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ATTRACTIVE POINT, WEAK AND STRONG CONVERGENCE THEOREMS FOR GENERIC 2-GENERALIZED HYBRID MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we prove attractive point and fixed point theorems for generic 2-generalized hybrid mappings in a Hilbert space. Next, we obtain weak convergence theorems of Mann's type iteration for finding attractive points and fixed points of the mappings in a Hilbert space. Finally, we prove strong convergence theorems of Halpern's type iteration for finding attractive points and fixed points of the mappings in a Hilbert space.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let T be a mapping from C into H, where C is a nonempty subset of H. The sets of fixed points and *attractive points* [24] of T are denoted by

$$F(T) = \{u \in C : Tu = u\},$$
$$A(T) = \{u \in H : \|Ty - u\| \leq \|y - u\| \text{ for all } y \in C\},$$

respectively. A mapping $T : C \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

For nonexpansive mappings, several approximation methods for finding fixed points have been proposed. The following iteration was introduced by Mann [16] in 1953:

$$x_1 \in C : \text{given},$$
$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) T x_n, \quad \forall n \in \mathbb{N},$$

where $T : C \to C$, $\mathbb{N}$ is the set of natural numbers, and $\{\lambda_n\} \subset [0,1]$. In 1967, Halpern proposed a different type of iteration [3]:

$$x_1 = x \in C : \text{given},$$
$$x_{n+1} = \lambda_n x + (1 - \lambda_n) T x_n, \quad \forall n \in \mathbb{N},$$

where $T : C \to C$, and $\{\lambda_n\} \subset [0,1]$. Mann's and Halpern's iterations yield weak and strong convergence, respectively; see Reich [18] and Wittmann [28].

Required conditions on mappings have been relaxed to include important classes of mappings. In 2010, Kocourek et al. [8] defined a wide class of mappings. A

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mapping $T : C \to H$ is called *generalized hybrid* [8] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta \|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. The class of generalized hybrid mappings simultaneously includes nonexpansive mappings, *nonspreading mappings* [9, 10], *hybrid mappings* [23], and λ-*hybrid mappings* [1] as special cases. The nonspreading mappings are deduced from resolvents of a maximal monotone operator; see [9, 10].

The class of generalized hybrid mappings has been further extended. A mapping $T : C \to C$ is called 2-*generalized hybrid* [17] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\alpha_1 \left\|T^2x - Ty\right\|^2 + \alpha_2 \|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2)\|x - Ty\|^2 \\ \le \beta_1 \left\|T^2x - y\right\|^2 + \beta_2 \|Tx - y\|^2 + (1 - \beta_1 - \beta_2)\|x - y\|^2$$

for all $x, y \in C$. Obviously, a 2-generalized hybrid mapping with $\alpha_1 = \beta_1 = 0$ is generalized hybrid. See Hojo et al. [6] for examples of 2-generalized hybrid mappings that are not generalized hybrid. Very recently, Kondo and Takahashi [13] introduced a class of mappings. A mapping $T : C \to C$ is called *generic* 2-*generalized hybrid* if there exist $\alpha_{ij}, \beta_i, \gamma_i \in \mathbb{R}$ $(i, j = 0, 1, 2)$ such that

$$\begin{aligned} &\alpha_{00}\|x - y\|^2 + \alpha_{01}\|x - Ty\|^2 + \alpha_{02}\|x - T^2y\|^2 \\ &+\alpha_{10}\|Tx - y\|^2 + \alpha_{11}\|Tx - Ty\|^2 + \alpha_{12}\left\|Tx - T^2y\right\|^2 \\ &+\alpha_{20}\left\|T^2x - y\right\|^2 + \alpha_{21}\left\|T^2x - Ty\right\|^2 + \alpha_{22}\left\|T^2x - T^2y\right\|^2 \\ &+\beta_0\|x - Tx\|^2 + \beta_1\left\|Tx - T^2x\right\|^2 + \beta_2\left\|T^2x - x\right\|^2 \\ &+\gamma_0\|y - Ty\|^2 + \gamma_1\left\|Ty - T^2y\right\|^2 + \gamma_2\left\|T^2y - y\right\|^2 \le 0 \end{aligned} \tag{1.1}$$

for all $x, y \in C$. We also refer such a mapping as $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-*generic* 2-*generalized hybrid*. In Theorem 6.1 and 7.1 of this paper, one of the following two conditions is assumed:

$$\begin{aligned} &(1)\ \alpha_{0\bullet} + \alpha_{1\bullet} \ge 0,\ \alpha_{2\bullet} \ge 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \ge 0; \\ &(2)\ \alpha_{\bullet 0} + \alpha_{\bullet 1} \ge 0,\ \alpha_{\bullet 2} \ge 0,\ \alpha_{\bullet 1} > 0,\ \gamma_0, \gamma_1, \gamma_2 \ge 0, \end{aligned} \tag{1.2}$$

where

$$\alpha_{i\bullet} \equiv \alpha_{i0} + \alpha_{i1} + \alpha_{i2} \text{ and } \alpha_{\bullet i} \equiv \alpha_{0i} + \alpha_{1i} + \alpha_{2i} \tag{1.3}$$

for $i = 0, 1, 2$. This type of mappings with (1.2) contains 2-generalized hybrid mappings. Indeed, set

$$\begin{aligned} &(1)\ \alpha_{2i} = 0,\ \alpha_{0\bullet} = -1,\ \alpha_{1\bullet} = 1,\ \beta_i = \gamma_i = 0; \\ &(2)\ \alpha_{i2} = 0,\ \alpha_{\bullet 0} = -1,\ \alpha_{\bullet 1} = 1,\ \beta_i = \gamma_i = 0 \end{aligned} \tag{1.4}$$

for $i = 0, 1, 2$ in (1.1). Then, the condition (1) (resp. (2)) of (1.4) is included by (1) (resp. (2)) of (1.2). Hence, Theorem 6.1 and 7.1 in this paper contain the case with 2-generalized hybrid mappings as special cases. Furthermore, if an $(\alpha_{ij}, \beta_i, \gamma_i;$

$i, j = 0, 1, 2$)-generic 2-generalized hybrid mapping T satisfies one of the following conditions

$$\begin{aligned}(1)\ &\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \alpha_{2i} = 0,\ \alpha_{1\bullet} > 0,\ \beta_i = \gamma_i = 0;\\ (2)\ &\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0,\ \alpha_{i2} = 0,\ \alpha_{\bullet 1} > 0,\ \beta_i = \gamma_i = 0;\end{aligned} \tag{1.5}$$

for $i = 0, 1, 2$, then T is *normally 2-generalized hybrid* [11]; see also [13].

In this paper, we prove attractive point and fixed point theorems for generic 2-generalized hybrid mappings in a Hilbert space. Next, we obtain weak convergence theorems of Mann's type iteration for finding attractive points and fixed points of the mappings in a Hilbert space. Finally, we prove strong convergence theorems of Halpern's type iteration for finding attractive points and fixed points of the mappings in a Hilbert space.

