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STRONGLY QUASINONEXPANSIVE MAPPINGS, III

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Abstract. This paper is devoted to the study of strongly quasinonexpansive mappings in a metric-like space. In particular, we give some characterizations of such mappings and show that the class of strongly quasinonexpansive mappings is closed under composition. Then we also study strongly quasinonexpansive mappings with respect to the Bregman distance in a Banach space.

1. Introduction

In 1974, Bruck and Reich [20] introduced a strongly nonexpansive mapping in a Banach space to study firmly nonexpansive mappings in the sense of Bruck [21]. They showed that the composition of strongly nonexpansive mappings is strongly nonexpansive, and that the fixed point set of the composition of strongly nonexpansive mappings which have a common fixed point is equal to the common fixed point set of the mappings.

After that, various kinds of strongly nonexpansive mappings were observed in several papers. For example, Bruck [19] introduced a strongly quasinonexpansive mapping in a metric space; Reich [26] studied a strongly nonexpansive mapping with respect to the set of asymptotic fixed points of the mapping in a Banach space; the authors [13–15] dealt with some properties and applications of a mapping of type (sr) which is a strongly-nonexpansive-type mapping with respect to the fixed point set and the nonnegative real-valued function ϕ on $E \times E$ defined by $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in E$, where E is a smooth Banach space and J is the normalized duality mapping on E; see also [3, 10–12, 25]. Moreover, some strong nonexpansiveness for a sequence of mappings and their applications to fixed point problems have also been investigated; see [2, 4–9, 16, 17] for more details.

Recently, a strongly quasinonexpansive mapping in an abstract space has been introduced and studied in [18]. Such a mapping is a generalization of mappings of type (sr) in the sense of [13–15] in a Banach space and strongly quasinonexpansive mappings in the sense of [3] in a metric space. However, the mapping is different from a strongly nonexpansive mapping due to Reich [26].

In this paper, in order to unify these mappings as above, we first devote the study of a strongly quasinonexpansive mapping in a metric-like space. In particular, we give some characterizations of such mappings, and moreover, we show that the composition of strongly quasinonexpansive mappings is strongly quasinonexpansive. Then, using these results, we obtain characterizations and properties of strongly

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quasinonexpansive mappings with respect to the Bregman distance and the set of asymptotic fixed points of the mappings in a Banach space.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers, $\mathbb{N}$ the set of positive integers, E a real Banach space, E^* the dual of E, $\|\cdot\|$ the norms of E or E^*, and $\langle x, x^*\rangle$ the value of $x^* \in E^*$ at $x \in E$. Strong convergence of a sequence $\{x_n\}$ in E to $x \in E$ is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Let $f\colon E \to \mathbb{R}$ be a function. We say that f is *convex* if $f\bigl(\lambda x + (1-\lambda)y\bigr) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $\lambda \in [0,1]$ and $x, y \in E$; f is *strictly convex* if $f\bigl(\lambda x + (1-\lambda)y\bigr) < \lambda f(x) + (1-\lambda)f(y)$ for all $\lambda \in (0,1)$ and $x, y \in E$ with $x \neq y$; f is *Gâteaux differentiable* at $x \in E$ if there exists $\xi^* \in E^*$ such that

$$\lim_{t\to 0} \frac{f(x+ty) - f(x)}{t} = \langle y, \xi^* \rangle \tag{2.1}$$

for all $y \in E$. In this case, ξ^* is called the *Gâteaux differential* of f at x and denoted by $\nabla f(x)$. We say that f is *Gâteaux differentiable* if f is Gâteaux differentiable at all $x \in E$; f is *bounded on bounded sets* if $\sup\{|f(x)| : x \in C\} < \infty$ for all bounded subsets C of E.

We know the following; see [22, Propositions 1.1.7, 1.1.10, and 1.1.11].

Lemma 2.1. *Let E be a Banach space, $f\colon E \to \mathbb{R}$ a continuous, convex, and Gâteaux differentiable function, and $D\colon E \times E \to \mathbb{R}$ a function defined by*

$$D(y,x) = f(y) - f(x) - \langle y - x, \nabla f(x)\rangle$$

for all $x, y \in E$. Then $D(y,x) \geq 0$ and

$$D(x,z) = D(x,y) + D(y,z) + \langle x - y, \nabla f(y) - \nabla f(z)\rangle \tag{2.2}$$

for all $x, y, z \in E$. Moreover, the following hold:

- *If f is strictly convex, then*

$$D(x,y) = 0 \Rightarrow x = y \tag{2.3}$$

  for all $x, y \in E$;
- *if f is bounded on bounded sets, then $\{\nabla f(x)\colon x \in C\}$ is bounded in E^* for all bounded subsets C of E.*

We say that the function D in Lemma 2.1 is the *Bregman distance* [22,23] corresponding to f.

