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SOME ALGORITHMS FOR THE CLOSEST POINT TO THE COMMON SOLUTION SET OF PSEUDOMONOTONE EQUILIBRIUM PROBLEMS AND FIXED POINTS OF QUASI-NONEXPANSIVE MAPPINGS PROBLEMS

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Abstract. This paper presents two hybrid extragradient algorithms for finding the closest point to the intersection of the solution set of equilibrium problems for pseudomonotone bifunctions and the set of fixed points of quasi-nonexpansive mappings in a real Hilbert space. Under some constraint qualifications of the scalar sequences, we show a strong convergence of the introduced algorithms. Some numerical experiments are also provided.

1. Introduction

The equilibrium problem and the fixed point problem are very useful tools for studying physics, chemistry, engineering and economics in different mathematical models, for instance, see [15, 19, 20, 23], and the references therein. Besides, they have applications in many important problems, such as optimization problems, variational inequality problems, minimax problems, Nash equilibrium problems, saddle point problems, and others, see [6, 7, 14, 34, 40], and the references therein. The equilibrium problem is a problem of finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \forall y \in C, \tag{1.1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, and $f : C \times C \to \mathbb{R}$ is a bifunction. The solution set of the equilibrium problem (1.1) will be represented by $EP(f, C)$. In the most appeared papers, the proposed method for solving the equilibrium problem is the proximal point method. This method was first introduced by Martinet [30] for solving variational inequality problems and it was extended by Moudafi [33] to monotone equilibrium problems. In the case of the proximal point method, at each iteration, we need to solve a regularized equilibrium problem:

$$find \quad x \in C \quad such\ that \quad f(x, y) + \frac{1}{r_k}\langle y - x, x - x_k\rangle \geq 0, \forall y \in C, \tag{1.2}$$

where $\{r_k\} \subset (0, \infty)$, $\{x_k\} \subset C$, and f is a monotone bifunction. Note that the existence of the solution of the problem (1.2) is guaranteed, see [7, 13]. However, if f satisfies a weaker assumption as pseudomonotone, the proximal point method cannot be applied in this situation. To overcome this drawback, the extragradient

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method was introduced for solving pseudomonotone equilibrium problems instead of the proximal point method. The extragradient method was first introduced by Korpelevich [28] for solving saddle point problems and it was extended by Noor [36] to pseudomonotone variational inequality problems. Later, Tran et al. [42] proposed the following extragradient method for solving the equilibrium problem when the bifunction f is pseudomonotone and Lipschitz-type continuous with positive constants L_1 and L_2:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min\{\rho f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ x_{k+1} = \arg\min\{\rho f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \end{cases} \tag{1.3}$$

where $0 < \rho < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$. They proved that the sequence $\{x_k\}$ generated by (1.3) converges weakly to a solution of the equilibrium problem (1.1).

On the other hand, for a nonempty closed convex subset C of H, and a mapping $T : C \to C$, the fixed point problem is a problem of finding a point $x \in C$ such that $Tx = x$. The set of fixed points of the mapping T will be denoted by $Fix(T)$.

A famous iterative method for finding fixed points of a nonexpansive mapping T was proposed by Mann [29] as followed:

$$\begin{cases} x_0 \in C, \\ x_{k+1} = (1 - \alpha_k)x_k + \alpha_k T x_k, \end{cases} \tag{1.4}$$

where $\{\alpha_k\} \subset (0, 1)$. In [39], the author proved that if T has a fixed point and $\sum_{k=0}^{\infty} \alpha_k(1-\alpha_k) = \infty$, then the sequence $\{x_k\}$ generated by (1.4) converges weakly to a fixed point of T. Besides, Park and Jeong [37] presented that if T is a quasi-nonexpansive mapping with $I - T$ demiclosed at 0, then the sequence which is generated by (1.4) also converges weakly to a fixed point of T.

In order to obtain a strong convergence result for Mann iterative method (1.4), Nakajo and Takahashi [35] proposed the following hybrid method for finding fixed points of a nonexpansive mapping T:

$$\begin{cases} x_0 \in C, \\ y_k = \alpha_k x_k + (1 - \alpha_k)T x_k, \\ C_k = \{x \in C : \|y_k - x\| \le \|x_k - x\|\}, \\ Q_k = \{x \in C : \langle x_0 - x_k, x - x_k\rangle \le 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0), \end{cases} \tag{1.5}$$

where $\{\alpha_k\} \subset [0, 1]$ such that $\alpha_k \le 1 - \overline{\alpha}$, for some $\overline{\alpha} \in (0, 1]$, and $P_{C_k \cap Q_k}$ is the metric projection onto $C_k \cap Q_k$. They proved that the sequence $\{x_k\}$ generated by (1.5) converges strongly to $P_{Fix(T)}(x_0)$.

Furthermore, Ishikawa [25] proposed the following method for finding fixed points of a Lipschitz pseudocontractive mapping T:

$$\begin{cases} x_0 \in C, \\ y_k = (1-\alpha_k)x_k + \alpha_k T x_k, \\ x_{k+1} = (1-\beta_k)x_k + \beta_k T y_k, \end{cases} \tag{1.6}$$

where $0 \le \beta_k \le \alpha_k \le 1$, $\lim_{k\to\infty} \alpha_k = 0$, and $\sum_{k=0}^{\infty} \alpha_k \beta_k = \infty$. In [25], the author proved that if C is a convex compact subset of H, then the sequence $\{x_k\}$ generated by (1.6) converges strongly to fixed points of T. It was noted that Mann iterative method may not, in general, be applicable for finding fixed points of a Lipschitz pseudocontractive mapping in a Hilbert space, for instance, see [12].

In recent years, many algorithms have been proposed for finding a point in the intersection of the solution set of the equilibrium problems and the solution set of the fixed point problems, for instance, see [1, 11, 18, 32] and the references therein. In 2016, by using the ideas of extragradient and hybrid methods together with Ishikawa iterative method, Dinh and Kim [15] proposed the following algorithm for finding the closest point to the intersection of the set of fixed points of a symmetric generalized hybrid mapping T and the solution set of equilibrium problem, when a bifunction f is pseudomonotone and Lipschitz-type continuous with positive constants L_1, L_2:

$$\begin{cases} x_0 \in C, \\ y_k = \arg\min\{\rho_k f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ z_k = \arg\min\{\rho_k f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ t_k = \alpha_k x_k + (1-\alpha_k) T x_k, \\ u_k = \beta_k t_k + (1-\beta_k) T z_k, \\ C_k = \{x \in H : \|x - u_k\| \le \|x - x_k\|\}, \\ Q_k = \{x \in H : \langle x - x_k, x_0 - x_k \rangle \le 0\}, \\ x_{k+1} = P_{C_k \cap Q_k \cap C}(x_0), \end{cases} \tag{1.7}$$

where $\{\rho_k\} \subset [\underline{\rho}, \overline{\rho}]$ with $0 < \underline{\rho} \le \overline{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0,1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, and $\{\beta_k\} \subset [0, 1-\overline{\beta}]$, for some $\overline{\beta} \in (0,1)$. They proved that the sequence $\{x_k\}$ generated by (1.7) converges strongly to $P_{EP(f,C)\cap Fix(T)}(x_0)$.