2. Preliminaries

Let H be a real Hilbert space. It is well-known that

$$2\langle x - y,\ y\rangle \leq \|x\|^2 - \|y\|^2 \leq 2\langle x - y,\ x\rangle \tag{2.1}$$

for all $x, y \in H$. It is also known that

$$2\langle x - y,\ z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.2}$$

for all $x, y, z, w \in H$. Let $\{x_n\}$ be a sequence in H, and let $x\,(\in H)$ be a point of H. Strong and weak convergence of $\{x_n\}$ to x are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. We know that a closed and convex subset C of H is weakly closed. For a bounded sequence $\{x_n\}$ in H, $\{x_n\}$ is weakly convergent if and only if every weakly convergent subsequence of $\{x_n\}$ has the same weak limit, that is,

$$x_n \rightharpoonup v \Longleftrightarrow [x_{n_i} \rightharpoonup u \text{ implies that } u = v], \tag{2.3}$$

where $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ and $u, v \in H$.

Let C be a nonempty, closed, and convex subset of H. For any $x \in H$, there exists a unique nearest point $u \in C$, that is, $\|x - u\| = \inf_{z \in C} \|x - z\|$. This correspondence is called the *metric projection* from H onto C, and is denoted by P_C. For the metric projection P_C from H onto C, it holds that

$$\langle x - P_C x,\ P_C x - z\rangle \geq 0 \tag{2.4}$$

for all $x \in H$ and $z \in C$. For more details, see Takahashi [21, 22].

Takahashi and Takeuchi [24] demonstrated that the set $A\,(T)$ of attractive points of a mapping which is defined in Introduction is closed and convex in a Hilbert space. Thus, if $A\,(T)$ is nonempty, the metric projection form H onto $A\,(T)$ is defined without any condition on the mapping T. A mapping $T : C \to H$ with $F\,(T) \neq \emptyset$ is said to be *quasi-nonexpansive* if $\|Tx - u\| \leq \|x - u\|$ for all $x \in C$ and $u \in F(T)$. Itoh and Takahashi [7] proved that the set of fixed points of a quasi-nonexpansive mapping is closed and convex.

Let l^∞ be the Banach space of bounded sequences of real numbers with the supremum norm, and let $(l^\infty)^*$ be its dual space. For $\mu \in (l^\infty)^*$, we denote $\mu\,(\{x_n\})$ by $\mu_n x_n$. A linear continuous functional $\mu \in (l^\infty)^*$ that satisfies the condition

$\mu(\{1,1,1,\cdots\}) = \|\mu\| = 1$ is called a *mean* on l^∞. We know that a mean μ preserves order relations, that is, $x_n \leq y_n$ ($\forall n \in \mathbb{N}$) implies that $\mu_n x_n \leq \mu_n y_n$. When a mean additionally satisfies $\mu_n x_n = \mu_n x_{n+1}$, it is called a *Banach limit* on l^∞. It is well-known that a Banach limit exists. For any $\{x_n\} \in l^\infty$, it holds that $\liminf_{n\to\infty} x_n \leq \mu_n x_n \leq \limsup_{n\to\infty} x_n$. Thus, if $x_n \to a\,(\in \mathbb{R})$, then $\mu_n x_n = a$. For more details, see Takahashi [21].

3. Lemmas

This section presents lemmas that are used in the proofs of the main theorems. The following lemma is utilized to show the existence of attractive points and fixed points in Section 5. See Lin and Takahashi [14] and Takahashi [20].

Lemma 3.1 ([14], [20])**.** *Let μ be a mean on l^∞, and let H be a real Hilbert space. Then, for any bounded sequence $\{x_n\}$ in H, there is a unique element $u \in \overline{co}\{x_n\}$ such that*

$$\mu_n \langle x_n,\ v\rangle = \langle u,\ v\rangle$$

for all $v \in H$, where $\overline{co}\{x_n\}$ is the closure of the convex hull of $\{x_n : n \in \mathbb{N}\}$.

The following lemma was proved by Takahashi and Toyoda [25] by using the parallelogram law. Basing on their proof, we provide an alternative proof.

Lemma 3.2 ([25])**.** *Let A be a nonempty, closed and convex subset of a real Hilbert space H, and let P_A be the metric projection from H onto A. Let $\{x_n\}$ be a sequence such that*

$$\|x_{n+1} - q\| \leq \|x_n - q\| \tag{3.1}$$

for all $q \in A$ and $n \in \mathbb{N}$. Then, the sequence $\{P_A x_n\}$ is convergent in A, in other words, there is an element $\overline{x}$ of A, and $P_A x_n \to \overline{x}$ as $n \to \infty$.

Proof. We show that $\{P_A x_n\}$ is a Cauchy sequence in A. Let $m, n \in \mathbb{N}$ with $m \geq n$. Since $P_A x_n \in A$, it holds from (2.4) that

$$2\langle x_m - P_A x_m,\ P_A x_m - P_A x_n\rangle \geq 0.$$

Using (2.2), we obtain

$$\|x_m - P_A x_n\|^2 - \|x_m - P_A x_m\|^2 - \|P_A x_m - P_A x_n\|^2 \geq 0.$$

From the assumption (3.1), we have that

$$\begin{aligned} \|x_m - P_A x_m\|^2 + \|P_A x_m - P_A x_n\|^2 &\leq \|x_m - P_A x_n\|^2 \\ &\leq \|x_n - P_A x_n\|^2. \end{aligned} \tag{3.2}$$

It follows from (3.2) that

$$\|x_m - P_A x_m\|^2 \leq \|x_n - P_A x_n\|^2$$

for all $m, n \in \mathbb{N}$ such that $m \geq n$. Therefore, $\left\{\|x_n - P_A x_n\|^2\right\}$ is convergent. Furthermore, we have from (3.2) that

$$\|P_A x_m - P_A x_n\|^2 \leq \|x_n - P_A x_n\|^2 - \|x_m - P_A x_m\|^2.$$

Since the right-hand side converges to 0 as $m, n \to \infty$, $\{P_A x_n\}$ is a Cauchy sequence. Since A is complete, $\{P_A x_n\}$ is convergent in A as claimed. □

In the following lemma, the part (a) was proved by Takahashi [22], which contains the parallelogram law as a special case with $a = b = 1/2$. The part (b) was established by Maruyama et al. [17] to deal with 2-generalized hybrid mappings.

Lemma 3.3 ([22], [17]). *Let $x, y, z \in H$ and $a, b, c \in \mathbb{R}$. Then, the following hold:*
(a) *If $a + b = 1$, then $\|ax + by\|^2 = a\|x\|^2 + b\|y\|^2 - ab\|x - y\|^2$.*
(b) *If $a + b + c = 1$, then*

$$\begin{aligned}\|ax + by + cz\|^2 &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 \\ &\quad - ab\|x - y\|^2 - bc\|y - z\|^2 - ca\|z - x\|^2.\end{aligned}$$

The following lemma was proved by Kondo and Takahashi [13].