Using Lemma 2.1, we obtain the following:

Lemma 2.2. *Let E be a Banach space, $f\colon E \to \mathbb{R}$ a continuous, convex, and Gâteaux differentiable function, $\{x_n\}$ a bounded sequence in E, and z a point in E. Suppose that f is bounded on bounded sets. Then there exists $M > 0$ such that $D(z, x_n) \leq M$ for all $n \in \mathbb{N}$, where D is the Bregman distance corresponding to f.*

Proof. Suppose that for any $m \in \mathbb{N}$ there exists $\tau(m) \in \mathbb{N}$ such that $D(z, x_{\tau(m)}) > m$. Since $\{x_n\}$ is bounded, it follows that $\{x_{\tau(m)} : m \in \mathbb{N}\}$ is bounded. Hence $\{f(x_{\tau(m)}) : m \in \mathbb{N}\}$ and $\{\nabla f(x_{\tau(m)}) : m \in \mathbb{N}\}$ are bounded by assumption and by Lemma 2.1, respectively. On the other hand, it follows from the definition of D that

$$m < D(z, x_{\tau(m)}) \leq |f(z)| + \left|f(x_{\tau(m)})\right| + \left\|z - x_{\tau(m)}\right\| \left\|\nabla f(x_{\tau(m)})\right\|$$

for all $m \in \mathbb{N}$, which is a contradiction. This completes the proof. $\square$

The following theorem is a direct consequence of [1, Theorem 1.8].

Theorem 2.3. *Let E be a Banach space and $f : E \to \mathbb{R}$ a Gâteaux differentiable function. Then*

$$|f(x) - f(y)| \leq \sup\left\{\left\|\nabla f\big(\lambda x + (1-\lambda)y\big)\right\| : \lambda \in [0,1]\right\} \|x - y\|$$

for all $x, y \in E$.

Using Lemma 2.1 and Theorem 2.3, we obtain the following:

Lemma 2.4. *Let E be a Banach space, C a nonempty bounded subset of E, and $f : E \to \mathbb{R}$ a continuous, convex, and Gâteaux differentiable function. Suppose that f is bounded on bounded sets. Then f is uniformly continuous on C.*

Proof. Since C is bounded, there exists $r > 0$ such that $C \subset B_r$, where $B_r = \{y \in E : \|y\| < r\}$. Then B_r is bounded and convex. Let $\{x_n\}$ and $\{y_n\}$ be sequences in C such that $\|x_n - y_n\| \to 0$. It is enough to show that $|f(x_n) - f(y_n)| \to 0$. Lemma 2.1 implies that $\{\nabla f(z) : z \in B_r\}$ is bounded. Thus it follows from Theorem 2.3 that

$$\begin{aligned} |f(x_n) - f(y_n)| &\leq \sup\left\{\left\|\nabla f\big(\lambda x_n + (1-\lambda)y_n\big)\right\| : \lambda \in [0,1]\right\} \|x_n - y_n\| \\ &\leq \sup\{\|\nabla f(z)\| : z \in B_r\} \|x_n - y_n\| \to 0 \end{aligned}$$

as $n \to \infty$. $\square$

Let $f : E \to \mathbb{R}$ be a function and C a nonempty convex subset of E. We say that f is *uniformly convex* on C [28] if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$f\big(\lambda x + (1-\lambda)y\big) \leq \lambda f(x) + (1-\lambda) f(y) - \lambda(1-\lambda)\delta$$

for all $\lambda \in [0,1]$ and $x, y \in C$ with $\|x - y\| \geq \epsilon$. It is clear that if f is uniformly convex on C, then f is convex on C. We say that f is *uniformly convex on bounded sets* if f is uniformly convex on all bounded convex subsets of E.

The following lemma is a direct consequence of [24, Lemma 3.1].

Lemma 2.5. *Let E be a Banach space and $f : E \to \mathbb{R}$ a continuous Gâteaux differentiable function. Suppose that f is uniformly convex on bounded sets. If both $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E and $D(x_n, y_n) \to 0$, then $\|x_n - y_n\| \to 0$, where D is the Bregman distance corresponding to f.*

Let $f\colon E \to \mathbb{R}$ be a function and S_E the unit sphere of E, that is, $S_E = \{x \in E\colon \|x\| = 1\}$. We say that f is *Fréchet differentiable* at $x \in E$ if there exists $\xi^* \in E^*$ such that the limit (2.1) is attained uniformly for $y \in S_E$. It is clear that f is Fréchet differentiable at $x \in E$, then f is Gâteaux differential there. Let U be a nonempty open subset of E. We say that a Gâteaux differential function f is *uniformly Fréchet differentiable* on U if the limit

$$\lim_{t\to 0} \frac{f(x+ty)-f(x)}{t} = \langle y, \nabla f(x)\rangle$$

attained uniformly for $x \in U$ and $y \in S_E$.

