^{-1}(Ju_n - \lambda_n Bu_n)\| \to 0$ and $\{u_{n_i}\}$ a subsequence of $\{u_n\}$ with $u_{n_i} \rightharpoonup u$. Let $v_n = \Pi_C J^{-1}(Ju_n - \lambda_n Bu_n)$. We have

$$\langle v_{n_i} - x, Ju_{n_i} - Jv_{n_i} - \lambda_{n_i} Bu_{n_i}\rangle \geq 0$$

for every $i \in \mathbb{N}$ and $x \in C$ which implies

$$\langle v_{n_i} - x, -Bx\rangle \geq \frac{1}{\lambda_{n_i}}\langle v_{n_i} - x, Jv_{n_i} - Ju_{n_i}\rangle + \langle v_{n_i} - x, Bu_{n_i} - Bv_{n_i}\rangle$$

$$
\begin{aligned}
&+ \langle v_{n_i} - x, Bv_{n_i} - Bx\rangle \\
(4.1) \qquad &\geq \frac{-1}{\lambda_{n_i}}\|v_{n_i} - x\| \cdot \|Jv_{n_i} - Ju_{n_i}\| + \langle v_{n_i} - x, Bu_{n_i} - Bv_{n_i}\rangle
\end{aligned}
$$

for every $i \in \mathbb{N}$ and $x \in C$. Since J is uniformly continuous on bounded subsets of E and $\|u_{n_i} - v_{n_i}\| \to 0$, $\|Ju_{n_i} - Jv_{n_i}\| \to 0$. And we get

$$
\begin{aligned}
|\langle v_{n_i} - x, Bu_{n_i} - Bv_{n_i}\rangle| &\leq \|v_{n_i} - x\| \cdot \|Bu_{n_i} - Bv_{n_i}\| \\
&\leq \|v_{n_i} - x\| \cdot L\|u_{n_i} - v_{n_i}\| \to 0.
\end{aligned}
$$

So, from $v_{n_i} \rightharpoonup u$, $\inf_{n\in\mathbb{N}} \lambda_n > 0$ and (4.1), we obtain

$$\langle u - x, -Bx\rangle \geq 0$$

for every $x \in C$. Since B is hemicontinuous, we have

$$\langle x - u, Bu\rangle \geq 0$$

for all $x \in C$, that is, $u \in VI(C, B)$. So, $\omega_w(u_n) \subset VI(C, B)$. Hence, the proof is complete by Theorem 3.2. □

In a real Hilbert space H, we have $c = 1$ in Theorem 2.2, $J = J^{-1} = I$, where I is the identity mapping and $\Pi_C = P_C$, where P_C is the metric projection of H onto C. So, we get new results in a real Hilbert space by Theorems 4.1 and 4.2.

Theorem 4.3. *Let A be a maximal monotone operator in $H \times H$ and B a monotone and Lipschitz continuous mapping of H into H with a Lipschitz constant $L > 0$ such that $F = (A + B)^{-1}0 \neq \emptyset$. Let $u \in H$ and $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
x_1 = x \in H, \\
y_n = J^A_{\lambda_n}(x_n - \lambda_n Bx_n - \alpha_n(x_n - u)), \\
x_{n+1} = y_n - \lambda_n(By_n - Bx_n)
\end{cases}
$$

for every $n \in \mathbb{N}$, where $0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < 1/L$ and $0 < \alpha_n \leq 1$ for all $n \in \mathbb{N}$ with $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_F u$.

Theorem 4.4. *Let C be a nonempty closed convex subset of H and B a monotone and Lipschitz continuous mapping of C into H with a Lipschitz constant $L > 0$ such that $VI(C, B) \neq \emptyset$. Let $u \in H$ and $\{x_n\}$ a sequence generated by*

$$
\begin{cases}
x_1 = x \in C, \\
y_n = P_C(x_n - \lambda_n Bx_n - \alpha_n(x_n - u)) \\
x_{n+1} = P_C(y_n - \lambda_n(By_n - Bx_n))
\end{cases}
$$

for every $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset (0, 1]$ are real sequences. Suppose that $0 < \inf_{n\in\mathbb{N}} \lambda_n \leq \sup_{n\in\mathbb{N}} \lambda_n < 1/L$, $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $P_{VI(C,B)}u$.

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