Lemma 3.4 ([13]). *Let C be a nonempty subset of a real Hilbert space H. Let T be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping from C into itself with $F(T) \neq \emptyset$. Suppose that T satisfies one of the following conditions:*

$$(1)\ \alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0;$$
$$(2)\ \alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0,\ \alpha_{\bullet 2} \geq 0,\ \alpha_{\bullet 1} > 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0,$$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3). *Then, T is quasi-nonexpansive.*

The next lemma will be utilized in the proofs of convergence theorems to points of $F(T^2)$.

Lemma 3.5. *Let C be a nonempty subset of a real Hilbert space H. Let T be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping from C into itself with $F(T^2) \neq \emptyset$. Suppose that T satisfies one of the following conditions:*

$$\begin{aligned}(1)\ &\alpha_{00} + \alpha_{02} + \alpha_{20} + \alpha_{22} \geq 0,\ \alpha_{10} + \alpha_{12} \geq 0,\ \alpha_{01}, \alpha_{11}, \alpha_{21} \geq 0, \\ &\alpha_{20} + \alpha_{22} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0,\ \gamma_0 + \gamma_1 \geq 0;\end{aligned}$$

$$\begin{aligned}(2)\ &\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0,\ \alpha_{01} + \alpha_{21} \geq 0,\ \alpha_{10}, \alpha_{11}, \alpha_{12} \geq 0, \\ &\alpha_{02} + \alpha_{22} > 0,\ \beta_0 + \beta_1 \geq 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0.\end{aligned}$$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3). *Then, T^2 is quasi-nonexpansive.*

Proof. **Case (2)**. Supposed that $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$, $\alpha_{01} + \alpha_{21} \geq 0$, $\alpha_{10}, \alpha_{11}, \alpha_{12} \geq 0$, $\alpha_{02} + \alpha_{22} > 0$, $\beta_0 + \beta_1 \geq 0$, and $\gamma_0, \gamma_1, \gamma_2 \geq 0$. Let $x \in C$ and $u \in F(T^2)$. We show that $\|T^2 x - u\| \leq \|x - u\|$. Since T is $(\alpha_{ij}, \beta_i, \gamma_i;$ $i, j = 0, 1, 2)$-generic 2-generalized hybrid, it holds that

$$\begin{aligned}(3.3)\qquad &\alpha_{00}\|u - x\|^2 + \alpha_{01}\|u - Tx\|^2 + \alpha_{02}\|u - T^2 x\|^2 \\ &+\alpha_{10}\|Tu - x\|^2 + \alpha_{11}\|Tu - Tx\|^2 + \alpha_{12}\|Tu - T^2 x\|^2 \\ &+\alpha_{20}\|T^2 u - x\|^2 + \alpha_{21}\|T^2 u - Tx\|^2 + \alpha_{22}\|T^2 u - T^2 x\|^2 \\ &+\beta_0\|u - Tu\|^2 + \beta_1\|Tu - T^2 u\|^2 + \beta_2\|T^2 u - u\|^2\end{aligned}$$

$$+\gamma_0 \|x - Tx\|^2 + \gamma_1 \left\|Tx - T^2x\right\|^2 + \gamma_2 \left\|T^2x - x\right\|^2 \leq 0.$$

Since $u \in F\left(T^2\right)$, we have that

$$\begin{aligned}
&\alpha_{00}\|u - x\|^2 + \alpha_{01}\|u - Tx\|^2 + \alpha_{02}\|u - T^2x\|^2 \\
&+\alpha_{10}\|Tu - x\|^2 + \alpha_{11}\|Tu - Tx\|^2 + \alpha_{12}\left\|Tu - T^2x\right\|^2 \\
&+\alpha_{20}\|u - x\|^2 + \alpha_{21}\|u - Tx\|^2 + \alpha_{22}\left\|u - T^2x\right\|^2 \\
&+\beta_0\|u - Tu\|^2 + \beta_1\|Tu - u\|^2 \\
&+\gamma_0\|x - Tx\|^2 + \gamma_1\left\|Tx - T^2x\right\|^2 + \gamma_2\left\|T^2x - x\right\|^2 \leq 0,
\end{aligned}$$

that is,

$$\begin{aligned}
&(\alpha_{00} + \alpha_{20})\|u - x\|^2 + (\alpha_{01} + \alpha_{21})\|u - Tx\|^2 + (\alpha_{02} + \alpha_{22})\|u - T^2x\|^2 \\
&+\alpha_{10}\|Tu - x\|^2 + \alpha_{11}\|Tu - Tx\|^2 + \alpha_{12}\left\|Tu - T^2x\right\|^2 \\
&+(\beta_0 + \beta_1)\|u - Tu\|^2 + \gamma_0\|x - Tx\|^2 + \gamma_1\left\|Tx - T^2x\right\|^2 + \gamma_2\left\|T^2x - x\right\|^2 \leq 0,
\end{aligned}$$

Since $\alpha_{10}, \alpha_{11}, \alpha_{12} \geq 0$, $\beta_0 + \beta_1 \geq 0$, and $\gamma_0, \gamma_1, \gamma_2 \geq 0$, we have that

$$(\alpha_{00} + \alpha_{20})\|u - x\|^2 + (\alpha_{01} + \alpha_{21})\|u - Tx\|^2 + (\alpha_{02} + \alpha_{22})\|u - T^2x\|^2 \leq 0.$$

Since $\alpha_{01} + \alpha_{21} \geq 0$, we obtain

$$(\alpha_{00} + \alpha_{20})\|u - x\|^2 + (\alpha_{02} + \alpha_{22})\|u - T^2x\|^2 \leq 0.$$

By using $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$, we have that

$$\begin{aligned}
(\alpha_{02} + \alpha_{22})\left\|u - T^2x\right\|^2 &\leq -(\alpha_{00} + \alpha_{20})\|u - x\|^2 \\
&\leq (\alpha_{02} + \alpha_{22})\|u - x\|^2.
\end{aligned}$$

Since $\alpha_{02} + \alpha_{22} > 0$,

$$\left\|u - T^2x\right\|^2 \leq \|u - x\|^2.$$

This means that T^2 is quasi-nonexpansive.

Case (1). Supposed that $\alpha_{00}+\alpha_{02}+\alpha_{20}+\alpha_{22} \geq 0$, $\alpha_{10}+\alpha_{12} \geq 0$, $\alpha_{01}, \alpha_{11}, \alpha_{21} \geq 0$, $\alpha_{20} + \alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$, and $\gamma_0 + \gamma_1 \geq 0$. Replacing u and x in (3.3), we can derive the desired result. □

The following two lemmas are exploited to derive the strong convergence theorems in Section 7.

