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APPROXIMATE FIXED POINTS OF NONEXPANSIVE MAPPINGS ON HYPERBOLIC SPACES

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Abstract. In this paper we consider a space of nonexpansive mappings, acting on a closed convex subset of a hyperbolic metric space, which is equipped with the topology of uniform convergence on bounded sets. We show the existence of an open and everywhere dense subset in this space such that every its element possesses an approximate fixed point, which is stable under small perturbations.

1. Introduction

During more than fifty-five years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [3, 5, 11, 13, 14, 17, 18, 19, 23, 24, 25, 26, 29, 30] and the references cited therein. This activity stems from Banach's classical theorem [1] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [2, 4, 6, 7, 8, 9, 10, 12, 15, 16, 20, 27, 28, 29, 30].

In the present paper we consider a space of nonexpansive mappings acting on a closed convex set, which is equipped with the topology of uniform convergence on bounded sets. We show the existence of an open and everywhere dense subset in this space such that every its element possesses an approximate fixed point, which is stable under small perturbations.

As a matter of fact, it turns out that our results also hold for nonexpansive self-mappings of closed and convex sets in complete hyperbolic spaces, an important class of metric spaces the definition of which we now recall.

Let (X, ρ) be a metric space and let R^1 denote the real line. We say that a mapping $c : R^1 \to X$ is a metric embedding of R^1 into X if $\rho(c(s), c(t)) = |s - t|$ for all real s and t. The image of R^1 under a metric embedding will be called a metric line. The image of a real interval $[a, b] = \{t \in R^1 : a \le t \le b\}$ under such a mapping will be called a metric segment.

Assume that (X, ρ) contains a family M of metric lines such that for each pair of distinct points x and y in X there is a unique metric line in M which passes through x and y. This metric line determines a unique metric segment joining x

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and y. We denote this segment by $[x, y]$. For each $0 \le t \le 1$ there is a unique point z in $[x, y]$ such that

$$\rho(x, z) = t\rho(x, y) \text{ and } \rho(z, y) = (1 - t)\rho(x, y).$$

This point will be denoted by $(1 - t)x \oplus ty$. We will say that X, or more precisely (X, ρ, M), is a hyperbolic space if

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}x \oplus \frac{1}{2}z\Big) \le \frac{1}{2}\rho(y, z)$$

for all x, y and z in X. An equivalent requirement is that

$$\rho\Big(\frac{1}{2}x \oplus \frac{1}{2}y, \frac{1}{2}w \oplus \frac{1}{2}z\Big) \le \frac{1}{2}(\rho(x, w) + \rho(y, z))$$

for all x, y, z and w in X. A set $K \subset X$ is called ρ-convex if $[x, y] \subset K$ for all x and y in K.

It is clear that all normed linear spaces are hyperbolic. A discussion of more examples of hyperbolic spaces and in particular of the Hilbert ball can be found, for example, in [21, 22].

2. The main result

Let (X, ρ, M) be a complete hyperbolic space and let K be a nonempty closed ρ-convex subset of X.

For each $x \in K$ and each $r > 0$ set

$$B(x, r) = \{y \in K : \ \rho(x, y) \le r\}.$$

Denote by $\mathcal{A}$ the set of all operators $A : K \to K$ such that

$$\rho(Ax, Ay) \le \rho(x, y) \text{ for all } x, y \in K. \tag{2.1}$$

Fix some $\theta \in K$.

We equip the set $\mathcal{A}$ with the uniformity determined by the base

$$\mathcal{U}(n) = \{(A, B) \in \mathcal{A} \times \mathcal{A} : \ \rho(Ax, Bx) \le n^{-1} \text{ for all } x \in B(\theta, n)\}, \tag{2.2}$$

where n is a natural number. Clearly the uniform space $\mathcal{A}$ is metrizable and complete.

The following theorem is our main result.

Theorem 2.1. *Let $\bar{r} > 0$, $\bar{M} > 0$ and $\epsilon \in (0, 1)$. Then there exists an open everywhere dense subset $\mathcal{F} \subset \mathcal{A}$ such that for each $B \in \mathcal{F}$ there exist $x_B \in K$, a natural number n_B, $\delta_B \in (0, \bar{r})$ and an open neighborhood $\mathcal{U}$ of B in $\mathcal{A}$ such that the following assertion holds.*