Using [27, Proposition 2.1], we obtain the following:

Lemma 2.6. *Let E be a reflexive Banach space, U a nonempty bounded open convex subset of E, and $f\colon E \to \mathbb{R}$ a continuous, convex, and Gâteaux differentiable function. Suppose that f is bounded on bounded sets and uniformly Fréchet differentiable on U. Then ∇f is uniformly norm-to-norm continuous on U.*

Let $f\colon E \to \mathbb{R}$ be a function. We say that f is *uniformly Fréchet differentiable on bounded sets* if f is uniformly Fréchet differentiable on all bounded open subsets of E.

Using lemmas above, we obtain the following:

Lemma 2.7. *Let E be a Banach space, $\{x_n\}$ and $\{y_n\}$ bounded sequences in E, and $f\colon E \to \mathbb{R}$ a continuous, convex, and Gâteaux differentiable function. Suppose that $x_n - y_n \to 0$ and f is bounded on bounded sets. Then $D(y_n, x_n) \to 0$, where D is the Bregman distance corresponding to f. Moreover, suppose that E is reflexive and f is uniformly Fréchet differentiable on bounded sets. Then $D(z, x_n) - D(z, y_n) \to 0$ for all $z \in E$.*

Proof. Lemmas 2.1 and 2.4 show that

$$D(y_n, x_n) \le |f(y_n) - f(x_n)| + \|y_n - x_n\|\,\|\nabla f(x_n)\| \to 0$$

as $n \to \infty$. Moreover, let $z \in E$ and suppose that E is reflexive and f is uniformly Fréchet differentiable on bounded sets. Then Lemma 2.6 implies that $\|\nabla f(y_n) - \nabla f(x_n)\| \to 0$. Therefore it follows from (2.2) that

$$|D(z, x_n) - D(z, y_n)| \le D(y_n, x_n) + \|z - y_n\|\,\|\nabla f(y_n) - \nabla f(x_n)\| \to 0$$

as $n \to \infty$. □

Using [18, Lemma 3.2], we obtain the following:

Lemma 2.8. *Let K be a nonempty set and both f and g functions of K into $\mathbb{R}_+$. Then the following are equivalent:*

(1) *For any $\epsilon > 0$ there exists $\delta > 0$ such that $x \in K$ and $g(x) < \delta$ imply $f(x) < \epsilon$;*
(2) *$f(x_n) \to 0$ whenever $\{x_n\}$ is a sequence in K and $g(x_n) \to 0$.*

Moreover, suppose that f and g are bounded above. Then (1) *or* (2) *is equivalent to the following:*

(3) *there exists a nondecreasing bounded function* $\gamma\colon [0,\alpha] \to \mathbb{R}_+$ *such that* $\gamma\big(f(x)\big) \le g(x)$ *for all* $x \in K$ *and* $\gamma(t) > 0$ *for all* $t \in (0,\alpha]$*, where* $\alpha = \sup\{f(x)\colon x \in K\}$.

Proof. It is not hard to verify that (1) and (2) are equivalent. The equivalence between (1) and (3) follows from [18, Lemma 3.2]. □

Let C be a nonempty subset of E and $T\colon C \to E$ a mapping. A point $p \in E$ is said to be an *asymptotic fixed point* of T [26] if there exists a sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ and $x_n \rightharpoonup p$. The set of asymptotic fixed points of T is denoted by $\hat{\mathrm{F}}(T)$. It is clear that $\mathrm{F}(T) \subset \hat{\mathrm{F}}(T)$, where $\mathrm{F}(T)$ is the fixed point set of T.

Remark 2.9. In [26], the notion of an asymptotic fixed point of a mapping $T\colon C \to E$ is studied when C is a convex subset of a Banach space E.

3. Strongly quasinonexpansive mappings in a metric-like space

In this section, we introduce a quasinonexpansive mapping and a strongly quasinonexpansive mapping in a metric-like space. Then we give some characterizations of a strongly quasinonexpansive mapping, and moreover, we show that the composition of two strongly quasinonexpansive mappings is also strongly quasinonexpansive.

Throughout this section, X denotes a nonempty set, σ a function of $X \times X$ into $\mathbb{R}_+$, and $\bar{B}(z,M)$ a subset of X defined by

$$\bar{B}(z,M) = \{x \in X\colon \sigma(z,x) \le M\}$$

for $z \in X$ and $M > 0$.