Lemma 3.6 ([2, 29]). *Let $\{X_n\}$ be a sequence of nonnegative real numbers, let $\{Y_n\}$ be a sequence of real numbers such that $\limsup_{n\to\infty} Y_n \leq 0$, and let $\{Z_n\}$ be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} Z_n < \infty$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1)$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. If $X_{n+1} \leq (1 - \lambda_n) X_n + \lambda_n Y_n + Z_n$ for all $n \in \mathbb{N}$, then $X_n \to 0$ as $n \to \infty$.*

Lemma 3.7 ([15]). *Let $\{X_n\}$ be a sequence of real numbers. Suppose that $\{X_n\}$ is not monotone decreasing for sufficiently large $n \in \mathbb{N}$, in other words, there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let n_0 be a* *natural number such that* $\{k \in \mathbb{N} : k \leq n_0,\ X_k < X_{k+1}\}$ *is nonempty. Define a sequence* $\{\tau(n)\}_{n \geq n_0}$ *of natural numbers as follows:*

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ X_k < X_{k+1}\} \quad \text{for all } n \geq n_0.$$

Then, the following hold:
(a) $\tau(n) \to \infty$ *as* $n \to \infty$;
(b) $X_n \leq X_{\tau(n)+1}$ *and* $X_{\tau(n)} < X_{\tau(n)+1}$ *for all* $n \geq n_0$.

4. The sets of attractive points

Let C be a nonempty subset of a real Hilbert space H, and let T be a mapping of C into H. In 2011, Takahashi and Takeuchi [24] introduced the concept of attractive points of T. In this paper, we denote the set of attractive points of T by

$$A_{10}(T) \equiv A(T) = \{u \in H : \|Ty - u\| \leq \|y - u\| \text{ for all } y \in C\}.$$

We also call it the set of $(1,0)$-*attractive points of* T. Similarly, we define

$$A_{20}(T) \equiv \left\{u \in H : \left\|T^2 y - u\right\| \leq \|y - u\| \text{ for all } y \in C\right\}.$$

We call it the set of $(2,0)$-*attractive points of* T. Obviously, $A_{20}(T) = A\left(T^2\right)$. Furthermore, we introduce the following set:

$$A_{21}(T) \equiv \left\{u \in H : \left\|T^2 y - u\right\| \leq \|Ty - u\| \text{ for all } y \in C\right\}.$$

We call it the set of $(2,1)$-*attractive points of* T. The following lemma is useful.

Lemma 4.1. *Let* C *be a nonempty subset of* H, *let* T *be a mapping from* C *into itself, and let* $u \in H$. *Then,*

$$u \in A_{10}(T) \Leftrightarrow \|Ty - y\|^2 + 2\langle Ty - y,\ y - u\rangle \leq 0,\ \forall y \in C; \tag{4.1}$$

$$u \in A_{20}(T) \Leftrightarrow \left\|T^2 y - y\right\|^2 + 2\left\langle T^2 y - y,\ y - u\right\rangle \leq 0,\ \forall y \in C; \tag{4.2}$$

$$u \in A_{21}(T) \Leftrightarrow \left\|T^2 y - Ty\right\|^2 + 2\left\langle T^2 y - Ty,\ Ty - u\right\rangle \leq 0,\ \forall y \in C. \tag{4.3}$$

Proof. The first part (4.1) was proved by Kondo and Takahashi [11]. The second part (4.2) directly follows from (4.1) since $A_{20}(T) = A_{10}\left(T^2\right)$. The part (4.3) is proved as follows:

$$\begin{aligned}
&u \in A_{21}(T) \\
&\quad \Leftrightarrow \left\|T^2 y - u\right\|^2 \leq \|Ty - u\|^2,\ \forall y \in C \\
&\quad \Leftrightarrow \left\|T^2 y - Ty\right\|^2 + 2\left\langle T^2 y - Ty,\ Ty - u\right\rangle + \|Ty - u\|^2 \leq \|Ty - u\|^2,\ \forall y \in C \\
&\quad \Leftrightarrow \left\|T^2 y - Ty\right\|^2 + 2\left\langle T^2 y - Ty,\ Ty - u\right\rangle \leq 0,\ \forall y \in C.
\end{aligned}$$

This completes the proof. □

We give examples to illustrate the concepts of $A_{10}(T)$, $A_{20}(T)$, and $A_{21}(T)$.

Example 4.2. Let $H = \mathbb{R}$, and let $C = \{-1, 1\}$. Define a nonexpansive mapping T from C into itself by

$$Tx = -x$$

for all $x \in C$. In this case, we have that $A_{10}(T) = A_{21}(T) = \{0\}$, $A_{20}(T) = \mathbb{R}$, $F(T) = \emptyset$, and $F(T^2) = \{-1, 1\}$.

Example 4.3. Let $H = \mathbb{R}$, and let $C = \left\{-1, 1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots\right\}$. Let $T : C \to C$ be a nonexpansive mapping defined by

$$Tx = \begin{cases} \frac{1}{2}x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{for all } x \in C.$$

Then, we have that $A_{10}(T) = \{0\}$, $A_{20}(T) = \left[-\frac{1}{4}, 0\right]$, $A_{21}(T) = (-\infty, 0]$, and $F(T) = F(T^2) = \emptyset$.

We know from Takahashi and Takeuchi [24] that the set of attractive points $A_{10}(T)$ of T is closed and convex. Thus, $A_{20}(T)$ is also closed and convex because $A_{20}(T) = A_{10}(T^2)$. Similarly, we can show that $A_{21}(T)$ is closed and convex.

Lemma 4.4. *Let C be a nonempty subset of H, and let T be a mapping from C into itself. Then, $A_{21}(T)$ is closed and convex.*

Proof. First, we prove that $A_{21}(T)$ is closed in H. Let $\{u_n\}$ be a sequence in $A_{21}(T)$ such that $u_n \to u\,(\in H)$. Let $y \in C$. It suffices to show that $\|T^2y - u\| \leq \|Ty - u\|$. Since $u_n \in A_{21}(T)$, it holds that $\|T^2y - u_n\| \leq \|Ty - u_n\|$ for all $n \in \mathbb{N}$. Therefore, we have that

$$\begin{aligned} \|T^2y - u\| &\leq \|T^2y - u_n\| + \|u_n - u\| \\ &\leq \|Ty - u_n\| + \|u_n - u\| \end{aligned}$$

for all $n \in \mathbb{N}$. Since $u_n \to u$, we obtain that $\|T^2y - u\| \leq \|Ty - u\|$ for all $y \in C$, which means that $u \in A_{21}(T)$.