Let $C \in \mathcal{U}$, n_1 be a natural number and let a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ be such that

$$\rho(x_0, \theta) \le \bar{M},$$

$$\rho(x_{i+1}, C(x_i)) \le \bar{r} \text{ for all integers } i \ge 0$$

and

$$\rho(x_{i+1}, C(x_i)) \le \delta_B \text{ for all integers } i \ge n_1.$$

Then

$$\rho(x_i, x_B) \le \epsilon \text{ for all integers } i \ge n_1 + n_B.$$

Corollary 2.2. *Let $\bar{r} > 0$, $\bar{M} > 0$ and $\epsilon \in (0, 1)$ and let an open everywhere dense subset $\mathcal{F} \subset \mathcal{A}$ be as guaranteed by Theorem 2.1. Assume that $B \in \mathcal{F}$, a point $x_B \in K$ and an open neighborhood $\mathcal{U}$ of B in $\mathcal{A}$ are as as guaranteed by Theorem 2.1.*

Let $C \in \mathcal{U}$ and a sequence $\{x_i\}_{i=0}^{\infty}$ be such that

$$\rho(x_0, \theta) \le \bar{M},$$

$$\rho(x_{i+1}, C(x_i)) \le \bar{r} \text{ for all integers } i \ge 0$$

and

$$\lim_{i \to \infty} \rho(x_{i+1}, C(x_i)) = 0.$$

Then

$$\rho(x_i, x_B) \le \epsilon \text{ for all sufficiently large natural numbers } i.$$

3. Proof of Theorem 2.1

Let $A \in \mathcal{A}$ and $\gamma \in (0, 1)$. Define

$$A_\gamma(x) = (1 - \gamma)A(x) \oplus \gamma\theta, \ x \in K. \tag{3.1}$$

Clearly, A_γ is a self-mapping of K. By (2.1) and (3.1), for each $x, y \in K$,

$$\begin{aligned} \rho(A_\gamma(x), A_\gamma(y)) &= \rho((1 - \gamma)A(x) \oplus \gamma\theta, (1 - \gamma)A(y) \oplus \gamma\theta) \\ &\le (1 - \gamma)\rho(A(x), A(y)) \le (1 - \gamma)\rho(x, y). \end{aligned} \tag{3.2}$$

In view of (3.2) and the Banach fixed point theorem, there exists a unique point $x(A, \gamma) \in K$ such that

$$A_\gamma(x(A, \gamma)) = x(A, \gamma). \tag{3.3}$$

Choose a positive number $M(A, \gamma)$ such that

$$M(A, \gamma) > 1 + \bar{M} + \rho(\theta, x(A, \gamma)) + (2\bar{r} + 1)\gamma^{-1}. \tag{3.4}$$

Clearly, if $x \in B(\theta, \bar{M})$, then it follows from (3.4) that

$$\begin{aligned} \rho(x, x(A, \gamma)) &\le \rho(x, \theta) + \rho(\theta, x(A, \gamma)) \\ &\le \bar{M} + \rho(\theta, x(A, \gamma)) < M(A, \gamma). \end{aligned}$$

This implies that

$$B(\theta, \bar{M}) \subset B(x(A, \gamma), M(A, \gamma)). \tag{3.5}$$

Assume that

$$x \in B(x(A, \gamma), M(A, \gamma)), \tag{3.6}$$

$$y \in K, \ \rho(y, A_\gamma(x)) \le 2\bar{r}. \tag{3.7}$$

By (3.2), (3.3), (3.4), (3.6) and (3.7),

$$\rho(y, x(A, \gamma)) \le \rho(y, A_\gamma(x)) + \rho(A_\gamma(x), x(A, \gamma))$$

$$\begin{aligned} &\leq 2\bar{r} + \rho(A_\gamma(x), A_\gamma(x(A,\gamma))) \\ &\leq 2\bar{r} + (1-\gamma)\rho(x, x(A,\gamma)) \\ &\leq 2\bar{r} + (1-\gamma)M(A,\gamma) \\ &= M(A,\gamma) + 2\bar{r} - \gamma M(A,\gamma) \\ &< M(A,\gamma). \end{aligned}$$

Thus we have shown that

$$\{y \in K : \text{ there is } x \in B(x(A,\gamma), M(A,\gamma)) \text{ such that } \rho(y, A_\gamma(x)) \leq 2\bar{r}\}$$

$$\subset B(x(A,\gamma), M(A,\gamma)). \tag{3.8}$$

Choose a number $\delta(A,\gamma) \in (0,\bar{r})$ such that

$$\delta(A,\gamma) < 4^{-1}\gamma\epsilon. \tag{3.9}$$

Denote by $\mathcal{U}(A,\gamma)$ an open neighborhood of A_γ in $\mathcal{A}$ such that

$$\mathcal{U}(A,\gamma) \subset \{C \in \mathcal{A} : \rho(C(z), A_\gamma(z)) \leq \delta(A,\gamma)$$

$$\text{for all } z \in B(x(A,\gamma), M(A,\gamma))\}. \tag{3.10}$$

By (2.1) and (3.1), for each $z \in K$,

$$\begin{aligned} \rho(A(z), A_\gamma(z)) &= \rho(A(z), (1-\gamma)A(z) \oplus \gamma\theta) \\ &\leq \gamma\rho(\theta, A(z)) \leq \gamma(\rho(\theta, A(\theta)) + \rho(A(\theta), A(z))) \\ &\leq \gamma(\rho(\theta, A(\theta)) + \rho(\theta, z)). \end{aligned}$$