In 2016, Hieu et al. [20] considered the following problem:

$$\begin{cases} \text{find a point } x^* \in C \quad \text{such that} \quad T_j x^* = x^*, j = 1, \ldots, M, \\ \text{and} \quad f_i(x^*, y) \ge 0, \forall y \in C, i = 1, \ldots, N, \end{cases} \tag{1.8}$$

where C is a nonempty closed convex subset of H, $T_j : C \to C$, $j = 1, \ldots, M$, are mappings, and $f_i : C \times C \to \mathbb{R}$, $i = 1, \ldots, N$, are bifunctions satisfying $f_i(x,x) = 0$, for each $x \in C$. By using the ideas of extragradient and hybrid methods together with Mann iterative method and parallel splitting-up techniques, see [2, 3], Hieu et al. [20] proposed the following algorithm for finding the closest point to the solution set of problem (1.8), when mappings are nonexpansive, and bifunctions are pseudomonotone and Lipschitz-type continuous with positive constants L_1 and L_2:

$$(1.9)\qquad \begin{cases} x_0 \in C, \\ y_k^i = \arg\min\{\rho f_i(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, i = 1, 2, \dots, N, \\ z_k^i = \arg\min\{\rho f_i(y_k^i, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, i = 1, 2, \dots, N, \\ \overline{z}_k = \arg\max\{\|z_k^i - x_k\| : i = 1, 2, \dots, N\}, \\ u_k^j = \alpha_k x_k + (1 - \alpha_k) T_j \overline{z}_k, j = 1, 2, \dots, M, \\ \overline{u}_k = \arg\max\{\|u_k^j - x_k\| : j = 1, 2, \dots, M\}, \\ C_k = \{x \in C : \|x - \overline{u}_k\| \leq \|x - x_k\|\}, \\ Q_k = \{x \in C : \langle x - x_k, x_0 - x_k \rangle \leq 0\}, \\ x_{k+1} = P_{C_k \cap Q_k}(x_0), \end{cases}$$

where $0 < \rho < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, and $\{\alpha_k\} \subset (0, 1)$ such that $\limsup_{k\to\infty} \alpha_k < 1$. They proved that the sequence $\{x_k\}$ generated by (1.9) converges strongly to $P_S(x_0)$, where $S := (\cap_{j=1}^M Fix(T_j)) \cap (\cap_{i=1}^N EP(f_i, C))$ is the solution set of problem (1.8). From now on, the algorithm (1.9) will be called PHMEM.

In this paper, we will still focus to the methods for finding the solutions of problem (1.8). That is, we will introduce some new iterative algorithms for finding the closest point to the intersection of the solution set of pseudomonotone equilibrium problems and the set of fixed points of quasi-nonexpansive mappings. Some numerical examples and comparison of the introduced methods with well-known algorithms will be considered.

This paper is organized as follows: In Section 2, some necessary definitions and properties will be reviewed. Section 3, two hybrid extragradient algorithms and prove their convergence will be considered. Finally, in Section 4, we discuss the performance of introduced algorithms and compare it with some appeared algorithms via the numerical experiments.

2. Preliminaries

This section will present the definitions and some important basic properties that will be used in this work. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and norm $\| \cdot \|$. The symbols $\to$ and $\rightharpoonup$ will be denoted for the strong convergence and the weak convergence in H, respectively.

First, we will recall the concerned definitions of nonlinear mappings.

Definition 2.1 ([9, 41])**.** A mapping $T : C \to C$ is said to be:

(i) pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C,$$

where I denotes the identity operator on C.

(ii) Lipschitzian if there exists $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C.$$

In particular, if $L = 1$, then T is said to be nonexpansive.

(iii) quasi-nonexpansive if $Fix(T)$ is a nonempty set and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, p \in Fix(T).$$

(iv) $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid if there exists $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + \beta(\|x - Ty\|^2 + \|y - Tx\|^2) + \gamma\|x - y\|^2$$
$$+\delta(\|x - Tx\|^2 + \|y - Ty\|^2) \leq 0, \forall x, y \in C.$$

Remark 2.2. A nonexpansive mapping with at least one fixed point is a quasi-nonexpansive mapping, but the converse is not true in general, for instance, see [16]. Moreover, $Fix(T)$ is closed and convex when T is a quasi-nonexpansive mapping, see [24].

Definition 2.3 (see [8])**.** Let C be a nonempty closed convex subset of H. A mapping $T : C \to H$ is said to be demiclosed at $y \in H$ if for any sequence $\{x_k\} \subset C$ with $x_k \rightharpoonup x^* \in C$ and $Tx_k \to y$ imply $Tx^* = y$.

Remark 2.4. It is well-known that if T is an $(\alpha, \beta, \gamma, \delta)$-symmetric generalized hybrid mapping satisfies (1) $\alpha + 2\beta + \gamma \geq 0$, (2) $\alpha + \beta > 0$, and (3) $\delta \geq 0$, then T is quasi-nonexpansive and $I - T$ demiclosed at 0, see [21, 27].

Now, we will recall the facts that are related to the equilibrium problems.

Definition 2.5 ([7, 31, 34])**.** A bifunction $f : C \times C \to \mathbb{R}$ is said to be:

(i) monotone on C if

$$f(x, y) + f(y, x) \leq 0, \forall x, y \in C;$$

(ii) pseudomonotone on C if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0, \forall x, y \in C;$$

(iii) Lipshitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$ if

$$f(x, y) + f(y, z) \geq f(x, z) - L_1\|x - y\|^2 - L_2\|y - z\|^2, \forall x, y, z \in C.$$

Remark 2.6. From Definition 2.5, we note that a monotone bifunction is a pseudomonotone bifunction. However, the converses may not be true, for instance, see [26].

Let C be a nonempty closed convex subset of H. The following assumptions on the bifunction $f : C \times C \to \mathbb{R}$ will be considered in this paper:

$(A1)$ f is weakly continuous on $C \times C$ in the sense that, if $x \in C$, $y \in C$, and $\{x_k\} \subset C$, $\{y_k\} \subset C$ are two sequences converge weakly to x and y respectively, then $f(x_k, y_k)$ converges to $f(x, y)$;

$(A2)$ $f(x, \cdot)$ is convex and subdifferentiable on C for each fixed $x \in C$;

$(A3)$ f is psuedomonotone on C and $f(x,x)=0$ for each $x \in C$;
$(A4)$ f is Lipshitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$.

Remark 2.7. It is well-known that the solution set $EP(f,C)$ is closed and convex when the bifunction f satisfies the assumptions $(A1)-(A3)$, see [5, 38, 42] and the references therein.

The following lemma will be useful in order to obtain the main results.

Lemma 2.8 ([1])**.** *Let $f : C \times C \to \mathbb{R}$ be satisfied $(A2)-(A4)$. Assume that $EP(f,C)$ is a nonempty set and $0 < \rho_0 < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$. Let $x_0 \in C$, and construct y_0 and z_0 by*

$$\begin{cases} y_0 = \arg\min\{\rho_0 f(x_0,w) + \frac{1}{2}\|w-x_0\|^2 : w \in C\}, \\ z_0 = \arg\min\{\rho_0 f(y_0,w) + \frac{1}{2}\|w-x_0\|^2 : w \in C\}. \end{cases}$$

Then,

(i) $\rho_0\ [f(x_0,w) - f(x_0,y_0)] \geq \langle y_0 - x_0, y_0 - w\rangle$, $\forall w \in C$;
(ii) $\|z_0 - q\|^2 \leq \|x_0 - q\|^2 - (1-2\rho_0 L_1)\|x_0 - y_0\|^2 - (1-2\rho_0 L_2)\|y_0 - z_0\|^2$, $\forall q \in EP(f,C)$.

We end this section by recalling some basic facts in the functional analysis which are needed in this paper.