Next, we demonstrate that $A_{21}(T)$ is convex. Let $u, v \in A_{21}(T)$ and $\lambda \in (0, 1)$. Define $w \equiv \lambda u + (1 - \lambda) v \in H$. We show that $w \in A_{21}(T)$. Let $y \in C$. Using Lemma 3.3, we have that

$$\begin{aligned} \|T^2y - w\|^2 &= \|T^2y - [\lambda u + (1-\lambda) v]\|^2 \\ &= \|\lambda (T^2y - u) + (1-\lambda)(T^2y - v)\|^2 \\ &= \lambda \|T^2y - u\|^2 + (1-\lambda)\|T^2y - v\|^2 - \lambda(1-\lambda)\|u - v\|^2 \\ &\leq \lambda \|Ty - u\|^2 + (1-\lambda)\|Ty - v\|^2 - \lambda(1-\lambda)\|u - v\|^2 \\ &= \|\lambda (Ty - u) + (1-\lambda)(Ty - v)\|^2 \\ &= \|Ty - [\lambda u + (1-\lambda) v]\|^2 \\ &= \|Ty - w\|^2 . \end{aligned}$$

Thus, we obtain that $\|T^2y - w\| \leq \|Ty - w\|$ for all $y \in C$, which means that $w \in A_{21}(T)$. □

We have the following result.

Lemma 4.5. *Let C be a nonempty subset of H, and let T be a mapping from C into itself. Then, it holds that $A_{10}(T) \subset A_{21}(T) \cap A_{20}(T)$.*

Proof. Let $u \in A_{10}(T)$ and $y \in C$. We demonstrate that $u \in A_{21}(T) \cap A_{20}(T)$. Since $T : C \to C$, we have that $Ty \in C$. Therefore,

$$\begin{aligned} \left\|T^2 y - u\right\| &= \|T(Ty) - u\| \\ &\le \|Ty - u\| \\ &\le \|y - u\|, \end{aligned}$$

which implies that $u \in A_{21}(T) \cap A_{20}(T)$. □

The following lemma was proved by Takahashi and Takeuchi [24].

Lemma 4.6 ([24])**.** *Let C be a nonempty subset of H, and let T be a mapping from C into H. Then, $A_{10}(T) \cap C \subset F(T)$.*

As a direct result, we have the following lemma because $A_{20}(T) = A_{10}\left(T^2\right)$.

Lemma 4.7. *Let C be a nonempty subset of H, and let T be a mapping on C. Then, $A_{20}(T) \cap C \subset F\left(T^2\right)$.*

The following three lemmas guarantee that a fixed point is an attractive point under certain conditions. Though the following lemma (Lemma 4.8) is deduced from Lemmas 3.4 and 4.5, for the sake of completeness, we give the proof.

Lemma 4.8. *Let C be a nonempty subset of H. Let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping with one of the following conditions:*

$$\begin{aligned} &(1)\ \alpha_{0\bullet} + \alpha_{1\bullet} \ge 0,\ \alpha_{2\bullet} \ge 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \ge 0; \\ &(2)\ \alpha_{\bullet 0} + \alpha_{\bullet 1} \ge 0,\ \alpha_{\bullet 2} \ge 0,\ \alpha_{\bullet 1} > 0,\ \gamma_0, \gamma_1, \gamma_2 \ge 0, \end{aligned}$$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3)*. Then, it holds that*

$$F(T) \subset A_{10}(T) = A_{10}(T) \cap A_{20}(T) \cap A_{21}(T).$$

Proof. It follows from Lemma 4.5 that $A_{10}(T) = A_{10}(T) \cap A_{20}(T) \cap A_{21}(T)$. Thus, it suffices to prove that $F(T) \subset A_{10}(T)$.

Case (2). Suppose that $\alpha_{\bullet 0} + \alpha_{\bullet 1} \ge 0$, $\alpha_{\bullet 2} \ge 0$, $\alpha_{\bullet 1} > 0$, and $\gamma_0, \gamma_1, \gamma_2 \ge 0$. Let $u \in F(T)$ and $y \in C$. We prove that

$$\|Ty - y\|^2 + 2\langle Ty - y,\ y - u\rangle \le 0.$$

Since T is $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid, it holds from (1.1) that

$$\begin{aligned} (4.4)\qquad &\alpha_{00}\|u - y\|^2 + \alpha_{01}\|u - Ty\|^2 + \alpha_{02}\|u - T^2 y\|^2 \\ &+\alpha_{10}\|Tu - y\|^2 + \alpha_{11}\|Tu - Ty\|^2 + \alpha_{12}\|Tu - T^2 y\|^2 \\ &+\alpha_{20}\|T^2 u - y\|^2 + \alpha_{21}\|T^2 u - Ty\|^2 + \alpha_{22}\|T^2 u - T^2 y\|^2 \\ &+\beta_0\|u - Tu\|^2 + \beta_1\|Tu - T^2 u\|^2 + \beta_2\|T^2 u - u\|^2 \end{aligned}$$

$$+\gamma_0\|y-Ty\|^2+\gamma_1\|Ty-T^2y\|^2+\gamma_2\|T^2y-y\|^2\leq 0.$$

Since $\gamma_0,\gamma_1,\gamma_2\geq 0$,

$$\alpha_{00}\left\|u-y\right\|^2+\alpha_{01}\left\|u-Ty\right\|^2+\alpha_{02}\left\|u-T^2y\right\|^2$$
$$+\alpha_{10}\left\|Tu-y\right\|^2+\alpha_{11}\left\|Tu-Ty\right\|^2+\alpha_{12}\left\|Tu-T^2y\right\|^2$$
$$+\alpha_{20}\left\|T^2u-y\right\|^2+\alpha_{21}\left\|T^2u-Ty\right\|^2+\alpha_{22}\left\|T^2u-T^2y\right\|^2$$
$$+\beta_0\left\|u-Tu\right\|^2+\beta_1\left\|Tu-T^2u\right\|^2+\beta_2\left\|T^2u-u\right\|^2\leq 0.$$

Since $u=Tu=T^2u$, we obtain

$$\alpha_{\bullet 0}\left\|u-y\right\|^2+\alpha_{\bullet 1}\left\|u-Ty\right\|^2+\alpha_{\bullet 2}\left\|u-T^2y\right\|^2\leq 0. \tag{4.5}$$

Since $\alpha_{\bullet 2}\geq 0$,

$$\alpha_{\bullet 0}\left\|u-y\right\|^2+\alpha_{\bullet 1}\left\|u-Ty\right\|^2\leq 0,$$

and thus,

$$\alpha_{\bullet 0}\left\|u-y\right\|^2+\alpha_{\bullet 1}\left(\left\|u-y\right\|^2+2\left\langle u-y\,,\ y-Ty\right\rangle+\left\|y-Ty\right\|^2\right)\leq 0.$$

We have from $\alpha_{\bullet 0}+\alpha_{\bullet 1}\geq 0$ that

$$\alpha_{\bullet 1}\left(2\left\langle u-y\,,\ y-Ty\right\rangle+\left\|y-Ty\right\|^2\right)\leq 0.$$

Since $\alpha_{\bullet 1}>0$, we obtain

$$2\left\langle u-y\,,\ y-Ty\right\rangle+\left\|y-Ty\right\|^2\leq 0.$$

This means from (4.1) that $u\in A_{10}(T)$.