This implies that for each neighborhood V of A in $\mathcal{A}$ there exists $\gamma_V \in (0,1)$ such that for each $\gamma \in (0,\gamma_V)$, $A_\gamma \in V$. Therefore

$$\{A_\gamma : A \in \mathcal{A}, \gamma \in (0,1)\}$$

is an everywhere dense subset of $\mathcal{A}$.

Set

$$\mathcal{F} = \cup\{\mathcal{U}(A,\gamma) : A \in \mathcal{A}, \gamma \in (0,1)\}. \tag{3.11}$$

Clearly, $\mathcal{F}$ is an open everywhere dense subset of $\mathcal{A}$.

Let

$$B \in \mathcal{F}. \tag{3.12}$$

By (3.11) and (3.12), there exist $A \in \mathcal{A}$ and $\gamma \in (0,1)$ such that

$$B \in \mathcal{U}(A,\gamma). \tag{3.13}$$

Set

$$x_B = x(A,\gamma), \tag{3.14}$$

$$\delta_B = \delta(A,\gamma) \tag{3.15}$$

and choose a natural number n_B such that

$$n_B > 2M(A,\gamma)(\gamma\epsilon)^{-1}. \tag{3.16}$$

Assume that

$$C \in \mathcal{U}(A,\gamma), \tag{3.17}$$

n_1 is a natural number and that a sequence $\{x_i\}_{i=0}^{\infty} \subset K$ satisfies

$$\rho(x_0,\theta) \le \bar{M}, \tag{3.18}$$

$$\rho(x_{i+1}, C(x_i)) \le \bar{r} \text{ for all integers } i \ge 0 \tag{3.19}$$

and

$$\rho(x_{i+1}, C(x_i)) \le \delta_B \text{ for all integers } i \ge n_1. \tag{3.20}$$

In order to complete the proof it is sufficient to show that

$$\rho(x_i, x_B) \le \epsilon \text{ for all integers } i \ge n_1 + n_B.$$

First we show that

$$\rho(x_i, x(A,\gamma)) \le M(A,\gamma) \tag{3.21}$$

for all integers $i \ge 0$. In view of (3.5) and (3.18), inequality (3.21) holds for $i = 0$.

Assume that an integer $j \ge 0$ and (3.21) holds for $i = j$. Thus

$$\rho(x_j, x(A,\gamma)) \le M(A,\gamma). \tag{3.22}$$

By (3.10), (3.17) and (3.22),

$$\rho(C(x_j), A_\gamma(x_j)) \le \delta(A,\gamma) < \bar{r}. \tag{3.23}$$

It follows from (3.2), (3.3), (3.4), (3.19), (3.22) and (3.23) that

$$\begin{aligned}\rho(x_{j+1}, x(A,\gamma)) &\le \rho(x_{j+1}, C(x_j)) + \rho(C(x_j), A_\gamma(x_j)) + \rho(A_\gamma(x_j), A_\gamma x(A,\gamma)) \\ &\le \bar{r} + \bar{r} + (1-\gamma)\rho(x_j, x(A,\gamma)) \\ &\le 2\bar{r} + (1-\gamma)M(A,\gamma) \le M(A,\gamma).\end{aligned}$$

Thus

$$\rho(x_{j+1}, x(A,\gamma)) \le M(A,\gamma)$$

and (3.21) holds for $i = j+1$. Therefore we conclude that (3.21) holds for all integers $i \ge 0$.