We deal with the following three conditions: We say that the pair (X,σ) satisfies the *condition (S)* if

$$x \neq y \Leftrightarrow \sigma(x,y) > 0$$

for all $x, y \in X$; (X,σ) satisfies the *condition (B)* if

$$\sup\{\sigma(x,y)\colon x, y \in \bar{B}(z,M)\} < \infty$$

for all $z \in X$ and $M > 0$; (X,σ) satisfies the *condition (T)* if for any $u \in X$, $M > 0$, and $\epsilon > 0$ there exists $\eta > 0$ such that

$$x, y, z \in \bar{B}(u,M),\ \sigma(x,y) < \eta,\ \sigma(y,z) < \eta \Rightarrow \sigma(x,z) < \epsilon; \tag{3.1}$$

see [18] for more information about these conditions. It is clear that if (X,σ) satisfies the condition (S), then $z \in \bar{B}(z,M)$ and hence $\bar{B}(z,M)$ is nonempty for all $z \in X$ and $M > 0$.

Remark 3.1. Suppose that σ is a metric on X, that is, (X,σ) is a metric space. Then it is obvious that (X,σ) satisfies the conditions (S), (B), and (T).

Lemma 3.2. *Suppose that the pair* (X,σ) *satisfies the condition (S). The following are equivalent:*

(1) (X,σ) *satisfies the condition (T);*

(2) *$\sigma(x_n, z_n) \to 0$ whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in $\bar{B}(u, M)$ for some $u \in X$ and $M > 0$ such that $\sigma(x_n, y_n) \to 0$ and $\sigma(y_n, z_n) \to 0$.*

Proof. Let $u \in X$ and $M > 0$ be given and set $K = \left[\bar{B}(u, M)\right]^3$. By virtue of the condition (S), we see that K is nonempty. Let f and g be functions of K into $\mathbb{R}_+$ defined by

$$f(x, y, z) = \sigma(x, z) \text{ and } g(x, y, z) = \sigma(x, y) + \sigma(y, z)$$

for $(x, y, z) \in K$. Using Lemma 2.8, we can get the conclusion. □

Let C and F be nonempty subsets of X and $T\colon C \to X$ a mapping. Inspired by [3,10,13,14,18,19], we introduce the following: T is said to be *quasinonexpansive with respect to* (σ, F) if $\sigma(z, Tx) \le \sigma(z, x)$ for all $z \in F$ and $x \in C$; T is said to be *strongly quasinonexpansive with respect to* (σ, F) if for any $\epsilon > 0$, $z \in F$, and $M > 0$ with $C \cap \bar{B}(z, M) \ne \emptyset$ there exists $\delta > 0$ such that

$$x \in C \cap \bar{B}(z, M),\ \sigma(z, x) - \sigma(z, Tx) < \delta \Rightarrow \sigma(Tx, x) < \epsilon.$$

In the rest of this section, the phrase "with respect to (σ, F)" will often be omitted if no possible confusion arises.

Example 3.3. Let C be a nonempty subset of X and $T\colon C \to X$ a mapping with a fixed point. Suppose that T is strongly quasinonexpansive with respect to $\big(\sigma, \mathrm{F}(T)\big)$, where $\mathrm{F}(T)$ is the fixed point set of T. Then we know the following:

- If σ is a metric on X, then T is strongly quasinonexpansive in the sense of [3];
- if X is a smooth Banach space and σ is defined by $\sigma(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for $x, y \in X$, then T is of type (sr) in the sense of [9,13,14], where J is the duality mapping of E.

A strongly quasinonexpansive mapping is quasinonexpansive as follows:

Lemma 3.4. *Let C and F be nonempty subsets of X and $T\colon C \to X$ a strongly quasinonexpansive mapping with respect to (σ, F). Suppose that (X, σ) satisfies the condition (S). Then T is quasinonexpansive with respect to (σ, F).*

Proof. Suppose that there exist $z \in F$ and $y \in C$ such that $\sigma(z, Ty) > \sigma(z, y)$. Then it is clear that $Ty \ne y$. Set $M = \sigma(z, y) + 1$ and $\epsilon = \sigma(Ty, y)$. We see that $M > 0$, $\epsilon > 0$, $y \in C \cap \bar{B}(z, M)$, and $\sigma(z, y) - \sigma(z, Ty) < 0$. Since T is strongly quasinonexpansive, we have $\sigma(Ty, y) < \epsilon$, which is a contradiction. Therefore, T is quasinonexpansive with respect to (σ, F). □

To prove the next theorem, we need the following lemmas:

Lemma 3.5. *Let C and F be nonempty subsets of X, $T\colon C \to X$ a quasinonexpansive mapping with respect to (σ, F), $z \in F$, and $M > 0$. Suppose that $K = C \cap \bar{B}(z, M) \ne \emptyset$. Let f and g be functions defined by*

$$f(x) = \sigma(Tx, x) \text{ and } g(x) = \sigma(z, x) - \sigma(z, Tx) \tag{3.2}$$

for $x \in K$. Then $f(K) \subset \mathbb{R}_+$, $g(K) \subset \mathbb{R}_+$, and g is bounded. Moreover, if (X, σ) satisfies the condition (B), then f is bounded.