Case (1). Suppose that $\alpha_{0\bullet}+\alpha_{1\bullet}\geq 0$, $\alpha_{2\bullet}\geq 0$, $\alpha_{1\bullet}>0$, and $\beta_0,\beta_1,\beta_2\geq 0$. Replacing u and y in (4.4), we can derive the desired result. □

Lemma 4.9. *Let C be a nonempty subset of H. Let $T:C\to C$ be an $(\alpha_{ij},\beta_i,\gamma_i;\ i,j=0,1,2)$-generic 2-generalized hybrid mapping with one of the following condition:*

$$(1)\ \alpha_{0\bullet}+\alpha_{2\bullet}\geq 0,\ \alpha_{1\bullet}\geq 0,\ \alpha_{2\bullet}>0,\ \beta_0,\beta_1,\beta_2\geq 0;$$
$$(2)\ \alpha_{\bullet 0}+\alpha_{\bullet 2}\geq 0,\ \alpha_{\bullet 1}\geq 0,\ \alpha_{\bullet 2}>0,\ \gamma_0,\gamma_1,\gamma_2\geq 0,$$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3)*. Then, it holds that*

$$F(T)\subset A_{20}(T).$$

Proof. **Case (2)**. Suppose that $\alpha_{\bullet 0}+\alpha_{\bullet 2}\geq 0$, $\alpha_{\bullet 1}\geq 0$, $\alpha_{\bullet 2}>0$, and $\gamma_0,\gamma_1,\gamma_2\geq 0$. Let $u\in F(T)$ and $y\in C$. From (4.2), it suffices to prove that

$$\left\|T^2y-y\right\|^2+2\left\langle T^2y-y,\ y-u\right\rangle\leq 0.$$

As in the proof of Lemma 4.8, we obtain the relationship (4.5):

$$\alpha_{\bullet 0}\left\|u-y\right\|^2+\alpha_{\bullet 1}\left\|u-Ty\right\|^2+\alpha_{\bullet 2}\left\|u-T^2y\right\|^2\leq 0$$

since $\gamma_0,\gamma_1,\gamma_2\geq 0$. Using $\alpha_{\bullet 1}\geq 0$, we have that

$$\alpha_{\bullet 0}\left\|u-y\right\|^2+\alpha_{\bullet 2}\left\|u-T^2y\right\|^2\leq 0.$$

Therefore,

$$\alpha_{\bullet 0}\left\|u-y\right\|^{2}+\alpha_{\bullet 2}\left(\left\|u-y\right\|^{2}+2\left\langle u-y\,,\ y-T^{2}y\right\rangle+\left\|y-T^{2}y\right\|^{2}\right)\leq 0.$$

We have from $\alpha_{\bullet 0}+\alpha_{\bullet 2}\geq 0$ that

$$\alpha_{\bullet 2}\left(2\left\langle u-y\,,\ y-T^{2}y\right\rangle+\left\|y-T^{2}y\right\|^{2}\right)\leq 0.$$

Since $\alpha_{\bullet 2}>0$,

$$2\left\langle u-y\,,\ y-T^{2}y\right\rangle+\left\|y-T^{2}y\right\|^{2}\leq 0,$$

which means that $u\in A_{20}\left(T\right)$.

Case (1). Suppose that $\alpha_{0\bullet}+\alpha_{2\bullet}\geq 0,\ \alpha_{1\bullet}\geq 0,\ \alpha_{2\bullet}>0$, and $\beta_0,\beta_1,\beta_2\geq 0$. As in the proof of Lemma 4.8, we can derive the desired result. □

Lemma 4.10. *Let C be a nonempty subset of H. Let $T:C\to C$ be an $(\alpha_{ij},\beta_i,\gamma_i;$ $i,j=0,1,2)$-generic 2-generalized hybrid mapping with one of the following conditions:*

$$(1)\ \alpha_{1\bullet}+\alpha_{2\bullet}\geq 0,\ \alpha_{0\bullet}\geq 0,\ \alpha_{2\bullet}>0,\ \beta_0,\beta_1,\beta_2\geq 0;$$
$$(2)\ \alpha_{\bullet 1}+\alpha_{\bullet 2}\geq 0,\ \alpha_{\bullet 0}\geq 0,\ \alpha_{\bullet 2}>0,\ \gamma_0,\gamma_1,\gamma_2\geq 0,$$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3). *Then, it holds that*

$$F\left(T\right)\subset A_{21}\left(T\right).$$

Proof. **Case (2)**. Suppose that $\alpha_{\bullet 1}+\alpha_{\bullet 2}\geq 0,\ \alpha_{\bullet 0}\geq 0, \alpha_{\bullet 2}>0$, and $\gamma_0,\gamma_1,\gamma_2\geq 0$. Let $u\in F\left(T\right)$ and $y\in C$. From (4.3), it suffices to prove that

$$\left\|T^{2}y-Ty\right\|^{2}+2\left\langle T^{2}y-Ty,\ Ty-u\right\rangle\leq 0.$$

As in the proof of Lemma4.8, we obtain the relationship (4.5):

$$\alpha_{\bullet 0}\left\|u-y\right\|^{2}+\alpha_{\bullet 1}\left\|u-Ty\right\|^{2}+\alpha_{\bullet 2}\left\|u-T^{2}y\right\|^{2}\leq 0$$

since $\gamma_0,\gamma_1,\gamma_2\geq 0$. By using $\alpha_{\bullet 0}\geq 0$, we have that

$$\alpha_{\bullet 1}\left\|u-Ty\right\|^{2}+\alpha_{\bullet 2}\left\|u-T^{2}y\right\|^{2}\leq 0.$$

Therefore,

$$\alpha_{\bullet 1}\left\|u-Ty\right\|^{2}+\alpha_{\bullet 2}\left(\left\|u-Ty\right\|^{2}+2\left\langle u-Ty\,,\ Ty-T^{2}y\right\rangle+\left\|Ty-T^{2}y\right\|^{2}\right)\leq 0.$$

We have from $\alpha_{\bullet 1}+\alpha_{\bullet 2}\geq 0$ that

$$\alpha_{\bullet 2}\left(2\left\langle u-Ty\,,\ Ty-T^{2}y\right\rangle+\left\|Ty-T^{2}y\right\|^{2}\right)\leq 0.$$

Since $\alpha_{\bullet 2}>0$,

$$2\left\langle u-Ty\,,\ Ty-T^{2}y\right\rangle+\left\|Ty-T^{2}y\right\|^{2}\leq 0,$$

which means that $u\in A_{21}\left(T\right)$.

Case (1). Suppose that $\alpha_{1\bullet}+\alpha_{2\bullet}\geq 0,\ \alpha_{0\bullet}\geq 0, \alpha_{2\bullet}>0$, and $\beta_0,\beta_1,\beta_2\geq 0$. As in the proof of Lemma 4.8, we can derive the desired result. □

We prove the following lemmas which are crucial for proving our main theorems. These lemmas have been developed by many authors; see, for example, [8], [26] and [17].