Let C be a nonempty closed convex subset of H. For each $x \in H$, we denote the metric projection of x onto C by $P_C(x)$, that is

$$\|x - P_C(x)\| \leq \|y - x\|, \forall y \in C.$$

Lemma 2.9 (see [10, 16])**.** *Let C be a nonempty closed convex subset of H. Then*

(i) *$P_C(x)$ is singleton and well-defined for each $x \in H$;*
(ii) *$z = P_C(x)$ if and only if $\langle x - z, y - z\rangle \leq 0$, $\forall y \in C$.*

Now, the weak limit set of the sequence $\{x_k\}$ will be denoted by $\omega_w(x_k)$, that is, $\omega_w(x_k) = \{x \in H :$ there is a subsequence $\{x_{k_n}\}$ of $\{x_k\}$ such that $x_{k_n} \rightharpoonup x\}$.

The following lemmas are very important in order to obtain the main results.

Lemma 2.10 ([43])**.** *Let C be a nonempty closed convex subset of H. Let $\{x_k\}$ be a sequence of H and $u \in H$. If $\|x_k - u\| \leq \|u - P_C(u)\|, \forall k \in \mathbb{N}$, and $\omega_w(x_k) \subset C$, then $x_k \to P_C(u)$.*

For a function $g : H \to \mathbb{R}$, the subdifferential of g at $z \in H$ is defined by

$$\partial g(z) = \{w \in H : g(y) - g(z) \geq \langle w, y - z\rangle, \forall y \in H\}.$$

The function g is said to be subdifferentiable at z if $\partial g(z) \neq \emptyset$.

Theorem 2.11 (see [10])**.** *For any $z \in H$, the subdifferentiable $\partial g(z)$ of a continuous convex function g is a nonempty, weakly closed and bounded convex set.*

3. Main results

In this section, we propose two hybrid extragradient algorithms for finding the closest point to the solution set of problem (1.8), when each mapping $T_j : C \to C$, $j = 1, 2, \ldots, M$, is quasi-nonexpansive with $I - T_j$ demiclosed at 0, and each bifunction f_i, $i = 1, 2, \ldots, N$, satisfies the assumptions $(A1) - (A4)$. We start with some observations. If each bifunction f_i, $i = 1, 2, \ldots, N$, is Lipshitz-type continuous on C with constants $L_1^i > 0$ and $L_2^i > 0$, then

$$\begin{aligned} f_i(x, y) + f_i(y, z) &\geq f_i(x, z) - L_1^i\|x - y\|^2 - L_2^i\|y - z\|^2 \\ &\geq f_i(x, z) - L_1\|x - y\|^2 - L_2\|y - z\|^2, \end{aligned}$$

where $L_1 = \max\{L_1^i : i = 1, 2, \ldots, N\}$, and $L_2 = \max\{L_2^i : i = 1, 2, \ldots, N\}$. This means the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipshitz-type continuous on C with constants $L_1 > 0$ and $L_2 > 0$.

From now on, for each $N \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$, a modulo function at k with respect to N will be denoted by $[k]_N$, that is,

$$[k]_N = k(mod\ N) + 1.$$

Now, the cyclic method is presented as following:

Cyclic Hybrid Extragradient Method (CHEM)

Initialization. Choose parameters $\{\rho_k\}$ with $0 < \inf \rho_k \leq \sup \rho_k < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0, 1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, and $\{\beta_k\} \subset [0, 1)$ with $0 \leq \inf \beta_k \leq \sup \beta_k < 1$. Pick $x_0 \in C$.

Step 1. Solve the strongly convex program

$$y_k = \arg\min\{\rho_k f_{[k]_N}(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}.$$

Step 2. Solve the strongly convex program

$$z_k = \arg\min\{\rho_k f_{[k]_N}(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}.$$

Step 3. Compute

$$\begin{aligned} t_k &= \alpha_k x_k + (1 - \alpha_k)T_{[k]_M} x_k, \\ u_k &= \beta_k t_k + (1 - \beta_k)T_{[k]_M} z_k. \end{aligned}$$

Step 4. Construct two closed convex subsets of C

$$C_k = \{x \in C : \|x - u_k\| \leq \|x - x_k\|\},$$

$$Q_k = \{x \in C : \langle x_0 - x_k, x - x_k\rangle \leq 0\}.$$

Step 5. The next approximation x_{k+1} is defined as the projection of x_0 onto $C_k \cap Q_k$, i.e.,

$$x_{k+1} = P_{C_k \cap Q_k}(x_0).$$

Step 6. Put $k := k + 1$ and go to **Step 1**.

Before going to prove the strong convergence of the CHEM Algorithm, we guarantee the well-definedness of the constructed sequence by the following lemma.

Lemma 3.1. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by CHEM Algorithm is well-defined.*

Proof. To get the conclusion, it suffices to show that $C_k \cap Q_k$ is a nonempty closed convex subset of H, for each $k \in \mathbb{N} \cup \{0\}$. First, we will assert the non-emptiness by showing that $S \subset C_k \cap Q_k$, for each $k \in \mathbb{N} \cup \{0\}$.

Let $k \in \mathbb{N} \cup \{0\}$ be fixed and let $q \in S$. Then, by Lemma 2.8 (ii), we have

$$\|z_k - q\|^2 \leq \|x_k - q\|^2 - (1 - 2\rho_k L_1)\|x_k - y_k\|^2 - (1 - 2\rho_k L_2)\|y_k - z_k\|^2.$$

This implies that

$$\|z_k - q\| \leq \|x_k - q\|. \tag{3.1}$$

Since for each $j \in \{1, 2, \ldots, M\}$, we also have $q \in Fix(T_j)$, it follows from the quasi-nonexpansivity of each T_j that

$$\begin{aligned} \|t_k - q\| &\leq \alpha_k\|x_k - q\| + (1 - \alpha_k)\|T_{[k]_M} x_k - q\| \\ &\leq \alpha_k\|x_k - q\| + (1 - \alpha_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \|u_k - q\| &\leq \beta_k\|t_k - q\| + (1 - \beta_k)\|T_{[k]_M} z_k - q\| \\ &\leq \beta_k\|t_k - q\| + (1 - \beta_k)\|z_k - q\|. \end{aligned}$$

Thus, in view of (3.1) and (3.2), we get

$$\begin{aligned} \|u_k - q\| &\leq \beta_k\|x_k - q\| + (1 - \beta_k)\|x_k - q\| \\ &= \|x_k - q\|. \end{aligned} \tag{3.3}$$

Using this relation, in view of the definition of C_k, we see that $q \in C_k$. Since $k \in \mathbb{N} \cup \{0\}$ is arbitrary, we can conclude that $S \subset C_k$, for each $k \in \mathbb{N} \cup \{0\}$.

Next, we will show that $S \subset Q_k$, for each $k \in \mathbb{N} \cup \{0\}$, by induction. Let $q \in S$. It is obvious $S \subset Q_0 = C$. Now, suppose that $S \subset Q_k$. Observe that, since $x_{k+1} = P_{C_k \cap Q_k}(x_0)$, by Lemma 2.9 (ii), we have

$$\langle x_0 - x_{k+1}, x - x_{k+1}\rangle \leq 0, \; \forall x \in C_k \cap Q_k.$$

It follows that

$$\langle x_0 - x_{k+1}, q - x_{k+1}\rangle \leq 0, \forall q \in S.$$

This implies that $q \in Q_{k+1}$, and so $S \subset Q_{k+1}$. Thus, by induction, we conclude that $S \subset Q_k$, for each $k \in \mathbb{N} \cup \{0\}$. Then, since S is a nonempty set, it follows that $C_k \cap Q_k$ is a nonempty closed convex subset, for each $k \in \mathbb{N} \cup \{0\}$. Consequently, we can guarantee that $\{x_k\}$ is well-defined. □

Now, we are ready to prove the strong convergence theorem of the sequence $\{x_k\}$ which is generated by the CHEM Algorithm.