Proof. By definition, $f(K) \subset \mathbb{R}_+$ is clear. Since T is quasinonexpansive, it follows that $0 \le g(x) \le \sigma(z, x) \le M$ for all $x \in K$. Hence g is bounded and $g(K) \subset \mathbb{R}_+$. Suppose that (X, σ) satisfies the condition (B). Since $Tx \in \bar{B}(z, M)$ for all $x \in K$, we have

$$\sup\{f(x) \colon x \in K\} \le \sup\{\sigma(y, x) \colon x, y \in \bar{B}(z, M)\} < \infty.$$

Therefore, f is bounded. □

Lemma 3.6. *Let C and F be nonempty subsets of X and $T \colon C \to X$ a mapping. Suppose that (X, σ) satisfies the condition (S). Then the following are equivalent:*

(1) *T is strongly quasinonexpansive with respect to (σ, F);*
(2) *T is quasinonexpansive with respect to (σ, F), and $\sigma(Tx_n, x_n) \to 0$ whenever $\{x_n\}$ is a sequence in $C \cap \bar{B}(z, M)$ and $\sigma(z, x_n) - \sigma(z, Tx_n) \to 0$ for some $z \in F$ and $M > 0$ with $C \cap \bar{B}(z, M) \neq \emptyset$.*

Proof. Let $z \in F$ and $M > 0$ be given. Suppose that $K = C \cap \bar{B}(z, M) \neq \emptyset$. Let f and g be functions defined by (3.2) for $x \in K$. Then Lemma 3.5 shows that $f(K) \subset \mathbb{R}_+$, and that $g(K) \subset \mathbb{R}_+$ if T is quasinonexpansive with respect to (σ, F). Therefore the conclusion follows from Lemmas 2.8 and 3.4. □

Using Lemmas 2.8, 3.5, and 3.6, we obtain the following characterizations of strongly quasinonexpansive mappings; see [18, Theorems 4.4 and 4.6] and [3, Theorem 3.7].

Theorem 3.7. *Let C and F be nonempty subsets of X and $T \colon C \to X$ a mapping. Suppose that (X, σ) satisfies the conditions (S) and (B). Then the following are equivalent:*

(1) *T is strongly quasinonexpansive with respect to (σ, F);*
(2) *T is quasinonexpansive with respect to (σ, F), and $\sigma(Tx_n, x_n) \to 0$ whenever $\{x_n\}$ is a sequence in $C \cap \bar{B}(z, M)$ and $\sigma(z, x_n) - \sigma(z, Tx_n) \to 0$ for some $z \in F$ and $M > 0$ with $C \cap \bar{B}(z, M) \neq \emptyset$;*
(3) *for any $z \in F$ and $M > 0$ with $C \cap \bar{B}(z, M) \neq \emptyset$ there exists a nondecreasing bounded function $\gamma \colon [0, \alpha] \to \mathbb{R}_+$ such that $\gamma(t) > 0$ for all $t \in (0, \alpha]$ and*
$$\gamma\big(\sigma(Tx, x)\big) \le \sigma(z, x) - \sigma(z, Tx)$$
for all $x \in K$, where $K = C \cap \bar{B}(z, M)$ and $\alpha = \sup\{\sigma(Tx, x) \colon x \in K\}$.

Proof. The equivalence between (1) and (2) follows from Lemma 3.6. Thus it is enough to show the equivalence between (1) and (3). Let $z \in F$ and $M > 0$ be given. Suppose that $K = C \cap \bar{B}(z, M) \neq \emptyset$. Let f and g be functions defined by (3.2) for $x \in K$. Lemma 3.5 shows that f and g are bounded functions of K into $\mathbb{R}_+$. Therefore Lemma 2.8 implies the equivalence between (1) and (3). □

We know that the class of strongly quasinonexpansive mappings in a metric space is closed under composition [3, Theorem 3.6]; see also [20, Proposition 1.1] and [18, Theorem 4.9]. The class of strongly quasinonexpansive mappings discussed in this section has a similar property as follows:

Theorem 3.8. *Let C_1, C_2, F_1, and F_2 be nonempty subsets of X, $T_1 \colon C_1 \to X$ a strongly quasinonexpansive mapping with respect to (σ, F_1), and $T_2 \colon C_2 \to X$ a strongly quasinonexpansive mapping with respect to (σ, F_2). Suppose that $F_1 \cap F_2 \neq \emptyset$, $T_1(C_1) \subset C_2$, and (X, σ) satisfies the conditions (S) and (T). Then T_2T_1 is strongly quasinonexpansive with respect to $(\sigma, F_1 \cap F_2)$.*