Lemma 4.11. *Let C be a nonempty subset of a real Hilbert space H. Let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\, i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping with one of the following conditions:*

$$(1)\ \alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \ \alpha_{2\bullet} \geq 0,\ \ \alpha_{1\bullet} > 0,\ \ \beta_0, \beta_1, \beta_2 \geq 0;$$
$$(2)\ \alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0,\ \ \alpha_{\bullet 2} \geq 0,\ \ \alpha_{\bullet 1} > 0,\ \ \gamma_0, \gamma_1, \gamma_2 \geq 0,$$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3). *Let $\{x_n\}$ be a sequence in C such that*

$$x_n - Tx_n \to 0, \qquad Tx_n - T^2x_n \to 0, \qquad T^2x_n - x_n \to 0. \tag{4.6}$$

If $x_n \rightharpoonup u$, then $u \in A_{10}(T)$.

Proof. **Case (2)**. Suppose that $\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0$, $\alpha_{\bullet 2} \geq 0$, $\alpha_{\bullet 1} > 0$, and $\gamma_0, \gamma_1, \gamma_2 \geq 0$. Let $y \in C$. From (4.1), it suffices to show that

$$\|Ty - y\| + 2\langle Ty - y,\ y - u\rangle \leq 0.$$

Since T is $(\alpha_{ij}, \beta_i, \gamma_i;\, i, j = 0, 1, 2)$-generic 2-generalized hybrid, it holds from (1.1) that

$$\begin{aligned}
&\alpha_{00}\|x_n - y\|^2 + \alpha_{01}\|x_n - Ty\|^2 + \alpha_{02}\|x_n - T^2y\|^2 \\
&+\alpha_{10}\|Tx_n - y\|^2 + \alpha_{11}\|Tx_n - Ty\|^2 + \alpha_{12}\|Tx_n - T^2y\|^2 \\
&+\alpha_{20}\|T^2x_n - y\|^2 + \alpha_{21}\|T^2x_n - Ty\|^2 + \alpha_{22}\|T^2x_n - T^2y\|^2 \\
&+\beta_0\|x_n - Tx_n\|^2 + \beta_1\|Tx_n - T^2x_n\|^2 + \beta_2\|T^2x_n - x_n\|^2 \\
&+\gamma_0\|y - Ty\|^2 + \gamma_1\|Ty - T^2y\|^2 + \gamma_2\|T^2y - y\|^2 \leq 0.
\end{aligned} \tag{4.7}$$

Since $\gamma_0, \gamma_1, \gamma_2 \geq 0$, we obtain

$$\begin{aligned}
&\alpha_{00}\left\|x_n - y\right\|^2 + \alpha_{01}\left\|x_n - Ty\right\|^2 + \alpha_{02}\left\|x_n - T^2y\right\|^2 \\
&+\alpha_{10}\left\|Tx_n - y\right\|^2 + \alpha_{11}\left\|Tx_n - Ty\right\|^2 + \alpha_{12}\left\|Tx_n - T^2y\right\|^2 \\
&+\alpha_{20}\left\|T^2x_n - y\right\|^2 + \alpha_{21}\left\|T^2x_n - Ty\right\|^2 + \alpha_{22}\left\|T^2x_n - T^2y\right\|^2 \\
&+\beta_0\left\|x_n - Tx_n\right\|^2 + \beta_1\left\|Tx_n - T^2x_n\right\|^2 + \beta_2\left\|T^2x_n - x_n\right\|^2 \leq 0,
\end{aligned}$$

and hence,

$$\begin{aligned}
&\alpha_{00}\left\|x_n - y\right\|^2 + \alpha_{01}\left\|x_n - Ty\right\|^2 + \alpha_{02}\left\|x_n - T^2y\right\|^2 \\
&+\alpha_{10}\left(\|Tx_n - x_n\|^2 + 2\langle Tx_n - x_n,\ x_n - y\rangle + \|x_n - y\|^2\right) \\
&+\alpha_{11}\left(\|Tx_n - x_n\|^2 + 2\langle Tx_n - x_n,\ x_n - Ty\rangle + \|x_n - Ty\|^2\right) \\
&+\alpha_{12}\left(\|Tx_n - x_n\|^2 + 2\left\langle Tx_n - x_n,\ x_n - T^2y\right\rangle + \left\|x_n - T^2y\right\|^2\right)
\end{aligned}$$

$$+\alpha_{20}\left(\left\|T^2x_n - x_n\right\|^2 + 2\left\langle T^2x_n - x_n,\ x_n - y\right\rangle + \|x_n - y\|^2\right)$$
$$+\alpha_{21}\left(\left\|T^2x_n - x_n\right\|^2 + 2\left\langle T^2x_n - x_n,\ x_n - Ty\right\rangle + \|x_n - Ty\|^2\right)$$
$$+\alpha_{22}\left(\left\|T^2x_n - x_n\right\|^2 + 2\left\langle T^2x_n - x_n,\ x_n - T^2y\right\rangle + \left\|x_n - T^2y\right\|^2\right)$$
$$+\beta_0\left\|x_n - Tx_n\right\|^2 + \beta_1\left\|Tx_n - T^2x_n\right\|^2 + \beta_2\left\|T^2x_n - x_n\right\|^2 \leq 0.$$

Applying a Banach limit μ, we obtain from (4.6) that

$$\alpha_{\bullet 0}\mu_n\|x_n - y\|^2 + \alpha_{\bullet 1}\mu_n\|x_n - Ty\|^2 + \alpha_{\bullet 2}\mu_n\left\|x_n - T^2y\right\|^2 \leq 0. \tag{4.8}$$

Since $\alpha_{\bullet 2} \geq 0$,

$$\alpha_{\bullet 0}\mu_n\|x_n - y\|^2 + \alpha_{\bullet 1}\mu_n\|x_n - Ty\|^2 \leq 0.$$

Thus,

$$\alpha_{\bullet 0}\mu_n\|x_n - y\|^2 + \alpha_{\bullet 1}\mu_n\left(\|x_n - y\|^2 + 2\langle x_n - y,\ y - Ty\rangle + \|y - Ty\|^2\right) \leq 0.$$

Since $\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0$,

$$\alpha_{\bullet 1}\mu_n\left(2\langle x_n - y,\ y - Ty\rangle + \|y - Ty\|^2\right) \leq 0.$$

Thus,

$$\alpha_{\bullet 1}\left(2\langle u - y,\ y - Ty\rangle + \|y - Ty\|^2\right) \leq 0.$$

Since $\alpha_{\bullet 1} > 0$,

$$2\langle u - y,\ y - Ty\rangle + \|y - Ty\|^2 \leq 0$$

for all $y \in C$. This means that $u \in A_{10}(T)$.