Theorem 3.2. *If the solution set S is nonempty, then the sequence $\{x_k\}$ which is generated by CHEM Algorithm converges strongly to $P_S(x_0)$.*

Proof. Let $q \in S$ be picked. By the definition of Q_k and Lemma 2.9 (ii), we observe that $x_k = P_{Q_k}(x_0)$, for each $k \in \mathbb{N} \cup \{0\}$. Thus, since $S \subset Q_k$, we have

$$\|x_k - x_0\| \leq \|q - x_0\|, \tag{3.4}$$

for each $k \in \mathbb{N} \cup \{0\}$. This implies that the sequence $\{x_k\}$ is bounded. Thus, by the relations (3.1), (3.2), and (3.3), we have $\{z_k\}$, $\{t_k\}$, and $\{u_k\}$ are also bounded.

Next, consider,

$$\begin{aligned}
\|x_{k+1} - x_k\|^2 &= \|x_{k+1} - x_0\|^2 + \|x_0 - x_k\|^2 + 2\langle x_{k+1} - x_0, x_0 - x_k\rangle \\
&= \|x_{k+1} - x_0\|^2 + \|x_0 - x_k\|^2 + 2\langle x_{k+1} - x_k, x_0 - x_k\rangle - 2\|x_0 - x_k\|^2 \\
&= \|x_{k+1} - x_0\|^2 - \|x_0 - x_k\|^2 + 2\langle x_{k+1} - x_k, x_0 - x_k\rangle, \qquad (3.5)
\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Note that, since $x_k = P_{Q_k}(x_0)$ and $x_{k+1} \in Q_k$, we have

$$\langle x_{k+1} - x_k, x_0 - x_k\rangle \leq 0,$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, from (3.5), we have

$$\|x_{k+1} - x_k\|^2 \leq \|x_{k+1} - x_0\|^2 - \|x_0 - x_k\|^2, \tag{3.6}$$

for each $k \in \mathbb{N} \cup \{0\}$. This implies that

$$\|x_k - x_0\| \leq \|x_{k+1} - x_0\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. This means that $\{\|x_k - x_0\|\}$ is a nondecreasing sequence. Consequently, by using this one together with the boundness property of $\{\|x_k - x_0\|\}$, we can conclude that $\{\|x_k - x_0\|\}$ is a convergent sequence. Thus, in view of (3.6), we also have

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.7}$$

By the definition of C_k and $x_{k+1} \in C_k$, we see that

$$\|x_{k+1} - u_k\| \leq \|x_{k+1} - x_k\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned}
\|u_k - x_k\| &\leq \|u_k - x_{k+1}\| + \|x_{k+1} - x_k\| \\
&\leq \|x_{k+1} - x_k\| + \|x_{k+1} - x_k\| \\
&= 2\|x_{k+1} - x_k\|,
\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, by applying (3.7) to the above inequality, we get

$$\lim_{k\to\infty} \|u_k - x_k\| = 0. \tag{3.8}$$

Next, for each $j \in \{1, 2, \dots, M\}$, by (3.2) and the quasi-nonexpansivity of T_j, we see that

$$\begin{aligned} \|u_k - q\|^2 &= \|\beta_k(t_k - q) + (1-\beta_k)(T_{[k]_M} z_k - q)\|^2 \\ &= \beta_k\|t_k - q\|^2 + (1-\beta_k)\|T_{[k]_M} z_k - q\|^2 - \beta_k(1-\beta_k)\|t_k - T_{[k]_M} z_k\|^2 \\ &\le \beta_k\|t_k - q\|^2 + (1-\beta_k)\|T_{[k]_M} z_k - q\|^2, \\ &\le \beta_k\|x_k - q\|^2 + (1-\beta_k)\|z_k - q\|^2, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. So, by applying Lemma 2.8 (ii) to the vector z_k, we have

$$\begin{aligned} \|u_k - q\|^2 &\le \beta_k\|x_k - q\|^2 + (1-\beta_k)[\|x_k - q\|^2 - (1-2\rho_k L_1)\|x_k - y_k\|^2 \\ &\quad -(1-2\rho_k L_2)\|y_k - z_k\|^2] \\ &\le \|x_k - q\|^2 - (1-\beta_k)[(1-2\rho_k L_1)\|x_k - y_k\|^2 \\ &\quad +(1-2\rho_k L_2)\|y_k - z_k\|^2], \end{aligned} \tag{3.9}$$

for each $k \in \mathbb{N} \cup \{0\}$. This means

$$\begin{aligned} (1-\beta_k)[(1-2\rho_k L_1)\|x_k - y_k\|^2 + (1-2\rho_k L_2)\|y_k - z_k\|^2] &\le \|x_k - u_k\|(\|x_k - q\| \\ &\quad +\|u_k - q\|), \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by (3.8) and the properties of the control sequences $\{\beta_k\}$, $\{\rho_k\}$, we obtain

$$\lim_{k\to\infty} \|x_k - y_k\| = 0, \tag{3.10}$$

and

$$\lim_{k\to\infty} \|y_k - z_k\| = 0. \tag{3.11}$$

These imply that

$$\lim_{k\to\infty} \|x_k - z_k\| = 0. \tag{3.12}$$

Using this one together with (3.7), we have

$$\lim_{k\to\infty} \|z_{k+1} - z_k\| = 0. \tag{3.13}$$

Now, for each fixed $j \in \{1, 2, \dots, M\}$, we consider

$$\|z_{k+j} - z_k\| \le \|z_{k+j} - z_{k+(j-1)}\| + \|z_{k+(j-1)} - z_{k+(j-2)}\| + \dots + \|z_{k+1} - z_k\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, by using (3.13), we have

$$\lim_{k\to\infty} \|z_{k+j} - z_k\| = 0, \tag{3.14}$$

for each fixed $j \in \{1, 2, \dots, M\}$.

From the definition of u_k, we see that

$$(1-\beta_k)\|T_{[k]_M} z_k - z_k\| = \|u_k - z_k - \beta_k(t_k - z_k)\|$$

$$
\begin{aligned}
&\leq \|u_k - z_k\| + \beta_k \|t_k - z_k\| \\
&\leq \|u_k - x_k\| + \beta_k \|t_k - x_k\| + (1+\beta_k)\|x_k - z_k\| \\
&= \|u_k - x_k\| + \beta_k \|\alpha_k x_k + (1-\alpha_k) T_{[k]_M} x_k - x_k\| \\
&\quad +(1+\beta_k)\|x_k - z_k\| \\
&= \|u_k - x_k\| + \beta_k (1-\alpha_k)\|x_k - T_{[k]_M} x_k\| \\
&\quad +(1+\beta_k)\|x_k - z_k\|,
\end{aligned}
$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by using the assumption on $\{\alpha_k\}$ together with (3.8), and (3.12), we get

$$\lim_{k\to\infty} \|T_{[k]_M} z_k - z_k\| = 0. \tag{3.15}$$

Next, since $\{x_k\}$ is a bounded sequence, we can find a subsequence $\{x_{k_m}\}$ of $\{x_k\}$ and $p \in H$ such that $x_{k_m} \rightharpoonup p$, as $m \to \infty$. We now show that $p \in S$.

First, we show that $p \in \cap_{j=1}^{M} Fix(T_j)$. We know that, by using (3.12), the subsequence $\{z_{k_m}\}$ of $\{z_k\}$ also weakly converges to p. This together with (3.14), for each $j \in \{1, 2, \ldots, M\}$, we have $z_{k_m+j} \rightharpoonup p$, as $m \to \infty$.