Proof. Let $\epsilon > 0$, $u \in F_1 \cap F_2$, and $M > 0$ be given. Suppose that $C_1 \cap \bar{B}(u, M) \neq \emptyset$. Since T_1 and T_2 are quasinonexpansive by Lemma 3.4 and $u \in F_1 \cap F_2$, we see that

$$\sigma(u, T_2T_1x) \leq \sigma(u, T_1x) \leq \sigma(u, x) \leq M \tag{3.3}$$

for all $x \in C_1 \cap \bar{B}(u, M)$. Thus $T_1x \in C_2 \cap \bar{B}(u, M)$ for all $x \in C_1 \cap \bar{B}(u, M)$, and hence $C_2 \cap \bar{B}(u, M) \neq \emptyset$. By the condition (T), there exists $\eta > 0$ such that (3.1) holds. Since $u \in F_1 \cap F_2$ and both T_1 and T_2 are strongly quasinonexpansive, there exists $\delta > 0$ such that

$$x \in C_1 \cap \bar{B}(u, M),\ \sigma(u, x) - \sigma(u, T_1x) < \delta \Rightarrow \sigma(T_1x, x) < \eta \tag{3.4}$$

and

$$y \in C_2 \cap \bar{B}(u, M),\ \sigma(u, y) - \sigma(u, T_2y) < \delta \Rightarrow \sigma(T_2y, y) < \eta. \tag{3.5}$$

Suppose that $x \in C_1 \cap \bar{B}(u, M)$ and $\sigma(u, x) - \sigma(u, T_2T_1x) < \delta$. It is enough to show that $\sigma(T_2T_1x, x) < \epsilon$. Taking into account (3.3), we have

$$\sigma(u, x) - \sigma(u, T_1x) < \delta \text{ and } \sigma(u, T_1x) - \sigma(u, T_2T_1x) < \delta.$$

Therefore it follows from (3.5) and (3.4) that $\sigma(T_2T_1x, T_1x) < \eta$ and $\sigma(T_1x, x) < \eta$. Thus, by virtue of the condition (T), we conclude that $\sigma(T_2T_1x, x) < \epsilon$. ☐

Using Theorem 3.8, we obtain the following:

Corollary 3.9 ([18, Theorem 4.9 (2)]). *Let C_1 and C_2 be nonempty subsets of X and both $T_1 \colon C_1 \to X$ and $T_2 \colon C_2 \to X$ mappings such that $\mathrm{F}(T_1) \cap \mathrm{F}(T_2) \neq \emptyset$ and $T_1(C_1) \subset C_2$. Suppose that (X, σ) satisfies the conditions (S) and (T), and that T_1 and T_2 are strongly quasinonexpansive with respect to $\big(\sigma, \mathrm{F}(T_1)\big)$ and $\big(\sigma, \mathrm{F}(T_2)\big)$, respectively. Then T_2T_1 is strongly quasinonexpansive with respect to $\big(\sigma, \mathrm{F}(T_2T_1)\big)$.*

Proof. From [18, Theorem 4.9 (1)], we know that $\mathrm{F}(T_1) \cap \mathrm{F}(T_2) = \mathrm{F}(T_2T_1)$. Thus Theorem 3.8 implies the conclusion. ☐

4. Strongly quasinonexpansive mappings in a Banach space

In this section, we apply the results of the previous section to the study of strongly quasinonexpansive mappings in a Banach space.

In what follows, we assume the following:

- E is a reflexive Banach space;
- $f \colon E \to \mathbb{R}$ is a continuous, strictly convex, and Gâteaux differentiable function;
- f is bounded on bounded sets;
- D is the Bregman distance corresponding to f, that is, $D(y, x) = f(y) - f(x) - \langle y - x, \nabla f(x) \rangle$ for all $x, y \in E$.

We also assume the following condition:

$$\bar{B}(z, M) = \{x \in E \colon D(z, x) \le M\} \tag{A}$$

is bounded for all $z \in E$ and $M > 0$.

Taking into account (2.3), we know that the pair (E, D) satisfies the condition (S), that is, $D(x, y) = 0 \Leftrightarrow x = y$ for all $x, y \in E$. We also know the following:

Lemma 4.1. *The pair (E, D) satisfies the condition (B), that is,*

$$\sup\{D(x, y) \colon x, y \in \bar{B}(z, M)\} < \infty$$

for all $z \in E$ and $M > 0$. Moreover, if f is uniformly convex on bounded sets, then (E, D) satisfies the condition (T), that is, $D(x_n, z_n) \to 0$ whenever $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are sequences in $\bar{B}(u, M)$ for some $u \in X$ and $M > 0$ such that $D(x_n, y_n) \to 0$ and $D(y_n, z_n) \to 0$.