We show that there exists an integer $i \in [n_1, n_1 + n_B]$ such that

$$\rho(x_i, x(A,\gamma)) \le \epsilon.$$

Assume the contrary. Then

$$\rho(x_i, x(A,\gamma)) > \epsilon \tag{3.24}$$

for all $i = n_1, \dots, n_1 + n_B$. Assume that an integer i satisfies

$$n_1 \le i \le n_1 + n_B - 1.$$

Then (3.24) holds. By (3.2), (3.3) and (3.20),

$$\begin{aligned}\rho(x_{i+1}, x(A,\gamma)) &\le \rho(x_{i+1}, C(x_i)) + \rho(C(x_i), A_\gamma(x_i)) + \rho(A_\gamma(x_i), x(A,\gamma)) \\ &\le \delta_B + \rho(C(x_i), A_\gamma(x_i)) + \rho(A_\gamma(x_i), A_\gamma(x(A,\gamma))) \\ &\le \delta_B + \rho(C(x_i), A_\gamma(x_i)) + (1-\gamma)\rho(x_i, x(A,\gamma)). \qquad (3.25)\end{aligned}$$

It follows from (3.10), (3.15), (3.17) and (3.21) that

$$\rho(C(x_i), A_\gamma(x_i)) \le \delta(A, \gamma) = \delta_B. \tag{3.26}$$

In view of (3.25) and (3.26),

$$\rho(x_{i+1}, x(A, \gamma)) \le 2\delta_B + (1-\gamma)\rho(x_i, x(A, \gamma)). \tag{3.27}$$

By (3.9), (3.15), (3.24) and (3.27),

$$\begin{aligned}\rho(x_i, x(A,\gamma)) - \rho(x_{i+1}, x(A,\gamma)) &\ge \gamma\rho(x_i, x(A,\gamma)) - 2\delta_B \\ &> \gamma\epsilon - 2\delta(A,\gamma) > \gamma\epsilon/2.\end{aligned}$$

Thus

$$\rho(x_i, x(A,\gamma)) - \rho(x_{i+1}, x(A,\gamma)) > \gamma\epsilon/2$$

for every integer i satisfying $n_1 \le i \le n_1 + n_B - 1$. Together with (3.21) this implies that

$$\begin{aligned}M(A,\gamma) &\ge \rho(x_{n_1}, x(A,\gamma)) - \rho(x_{n_1+n_B}, x(A,\gamma)) \\ &= \sum_{i=n_1}^{n_1+n_B-1} (\rho(x_i, x(A,\gamma)) - \rho(x_{i+1}, x(A,\gamma))) \\ &> n_B\gamma\epsilon/2\end{aligned}$$

and

$$n_B < 2(\gamma\epsilon)^{-1}M(A,\gamma).$$

This contradicts (3.16). The contradiction we have reached proves that there exists an integer

$$j \in [n_1, n_1 + n_B] \tag{3.28}$$

such that

$$\rho(x_j, x(A,\gamma)) \le \epsilon. \tag{3.29}$$

Assume that an integer i satisfies

$$i \ge n_1, \ \rho(x_i, x(A,\gamma)) \le \epsilon. \tag{3.30}$$

We show that

$$\rho(x_{i+1}, x(A,\gamma)) \le \epsilon.$$

By (3.2), (3.3), (3.20) and (3.30),

$$\begin{aligned}\rho(x_{i+1}, x(A,\gamma)) &\le \rho(x_{i+1}, C(x_i)) + \rho(C(x_i), A_\gamma(x_i)) + \rho(A_\gamma(x_i), A_\gamma(x(A,\gamma))) \\ &\le \delta_B + \rho(C(x_i), A_\gamma(x_i)) + (1-\gamma)\rho(x_i, x(A,\gamma)).\end{aligned} \tag{3.31}$$

In view of (3.10), (3.17) and (3.21),

$$\rho(C(x_i), A_\gamma(x_i)) \le \delta(A,\gamma). \tag{3.32}$$

There are two cases:

$$\rho(x_i, x(A,\gamma)) \le \epsilon/2; \tag{3.33}$$

$$\rho(x_i, x(A,\gamma)) > \epsilon/2. \tag{3.34}$$

Assume that (3.33) holds. By (3.9), (3.15), (3.31), (3.32) and (3.33),

$$\rho(x_{i+1}, x(A,\gamma)) \le 2\delta(A,\gamma) + \rho(x_i, x(A,\gamma)) \le 2\delta(A,\gamma) + \epsilon/2 < \epsilon.$$

Assume that (3.34) holds. It follows from (3.9), (3.15), (3.30), (3.31) and (3.32) that

$$\rho(x_{i+1}, x(A,\gamma)) \le 2\delta(A,\gamma) + (1-\gamma)\epsilon < \epsilon.$$

Thus in the both cases

$$\rho(x_{i+1}, x(A,\gamma)) \le \epsilon. \tag{3.35}$$

In view of (3.28) and (3.29), we have shown by induction that for all integers $i \ge j \in [n_1, n_1 + n_B]$, inequality (3.35) holds. This completes the proof of Theorem 2.1.

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