Now, let $j \in \{1, 2, \ldots, M\}$ be fixed. For $m = 0$, we see that there is $\triangle_0^j \in \{1, 2, \ldots, M\}$ such that $[k_0 + \triangle_0^j]_M = j$. Put $r_0^j = k_0 + \triangle_0^j$. Again, for $m \geq 1$, there is $\triangle_m^j \in \{1, 2, \ldots, M\}$ such that $[k_m + \triangle_m^j]_M = j$. Put $r_m^j = \min A_{m-1}^j$, where $A_{m-1}^j = \{k_l + \triangle_l^j : k_l + \triangle_l^j > r_{m-1}^j \text{ and } l > m-1\}$. Then, for each $j \in \{1, 2, \ldots, M\}$, we can choose a subsequence $\{r_m^j\}$ such that $[r_m^j]_M = j$, and $z_{r_m^j} \rightharpoonup p$, as $m \to \infty$. This together with (3.15) implies that

$$0 = \lim_{m\to\infty} \|T_{[r_m^j]_M} z_{r_m^j} - z_{r_m^j}\| = \lim_{m\to\infty} \|T_j z_{r_m^j} - z_{r_m^j}\|, \tag{3.16}$$

for each $j \in \{1, 2, \ldots, M\}$. Combining with $z_{r_m^j} \rightharpoonup p$, as $m \to \infty$, by the demiclosedness at 0 of $I - T_j$, implies that

$$T_j p = p,$$

for each $j = 1, 2, \ldots, M$.

Next, we show that $p \in \cap_{i=1}^{N} EP(f_i, C)$. Similarly, by using (3.7), for each fixed $i \in \{1, 2, \ldots, N\}$, we get that $\lim_{k\to\infty} \|x_{k+i} - x_k\| = 0$. It follows from $x_{k_m} \rightharpoonup p$, as $m \to \infty$, that for each $i \in \{1, 2, \ldots, N\}$, we have $x_{k_m+i} \rightharpoonup p$, as $m \to \infty$. Then, for each $i \in \{1, 2, \ldots, N\}$, we can choose a subsequence $\{r_n^i\}$ such that $[r_n^i]_N = i$, and $x_{r_n^i} \rightharpoonup p$, as $n \to \infty$. This together with (3.10) implies that for each $i \in \{1, 2, \ldots, N\}$, we obtain $y_{r_n^i} \rightharpoonup p$, as $n \to \infty$. By Lemma 2.8 (i), for each $i \in \{1, 2, \ldots, N\}$, we have

$$\rho_{r_n^i}[f_{[r_n^i]_N}(x_{r_n^i}, y) - f_{[r_n^i]_N}(x_{r_n^i}, y_{r_n^i})] \geq \langle y_{r_n^i} - x_{r_n^i}, y_{r_n^i} - y\rangle, \forall y \in C.$$

This implies that, for each $i \in \{1, 2, \ldots, N\}$, we have

$$f_{[r_n^i]_N}(x_{r_n^i}, y) - f_{[r_n^i]_N}(x_{r_n^i}, y_{r_n^i}) \geq -\frac{1}{\rho_{r_n^i}} \|y_{r_n^i} - x_{r_n^i}\| \|y_{r_n^i} - y\|, \forall y \in C.$$

By using (3.10) and the weak continuity of each f_i $(i \in \{1, 2, \dots, N\})$, we obtain that

$$f_i(p, y) \geq 0, \forall y \in C,$$

for each $i = 1, 2, \dots, N$. Then, we had shown that $p \in S$, and so $\omega_w(x_k) \subset S$.

Finally, we show that the sequence $\{x_k\}$ converges strongly to $P_S(x_0)$.

In fact, since $x_k = P_{Q_k}(x_0)$, it follows from $P_S(x_0) \in S \subset Q_k$ that

$$\|x_k - x_0\| \leq \|P_S(x_0) - x_0\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by Lemma 2.10, we can conclude that the sequence $\{x_k\}$ converges strongly to $P_S(x_0)$. This completes the proof. □

Next, we consider the parallel type method as following:

Parallel Hybrid Extragradient Method (PHEM)

Initialization. Choose parameters $\{\rho_k^i\}$ with $0 < \inf \rho_k^i \leq \sup \rho_k^i < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $i = 1, 2, \dots, N$, $\{\alpha_k\} \subset [0, 1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, and $\{\beta_k\} \subset [0, 1)$ with $0 \leq \inf \beta_k \leq \sup \beta_k < 1$. Pick $x_0 \in C$.

Step 1. Solve N strongly convex programs

$$y_k^i = \arg\min\{\rho_k^i f_i(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, i = 1, 2, \dots, N.$$

Step 2. Solve N strongly convex programs

$$z_k^i = \arg\min\{\rho_k^i f_i(y_k^i, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, i = 1, 2, \dots, N.$$

Step 3. Find the farthest element from x_k among z_k^i, $i = 1, 2, \dots, N$, i.e.,

$$\overline{z}_k = \arg\max\{\|z_k^i - x_k\| : i = 1, 2, \dots, N\}.$$

Step 4. Compute

$$t_k^j = \alpha_k x_k + (1 - \alpha_k)T_j x_k, j = 1, 2, \dots, M,$$
$$u_k^j = \beta_k t_k^j + (1 - \beta_k)T_j \overline{z}_k, j = 1, 2, \dots, M.$$

Step 5. Find the farthest element from x_k among u_k^j, $j = 1, 2, \dots, M$, i.e.,

$$\overline{u}_k = \arg\max\{\|u_k^j - x_k\| : j = 1, 2, \dots, M\}.$$

Step 6. Construct two closed convex subsets of C

$$C_k = \{x \in C : \|x - \overline{u}_k\| \leq \|x - x_k\|\},$$
$$Q_k = \{x \in C : \langle x_0 - x_k, x - x_k\rangle \leq 0\}.$$

Step 7. The next approximation x_{k+1} is defined as the projection of x_0 onto $C_k \cap Q_k$, i.e.,

$$x_{k+1} = P_{C_k \cap Q_k}(x_0).$$

Step 8. Put $k := k + 1$ and go to **Step 1**.

Theorem 3.3. *Suppose that the solution set S is nonempty. Then, the sequence $\{x_k\}$ which is generated by PHEM Algorithm converges strongly to $P_S(x_0)$.*

Proof. Let $q \in S$. By the definition of $\overline{z}_k$, we suppose that $i_k \in \{1, 2, \ldots, N\}$ such that $z_k^{i_k} = \overline{z}_k = \arg\max\{\|z_k^i - x_k\| : i = 1, 2, \ldots, N\}$. Then, by Lemma 2.8 (ii), we have

$$\|\overline{z}_k - q\|^2 \leq \|x_k - q\|^2 - (1 - 2\rho_k^{i_k} L_1)\|x_k - y_k^{i_k}\|^2 - (1 - 2\rho_k^{i_k} L_2)\|y_k^{i_k} - \overline{z}_k\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. This implies that

$$\|\overline{z}_k - q\| \leq \|x_k - q\|, \tag{3.17}$$

for each $k \in \mathbb{N} \cup \{0\}$. Since for each $j \in \{1, 2, \ldots, M\}$, we also have $q \in Fix(T_j)$, it follows from the quasi-nonexpansivity of each T_j that

$$\begin{aligned} \|t_k^j - q\| &\leq \alpha_k \|x_k - q\| + (1 - \alpha_k)\|T_j x_k - q\| \\ &\leq \alpha_k \|x_k - q\| + (1 - \alpha_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{3.18}$$