Proof. Let $z \in E$ and $M > 0$ be given. Then, by assumption, $\bar{B}(z, M)$ and $\{|f(x) - f(y)| : x, y \in \bar{B}(z, M)\}$ are bounded. Moreover, Lemma 2.1 shows that $\{\|\nabla f(y)\| : y \in \bar{B}(z, M)\}$ is also bounded. On the other hand, it follows from the definition of D that

$$D(x, y) \le |f(x) - f(y)| + \|x - y\| \, \|\nabla f(y)\|$$

for all $x, y \in \bar{B}(z, M)$. Therefore, (E, D) satisfies the condition (B).

We next suppose that f is uniformly convex on bounded sets. Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be sequences in $\bar{B}(u, M)$ for some $u \in E$ and $M > 0$ such that $D(x_n, y_n) \to 0$ and $D(y_n, z_n) \to 0$. It is enough to show that $D(x_n, z_n) \to 0$. Since $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ are bounded by the assumption (A), we deduce that $\|x_n - y_n\| \to 0$ by Lemma 2.5 and $\{\nabla f(y_n) - \nabla f(z_n)\}$ is bounded by Lemma 2.1. Using (2.2), we conclude that

$$D(x_n, z_n) \le D(x_n, y_n) + D(y_n, z_n) + \|x_n - y_n\| \, \|\nabla f(y_n) - \nabla f(z_n)\| \to 0$$

as $n \to 0$. □

Let C be a nonempty subset of E and $T \colon C \to E$ a mapping. Recall that a point $p \in E$ is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ and $x_n \rightharpoonup p$. The set of asymptotic fixed points of T is denoted by $\hat{\mathrm{F}}(T)$. From now on we assume that $\hat{\mathrm{F}}(T)$ is nonempty. Recall that a mapping T is said to be *quasinonexpansive with respect to* $(D, \hat{\mathrm{F}}(T))$ if $D(z, Tx) \le D(z, x)$ for all $z \in \hat{\mathrm{F}}(T)$ and $x \in C$; T is said to be *strongly quasinonexpansive with respect to* $(D, \hat{\mathrm{F}}(T))$ if for any $\epsilon > 0$, $z \in \hat{\mathrm{F}}(T)$, and $M > 0$ with $C \cap \bar{B}(z, M) \neq \emptyset$ there exists $\delta > 0$ such that

$$x \in C \cap \bar{B}(z, M), \; D(z, x) - D(z, Tx) < \delta \Rightarrow D(Tx, x) < \epsilon.$$

We often omit the phrase "with respect to $(D, \hat{\mathrm{F}}(T))$" if no possible confusion arises.

Using Theorem 3.7 and Lemma 4.1, we obtain the following characterizations of strongly quasinonexpansive mappings.

Theorem 4.2. *Let C be a nonempty subset of E and $T\colon C \to E$ a mapping such that $\hat{\mathrm{F}}(T)$ is nonempty. Then the following are equivalent:*

1. *T is strongly quasinonexpansive with respect to $(D, \hat{\mathrm{F}}(T))$;*
2. *T is quasinonexpansive with respect to $(D, \hat{\mathrm{F}}(T))$ and $D(Tx_n, x_n) \to 0$ whenever $\{x_n\}$ is a sequence in $C \cap \bar{B}(z, M)$ and $D(z, x_n) - D(z, Tx_n) \to 0$ for some $z \in \hat{\mathrm{F}}(T)$ and $M > 0$ with $C \cap \bar{B}(z, M) \neq \emptyset$;*
3. *for any $z \in \hat{\mathrm{F}}(T)$ and $M > 0$ with $C \cap \bar{B}(z, M) \neq \emptyset$ there exists a nondecreasing bounded function $\gamma\colon [0, \alpha] \to \mathbb{R}_+$ such that $\gamma(t) > 0$ for all $t \in (0, \alpha]$ and*
$$\gamma\big(D(Tx, x)\big) \leq D(z, x) - D(z, Tx)$$
for all $x \in K$, where $K = C \cap \bar{B}(z, M)$ and $\alpha = \sup\{D(Tx, x)\colon x \in K\}$.

Remark 4.3. Under the assumptions of Theorem 4.2, by virtue of Lemma 2.2 and the condition (A), we can check that $\{x_n\}$ is a bounded sequence in C if and only if $\{x_n\}$ is a sequence in $C \cap \bar{B}(z, M)$ for some $z \in E$ and $M > 0$. Therefore a mapping T which satisfies the condition (2) in Theorem 4.2 is strongly nonexpansive in the sense of [26].

To prove the next theorem, we need the following; see [26, Lemma 1] and [18, Theorem 4.9 (1)].