for each $k \in \mathbb{N} \cup \{0\}$. Besides, by the definition of $\overline{u}_k$, we suppose that $j_k \in \{1, 2, \ldots, M\}$ such that $u_k^{j_k} = \overline{u}_k = \arg\max\{\|u_k^j - x_k\| : j = 1, 2, \ldots, M\}$. It follows from the quasi-nonexpansivity of each T_j, $j \in \{1, 2, \ldots, M\}$, that

$$\begin{aligned} \|\overline{u}_k - q\| &\leq \beta_k \|t_k^{j_k} - q\| + (1 - \beta_k)\|T_{j_k}\overline{z}_k - q\| \\ &\leq \beta_k \|t_k^{j_k} - q\| + (1 - \beta_k)\|\overline{z}_k - q\|, \end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, in view of (3.17), and (3.18), we get

$$\begin{aligned} \|\overline{u}_k - q\| &\leq \beta_k \|x_k - q\| + (1 - \beta_k)\|x_k - q\| \\ &= \|x_k - q\|, \end{aligned} \tag{3.19}$$

for each $k \in \mathbb{N} \cup \{0\}$. Following the proof of Lemma 3.1 and Theorem 3.2, we can show that $S \subset C_k \cap Q_k$, for each $k \in \mathbb{N} \cup \{0\}$. Moreover, we can check that the sequence $\{x_k\}$ is bounded, and

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{3.20}$$

By the definition of C_k and $x_{k+1} \in C_k$, we see that

$$\|x_{k+1} - \overline{u}_k\| \leq \|x_{k+1} - x_k\|,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that

$$\begin{aligned} \|\overline{u}_k - x_k\| &\leq \|\overline{u}_k - x_{k+1}\| + \|x_{k+1} - x_k\| \\ &\leq \|x_{k+1} - x_k\| + \|x_{k+1} - x_k\| \\ &= 2\|x_{k+1} - x_k\|, \end{aligned} \tag{3.21}$$

for each $k \in \mathbb{N} \cup \{0\}$. Thus, applying (3.20) to the above inequality, we get

$$\lim_{k\to\infty} \|\overline{u}_k - x_k\| = 0.$$

From the definition of $\overline{u}_k$, we have

$$\lim_{k\to\infty} \|u_k^j - x_k\| = 0, \tag{3.22}$$

for each $j = 1, 2, \ldots, M$.

Next, for each $j = 1, 2, \ldots, M$, by (3.18) and the quasi-nonexpansivity of T_j, we see that

$$\begin{aligned}
\|u_k^j - q\|^2 &= \|\beta_k(t_k^j - q) + (1-\beta_k)(T_j\overline{z}_k - q)\|^2 \\
&= \beta_k\|t_k^j - q\|^2 + (1-\beta_k)\|T_j\overline{z}_k - q\|^2 - \beta_k(1-\beta_k)\|t_k^j - T_j\overline{z}_k\|^2 \\
&\leq \beta_k\|t_k^j - q\|^2 + (1-\beta_k)\|T_j\overline{z}_k - q\|^2 \\
&\leq \beta_k\|x_k - q\|^2 + (1-\beta_k)\|\overline{z}_k - q\|^2,
\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. So, by applying Lemma 2.8 (ii) to the vector $\overline{z}_k$, we have

$$\begin{aligned}
\|u_k^j - q\|^2 &\leq \beta_k\|x_k - q\|^2 + (1-\beta_k)[\|x_k - q\|^2 - (1-2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 \\
&\quad -(1-2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2] \\
&= \|x_k - q\|^2 - (1-\beta_k)[(1-2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 \\
&\quad +(1-2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2],
\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. This means

$$\begin{aligned}
(1-\beta_k)[(1-2\rho_k^{i_k}L_1)\|x_k - y_k^{i_k}\|^2 + (1-2\rho_k^{i_k}L_2)\|y_k^{i_k} - \overline{z}_k\|^2] &\leq \|x_k - u_k^j\|(\|x_k - q\| \\
&\quad +\|u_k^j - q\|),
\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by (3.22) and the properties of the control sequences $\{\beta_k\}$, $\{\rho_k^i\}$, we obtain

$$\lim_{k\to\infty} \|x_k - y_k^{i_k}\| = 0, \tag{3.23}$$

and

$$\lim_{k\to\infty} \|y_k^{i_k} - \overline{z}_k\| = 0. \tag{3.24}$$

These imply that

$$\lim_{k\to\infty} \|x_k - \overline{z}_k\| = 0. \tag{3.25}$$

Then, by the definition of $\overline{z}_k$, we have

$$\lim_{k\to\infty} \|x_k - z_k^i\| = 0, \tag{3.26}$$

for each $i = 1, 2, \ldots, N$. Moreover, by Lemma 2.8 (ii), for each $i = 1, 2, \ldots, N$, we get that

$$\|z_k^i - q\|^2 \leq \|x_k - q\|^2 - (1-2\rho_k^i L_1)\|x_k - y_k^i\|^2 - (1-2\rho_k^i L_2)\|y_k^i - z_k^i\|^2,$$

for each $k \in \mathbb{N} \cup \{0\}$. It follows that, for each $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
(1-2\rho_k^i L_1)\|x_k - y_k^i\|^2 + (1-2\rho_k^i L_2)\|y_k^i - z_k^i\|^2 &\leq \|x_k - q\|^2 - \|z_k^i - q\|^2 \\
&= \|x_k - z_k^i\|(\|x_k - q\| + \|z_k^i - q\|),
\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. This together with (3.26) implies that

$$\lim_{k\to\infty} \|x_k - y_k^i\| = 0, \tag{3.27}$$

and

$$\lim_{k\to\infty} \|y_k^i - z_k^i\| = 0, \tag{3.28}$$

for each $i = 1, 2, \ldots, N$. From the definition of u_k^j, for each $j = 1, 2, \ldots, M$, we see that

$$\begin{aligned}(1-\beta_k)\|T_j\overline{z}_k - \overline{z}_k\| &= \|u_k^j - \overline{z}_k - \beta_k(t_k^j - \overline{z}_k)\| \\ &\leq \|u_k^j - \overline{z}_k\| + \beta_k\|t_k^j - \overline{z}_k\| \\ &\leq \|u_k^j - x_k\| + \beta_k\|t_k^j - x_k\| + (1+\beta_k)\|x_k - \overline{z}_k\| \\ &= \|u_k^j - x_k\| + \beta_k\|\alpha_k x_k + (1-\alpha_k)T_j x_k - x_k\| \\ &\quad +(1+\beta_k)\|x_k - \overline{z}_k\| \\ &= \|u_k^j - x_k\| + \beta_k(1-\alpha_k)\|x_k - T_j x_k\| + (1+\beta_k)\|x_k - \overline{z}_k\|,\end{aligned}$$

for each $k \in \mathbb{N} \cup \{0\}$. Then, by using the assumption on $\{\alpha_k\}$ together with (3.22) and (3.25), we get

$$\lim_{k\to\infty} \|T_j\overline{z}_k - \overline{z}_k\| = 0, \tag{3.29}$$

for each $j = 1, 2, \ldots, M$.

Next, since $\{x_k\}$ is a bounded sequence, we can find a subsequence $\{x_{k_m}\}$ of $\{x_k\}$ and $p \in H$ such that $x_{k_m} \rightharpoonup p$, as $m \to \infty$. We now show that $p \in S$.

We know that, by using (3.25), the subsequence $\{\overline{z}_{k_m}\}$ of $\{\overline{z}_k\}$ also weakly converges to p. This together with (3.29), by the demiclosedness at 0 of $I - T_j$, implies that

$$T_j p = p,$$

for each $j = 1, 2, \ldots, M$.