Lemma 4.4. *Let C_1 and C_2 be nonempty subsets of E, $T_1\colon C_1 \to E$ a quasinonexpansive mapping with respect to $\big(D, \hat{\mathrm{F}}(T_1)\big)$, and $T_2\colon C_2 \to E$ a quasinonexpansive mapping with respect to $\big(D, \hat{\mathrm{F}}(T_2)\big)$. Suppose that f is both uniformly convex and uniformly Fréchet differentiable on bounded sets, $T_1(C_1) \subset C_2$, and both $\hat{\mathrm{F}}(T_1) \cap \hat{\mathrm{F}}(T_2)$ and $\hat{\mathrm{F}}(T_2T_1)$ are nonempty. If T_1 is strongly quasinonexpansive with respect to $\big(D, \hat{\mathrm{F}}(T_1)\big)$* or *$T_2$ is strongly quasinonexpansive with respect to $\big(D, \hat{\mathrm{F}}(T_2)\big)$, then $\hat{\mathrm{F}}(T_2T_1) \subset \hat{\mathrm{F}}(T_1) \cap \hat{\mathrm{F}}(T_2)$.*

Proof. Let $z \in \hat{\mathrm{F}}(T_2T_1)$ and $w \in \hat{\mathrm{F}}(T_1) \cap \hat{\mathrm{F}}(T_2)$ be given. Then there exists a sequence $\{x_n\}$ in C_1 such that $x_n \rightharpoonup z$ and $\|x_n - T_2T_1x_n\| \to 0$. Since $\{x_n\}$ is bounded, Lemma 2.2 implies that there exists $M > 0$ such that $x_n \in \bar{B}(w, M)$ for all $n \in \mathbb{N}$. Moreover, Lemma 2.7 implies that
$$D(w, x_n) - D(w, T_2T_1x_n) \to 0.$$
Since both T_1 and T_2 are quasinonexpansive and $w \in \hat{\mathrm{F}}(T_1) \cap \hat{\mathrm{F}}(T_2)$, we have
$$D(w, T_2T_1x_n) \leq D(w, T_1x_n) \leq D(w, x_n) \leq M.$$
This shows that $T_1x_n, T_2T_1x_n \in \bar{B}(w, M)$ for all $n \in \mathbb{N}$, and hence $\{T_1x_n\}$ and $\{T_2T_1x_n\}$ are bounded by the condition (A). Moreover, we see that
$$0 \leq D(w, x_n) - D(w, T_1x_n) \leq D(w, x_n) - D(w, T_2T_1x_n),$$
and
$$0 \leq D(w, T_1x_n) - D(w, T_2T_1x_n) \leq D(w, x_n) - D(w, T_2T_1x_n)$$
for all $n \in \mathbb{N}$. As a result, we deduce that
$$D(w, x_n) - D(w, T_1x_n) \to 0 \tag{4.1}$$

and

$$D(w, T_1 x_n) - D(w, T_2 T_1 x_n) \to 0. \tag{4.2}$$

Now suppose that T_1 is strongly quasinonexpansive. Then, according to (4.1) and Theorem 4.2, we see that $D(T_1 x_n, x_n) \to 0$, and hence $\|T_1 x_n - x_n\| \to 0$ by Lemma 2.5. Thus $z \in \hat{\mathrm{F}}(T_1)$, $T_2 T_1 x_n - T_1 x_n = T_2 T_1 x_n - x_n + x_n - T_1 x_n \to 0$, and $T_1 x_n = T_1 x_n - x_n + x_n \rightharpoonup z$. Hence $z \in \hat{\mathrm{F}}(T_2)$. Consequently, $z \in \hat{\mathrm{F}}(T_1) \cap \hat{\mathrm{F}}(T_2)$. On the other hand, suppose that T_2 is strongly quasinonexpansive. Then it follows from (4.2) and Theorem 4.2 that $D(T_2 T_1 x_n, T_1 x_n) \to 0$, and hence $\|T_2 T_1 x_n - T_1 x_n\| \to 0$ by Lemma 2.5. Thus, $T_1 x_n - x_n = T_1 x_n - T_2 T_1 x_n + T_2 T_1 x_n - x_n \to 0$ and $T_1 x_n = T_1 x_n - x_n + x_n \rightharpoonup z$. Therefore, $z \in \hat{\mathrm{F}}(T_1) \cap \hat{\mathrm{F}}(T_2)$. □

Using Theorem 3.8, Lemmas 4.1, and 4.4, we obtain the following theorem; see [26, Lemma 2].

Theorem 4.5. *In addition to the assumptions of Theorem 4.4, suppose that T_1 and T_2 are strongly quasinonexpansive with respect to $\big(D, \hat{\mathrm{F}}(T_1)\big)$ and $\big(D, \hat{\mathrm{F}}(T_2)\big)$, respectively. Then $T_2 T_1$ is strongly quasinonexpansive with respect to $\big(D, \hat{\mathrm{F}}(T_1) \cap \hat{\mathrm{F}}(T_2)\big)$. In particular, $T_2 T_1$ is strongly quasinonexpansive with respect to $\big(D, \hat{\mathrm{F}}(T_2 T_1)\big)$.*

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