On the other hand, by using (3.27), for each $i \in \{1, 2, \ldots, N\}$, we get that $y_{k_m}^i \rightharpoonup p$, as $m \to \infty$. Thus, by Lemma 2.8 (i), for each $i \in \{1, 2, \ldots, N\}$, we have

$$\rho_{k_m}^i[f_i(x_{k_m}, y) - f_i(x_{k_m}, y_{k_m}^i)] \geq \langle y_{k_m}^i - x_{k_m}, y_{k_m}^i - y\rangle, \forall y \in C.$$

This implies that, for each $i = 1, 2, \ldots, N$, we get

$$f_i(x_{k_m}, y) - f_i(x_{k_m}, y_{k_m}^i) \geq -\frac{1}{\rho_{k_m}^i}\|y_{k_m}^i - x_{k_m}\|\|y_{k_m}^i - y\|, \forall y \in C.$$

It follows from (3.27) and the weak continuity of each f_i ($i \in \{1, 2, \ldots, N\}$) that

$$f_i(p, y) \geq 0, \forall y \in C,$$

for each $i = 1, 2, \ldots, N$. Then, we had shown that $p \in S$, and so $\omega_w(x_k) \subset S$.

The rest of the proof is similar to the arguments in the proof of Theorem 3.2, and it leads to the conclusion that the sequence $\{x_k\}$ converges strongly to $P_S(x_0)$. $\square$

Remark 3.4. We observe that if $\alpha_k = 1$, for each $k \in \mathbb{N} \cup \{0\}$, then the PHEM Algorithm reduces to the PHMEM Algorithm, which was presented in [20]. We point out that, by Remark 2.2, we know that the class of quasi-nonexpansive mapping is larger than the class of nonexpansive mapping. The PHEM Algorithm can solve quasi-nonexpansive mappings meanwhile the PHMEM Algorithm may not be applied in this situation.

The next result is an improvement version of Algorithm (1.7) in the reference [15]. Notice that, in this paper, we consider the class of quasi-nonexpansive mapping while in [15] the authors considerd the class of symmetric generalized hybrid mapping.

Corollary 3.5. *Let T be a quasi-nonexpansive self-mapping on C with $I-T$ demi-closed at 0 and let f be a bifunction satisfies the assumptions $(A1)-(A4)$. Suppose that the solution set $S = EP(f,C) \cap Fix(T)$ is nonempty. Pick $x_0 \in C$, choose parameters $\{\rho_k\}$ with $0 < \inf \rho_k \leq \sup \rho_k < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0,1]$ such that $\lim_{k\to\infty} \alpha_k = 1$, $\{\beta_k\} \subset [0,1)$ with $0 \leq \inf \beta_k \leq \sup \beta_k < 1$, and the sequences $\{x_k\}$, $\{y_k\}$, $\{z_k\}$, $\{t_k\}$, $\{u_k\}$ are defined by*

$$
\begin{cases}
y_k = \arg\min\{\rho_k f(x_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\
z_k = \arg\min\{\rho_k f(y_k, y) + \frac{1}{2}\|y - x_k\|^2 : y \in C\}, \\
t_k = \alpha_k x_k + (1-\alpha_k) T x_k, \\
u_k = \beta_k t_k + (1-\beta_k) T z_k, \\
C_k = \{x \in C : \|x - u_k\| \leq \|x - x_k\|\}, \\
Q_k = \{x \in C : \langle x_0 - x_k, x - x_k \rangle \leq 0\}, \\
x_{k+1} = P_{C_k \cap Q_k}(x_0).
\end{cases}
\tag{3.30}
$$

Then, the sequence $\{x_k\}$ converges strongly to $P_S(x_0)$.

From now on, the algorithm (3.30) will be called Hybrid Extragradient Method (HEM).

4. Numerical experiments

In this section, we consider some examples and numerical results to support the main theorems. Additionally, we will compare the two introduced algorithms, CHEM and PHEM, with the PHMEM Algorithm, which was presented in [20]. In the case $M = N = 1$, we will compare the HEM Algorithm (3.30) with the algorithm that was presented in [19]. The numerical experiments are written in Matlab R2015b and performed on a Desktop with AMD Dual Core R3-2200U CPU @ 2.50GHz and RAM 4.00 GB.

Example 4.1. Consider a real Hilbert space $H = \mathbb{R}^n$, and $C = H$. The bifunctions f_i, $i = 1, 2, \ldots, N$, which are given by the form of Nash-Cournot equilibrium model

[42], are defined by

$$f_i(x, y) = \langle P_i x + Q_i y, y - x\rangle,\ \forall x, y \in \mathbb{R}^n,\ i = 1, 2, \ldots, N,$$

where $P_i \in \mathbb{R}^{n\times n}$, and $Q_i \in \mathbb{R}^{n\times n}$ are symmetric positive semidefinite matrices such that $P_i - Q_i$ are also positive semidefinite matrices. We know that the bifunctions f_i, $i = 1, 2, \ldots, N$, satisfy conditions $(A1) - (A4)$, see [42]. Notice that the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipshitz-type continuous with constants $L_1^i = L_2^i = \frac{1}{2}\|P_i - Q_i\|$. Choose $L_1 = L_2 = \max\{L_1^i : i = 1, 2, \ldots, N\}$. Then, the bifunctions f_i, $i = 1, 2, \ldots, N$, are Lipshitz-type continuous with constants L_1 and L_2. On the other hand, for the boxes D_j, $j = 1, 2, \ldots, M$, which are given by

$$D_j = \{x \in \mathbb{R}^n : -d_j \le x_l \le d_j, \forall l = 1, 2, \ldots, n\},\ j = 1, 2, \ldots, M,$$

where d_j are the positive real numbers, we will consider the nonexpansive mappings T_j, $j = 1, 2, \ldots, M$, which are defined by

$$T_j = P_{D_j},\ j = 1, 2, \ldots, M.$$

The numerical experiment is considered under the following setting: for each $i = 1, 2, \ldots, N$, the matrices P_i, and Q_i are randomly chosen from the interval $[-5, 5]$ such that they satisfy the above required properties. Besides, for each $j = 1, 2, \ldots, M$, the real numbers d_j are randomly chosen from the interval $(0, 3)$. We will concern with these parameters: $\rho_k = \frac{0.49}{L_1}$, for the CHEM Algorithm, and $\rho_k^i = \frac{0.49}{L_1}$, $i = 1, 2, \ldots, N$, for the PHEM Algorithm, when $n = 10$, $N = 10$, and $M = 20$. The following five cases of the parameters α_k and β_k are considered:

Case 1. $\alpha_k = 1 - \dfrac{1}{\ln(k+3)}$, $\beta_k = \dfrac{1}{k+2}$.

Case 2. $\alpha_k = 1 - \dfrac{1}{\ln(k+3)}$, $\beta_k = 0.5 + \dfrac{1}{k+3}$.

Case 3. $\alpha_k = 1$, $\beta_k = \dfrac{1}{k+2}$.

Case 4. $\alpha_k = 1$, $\beta_k = 0.5 + \dfrac{1}{k+3}$.

Case 5. $\alpha_k = 1$, $\beta_k = 0$.

The function *quadprog* in Matlab Optimization Toolbox was used to solve vectors y_k, z_k, for the CHEM Algorithm; y_k^i, z_k^i, $i = 1, 2, \ldots, N$, for the PHEM Algorithm. Note that the solution set S is nonempty because of $0 \in S$. The PHMEM Algorithm was tested by using the starting point x_0 as $(1, 1, \ldots, 1)^T \in \mathbb{R}^n$, and the stopping criteria $\|x_{k+1} - x_k\| < 10^{-4}$ for approximating solution $x^* \in S$. After that, the CHEM and PHEM algorithms were tested along with the PHMEM Algorithm by using the starting point x_0 as $(1, 1, \ldots, 1)^T \in \mathbb{R}^n$, and the stopping criteria $\|x_k - x^*\| < 10^{-4}$. Notice that the metric projection of a point x_0 onto the set $C_k \cap Q_k$ was computed by using the explicit formula as in [17].

Table 1 shows that the number of iterations of the PHEM Algorithm in case 1 is better than other all considered cases. Meanwhile, the CPU times of the CHEM Algorithm in case 3 is better than other all considered cases. We would like to

	CPU times (sec)			Number of iterations		
Cases	CHEM	PHEM	PHMEM	CHEM	PHEM	PHMEM
1	147.8594	218.0156	216.6094	13255	5824	6154
2	471.0000	777.5313	818.4531	43265	21811	21577
3	108.0156	216.6094	216.6094	13631	6154	6154
4	343.0000	818.4531	818.4531	42921	21577	21577
5	108.7500	217.9375	-	13481	5925	-

Table 1. The numerical results for five different cases parameters α_k and β_k

remind that we solve y_k^i, z_k^i, $i = 1, 2, \ldots, N$, by using N bifunctions and compute t_k^j, u_k^j, $j = 1, 2, \ldots, M$, by using M mappings for the PHEM Algorithm. On the other hand, we solve only y_k, z_k, by using a bifunction and compute only t_k, u_k, by using a mapping for the CHEM Algorithm. This should be a reason for the results that the number of iterations of the PHEM Algorithm is better than the CHEM Algorithm, while the CPU times of the CHEM Algorithm is better than the PHEM Algorithm in all considered cases.

Example 4.2. In the case $M = N = 1$, we will compare the HEM Algorithm (3.30) with the following algorithm (4.1), which was presented by Hieu [19], when T is a quasi-nonexpansive mapping and f is a pseudomonotone and Lipschitz-type continuous bifunction with positive constants L_1, L_2:

$$\begin{cases} x_0 \in H, \\ y_k = \arg\min\{\rho_k f(x_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ z_k = \arg\min\{\rho_k f(y_k, y) + \frac{1}{2}\|x_k - y\|^2 : y \in C\}, \\ x_{k+1} = (1 - \alpha_k - \beta_k) z_k + \beta_k T z_k, \end{cases} \tag{4.1}$$

where $\{\rho_k\} \subset [\underline{\rho}, \overline{\rho}]$ with $0 < \underline{\rho} \leq \overline{\rho} < \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\{\alpha_k\} \subset [0,1]$ such that $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=1}^{\infty} \alpha_k = +\infty$, and $\{\beta_k\} \subset [\underline{\beta}, \overline{\beta}] \subset (0,1)$, for some $\overline{\beta} > \underline{\beta} > 0$. Hieu [19] proved that the sequence $\{x_k\}$ generated by (4.1) converges strongly to an element in the solution set $S = EP(f, C) \cap Fix(T)$. In this paper, the algorithm (4.1) will be called NH Algorithm.

Consider a real Hilbert space $H = \mathbb{R}^n$, and $C = H$. Recall that the quadratic function $h : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$h(x) = \frac{1}{2} x^T Q x + b^T x,$$

where $b \in \mathbb{R}^n$, $Q \in \mathbb{R}^{n \times n}$ is a symmetric positive semidefinite matrix. Here, we will focus on the case $(I_n - QQ^+)b = 0$, when I_n is the identity matrix, and $Q^+ \in \mathbb{R}^{n\times n}$ is a pseudoinverse matrix of Q. We consider the bifunction f, which is defined by

$$f(x, y) = h(y) - h(x), \ \forall x, y \in \mathbb{R}^n.$$

It is clear that

$$f(x, y) + f(y, x) = 0, \ \forall x, y \in \mathbb{R}^n.$$

Thus, the bifunction f is monotone, and so is pseudomonotone. Moreover, it is easy to see that

$$f(x,y)+f(y,z)-f(x,z)\geq -\|x-y\|^2-\|y-z\|^2,\ \forall x,y,z\in\mathbb{R}^n.$$

Then, the bifunction f is Lipschitz-type continuous with constants $L_1=L_2=1$.

On the other hand, for a convex function $g:\mathbb{R}^n\to\mathbb{R}$ such that there is $x\in\mathbb{R}^n$ satisfied $g(x)\leq 0$, we consider a mapping $T:\mathbb{R}^n\to\mathbb{R}^n$, which is defined by

$$Tx=\begin{cases} x-\frac{g(x)}{\|z_x\|^2}z_x, & \text{if } g(x)>0,\\ x, & \text{otherwise,}\end{cases}$$

where $z_x\in\partial g(x)$. Then we know that T is a quasi-nonexpansive mapping with $I-T$ demiclosed at 0, and $Fix(T)=\{x\in\mathbb{R}^n : g(x)\leq 0\}$, see [4, 22].

The numerical experiment is considered under the following setting: $Q_1\in\mathbb{R}^{n\times n}$ is an orthogonal matrix and its entries are randomly chosen from the interval $(0,5)$. The matrix $Q_2=(q_{ij})\in\mathbb{R}^{n\times n}$ is defined by $q_{ij}=a$, if $i=j=1$; $q_{ij}=0$, otherwise, where the real number a is randomly chosen from the interval $(4,5)$. The positive semidefinite matrix Q is constructed by $Q=Q_1Q_2Q_1^T$. Besides, we consider $g(x)=\max\{0,\langle c,x\rangle+d\}$, where the real number d is randomly chosen from the interval $(-2,-3)$, and the vector $c\in\mathbb{R}^n$ is randomly chosen from the interval $(0,2)$. Note that the solution set S is nonempty because of $-Q^+b\in S$. We will concern with these parameters: $\rho_k=\frac{1}{5}$, $\alpha_k=1-\frac{1}{k+2}$, and $\beta_k=0.5+\frac{1}{k+3}$, when $n=10$. The function *quadprog* in Matlab Optimization Toolbox was used to solve vectors y_k, and z_k. Again, the metric projection of a point x_0 onto the set $C_k\cap Q_k$ was computed by using the explicit formula as in [17]. The HEM Algorithm is compared with the NH Algorithm by using the starting point x_0 as $(0,0,\ldots,0)^T\in\mathbb{R}^n$, and the stopping criteria $\|x_{k+1}-x_k\|<10^{-6}$. The following results were presented as averages calculated from 10 tested problems.

Average CPU Times (sec)		Average Iterations	
HEM	NH	HEM	NH
0.2953	2.1360	91.8	805.3

TABLE 2. The numerical results for $N=1$ and $M=1$

Table 2 shows that the HEM Algorithm yields better both the CPU times and the number of iterations than the NH Algorithm. We notice that, in this experiment, the starting point $x_0:=0\in\mathbb{R}^n$ means that the solution $P_S(0)$ has the minimum norm over the set S. Furthermore, we observe that, if the starting point $x_0\in S$, the HEM Algorithm will be stopped at the iteration x_1, but the NH Algorithm may not.

5. CONCLUSION

We present two algorithms for finding the closest point to the intersection of the set of fixed points of a finite family for quasi-nonexpansive mappings and the solution set of equilibrium problems of a finite family for pseudomonotone bifunctions in a real Hilbert space. We consider both extragradient and hybrid methods together with Ishikawa iterative method for introducing sequence which is strongly convergent to a solution of the considered problems. Some numerical experiments are performed to illustrate the convergence of introduced algorithms and compare them with some appeared algorithms.

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