Case (1). Suppose that $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} > 0$, and $\beta_0, \beta_1, \beta_2 \geq 0$. Replacing x_n and y in (4.7), we can derive the desired result. ☐

Lemma 4.12. *Let C be a nonempty subset of a real Hilbert space H. Let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i,j = 0,1,2)$-generic 2-generalized hybrid mapping with one of the following conditions:*

(1) $\alpha_{00} + \alpha_{02} + \alpha_{20} + \alpha_{22} \geq 0,\ \alpha_{10} + \alpha_{12} \geq 0,\ \alpha_{01}, \alpha_{11}, \alpha_{21} \geq 0,$
$\alpha_{20} + \alpha_{22} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0,\ \gamma_0, \gamma_1 \geq 0;$

(2) $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0,\ \alpha_{01} + \alpha_{21} \geq 0,\ \alpha_{10}, \alpha_{11}, \alpha_{12} \geq 0,$
$\alpha_{02} + \alpha_{22} > 0,\ \beta_0, \beta_1 \geq 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0.$

Let $\{x_n\}$ be a sequence in C such that

$$T^2x_n - x_n \to 0. \tag{4.9}$$

If $x_n \rightharpoonup u$, then $u \in A_{20}(T)$.

Proof. **Case (2)**. Suppose that $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0,\ \alpha_{01} + \alpha_{21} \geq 0,\ \alpha_{10}, \alpha_{11}, \alpha_{12} \geq 0,\ \alpha_{02} + \alpha_{22} > 0,\ \beta_0, \beta_1 \geq 0$, and $\gamma_0, \gamma_1, \gamma_2 \geq 0$. Let $y \in C$. From (4.2), it suffices to show that

$$\left\|T^2y - y\right\|^2 + 2\left\langle T^2y - y,\ y - u\right\rangle \leq 0.$$

Since T is $(\alpha_{ij}, \beta_i, \gamma_i;\, i, j = 0, 1, 2)$-generic 2-generalized hybrid, it holds that

$$
\begin{aligned}
&\alpha_{00} \|x_n - y\|^2 + \alpha_{01} \|x_n - Ty\|^2 + \alpha_{02} \left\|x_n - T^2 y\right\|^2 \\
&+\alpha_{10} \|Tx_n - y\|^2 + \alpha_{11} \|Tx_n - Ty\|^2 + \alpha_{12} \left\|Tx_n - T^2 y\right\|^2 \\
&+\alpha_{20} \left\|T^2 x_n - y\right\|^2 + \alpha_{21} \left\|T^2 x_n - Ty\right\|^2 + \alpha_{22} \left\|T^2 x_n - T^2 y\right\|^2 \\
&+\beta_0 \|x_n - Tx_n\|^2 + \beta_1 \left\|Tx_n - T^2 x_n\right\|^2 + \beta_2 \left\|T^2 x_n - x_n\right\|^2 \\
&+\gamma_0 \|y - Ty\|^2 + \gamma_1 \left\|Ty - T^2 y\right\|^2 + \gamma_2 \left\|T^2 y - y\right\|^2 \leq 0.
\end{aligned} \tag{4.10}
$$

Since $\alpha_{10}, \alpha_{11}, \alpha_{12} \geq 0$, $\beta_0, \beta_1 \geq 0$, and $\gamma_0, \gamma_1, \gamma_2 \geq 0$, we obtain

$$
\begin{aligned}
&\alpha_{00} \|x_n - y\|^2 + \alpha_{01} \|x_n - Ty\|^2 + \alpha_{02} \left\|x_n - T^2 y\right\|^2 \\
&+\alpha_{20} \left\|T^2 x_n - y\right\|^2 + \alpha_{21} \left\|T^2 x_n - Ty\right\|^2 + \alpha_{22} \left\|T^2 x_n - T^2 y\right\|^2 \\
&+\beta_2 \left\|T^2 x_n - x_n\right\|^2 \leq 0,
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&\alpha_{00} \|x_n - y\|^2 + \alpha_{01} \|x_n - Ty\|^2 + \alpha_{02} \left\|x_n - T^2 y\right\|^2 \\
&+\alpha_{20} \left(\left\|T^2 x_n - x_n\right\|^2 + 2 \left\langle T^2 x_n - x_n,\ x_n - y\right\rangle + \|x_n - y\|^2\right) \\
&+\alpha_{21} \left(\left\|T^2 x_n - x_n\right\|^2 + 2 \left\langle T^2 x_n - x_n,\ x_n - Ty\right\rangle + \|x_n - Ty\|^2\right) \\
&+\alpha_{22} \left(\left\|T^2 x_n - x_n\right\|^2 + 2 \left\langle T^2 x_n - x_n,\ x_n - T^2 y\right\rangle + \left\|x_n - T^2 y\right\|^2\right) \\
&+\beta_2 \left\|T^2 x_n - x_n\right\|^2 \leq 0.
\end{aligned}
$$

Applying a Banach limit μ, we obtain from (4.9) that

$$
\begin{aligned}
&(\alpha_{00}+\alpha_{20})\mu_n\|x_n - y\|^2 \\
&\quad + (\alpha_{01} + \alpha_{21})\mu_n\|x_n - Ty\|^2 + (\alpha_{02} + \alpha_{22})\mu_n\|x_n - T^2 y\|^2 \leq 0.
\end{aligned} \tag{4.11}
$$

Since $\alpha_{01} + \alpha_{21} \geq 0$,

$$
(\alpha_{00} + \alpha_{20})\mu_n\|x_n - y\|^2 + (\alpha_{02} + \alpha_{22})\mu_n\|x_n - T^2 y\|^2 \leq 0.
$$

Thus, we obtain that

$$
\begin{aligned}
&(\alpha_{00} + \alpha_{20})\mu_n\|x_n - y\|^2 \\
&\quad + (\alpha_{02} + \alpha_{22})\mu_n \left(\|x_n - y\|^2 + 2 \left\langle x_n - y,\ y - T^2 y\right\rangle + \|y - T^2 y\|^2\right) \leq 0.
\end{aligned}
$$

Since $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$,

$$
(\alpha_{02} + \alpha_{22})\mu_n \left(2 \left\langle x_n - y,\ y - T^2 y\right\rangle + \left\|y - T^2 y\right\|^2\right) \leq 0.
$$

Since $x_n \rightharpoonup u$, we have that

$$
(\alpha_{02} + \alpha_{22}) \left(2 \left\langle u - y,\ y - T^2 y\right\rangle + \left\|y - T^2 y\right\|^2\right) \leq 0.
$$

Since $\alpha_{02} + \alpha_{22} > 0$,

$$
2 \left\langle u - y,\ y - T^2 y\right\rangle + \left\|y - T^2 y\right\|^2 \leq 0
$$

for all $y \in C$. This means that $u \in A_{20}(T)$.

Case (1). Suppose that $\alpha_{00}+\alpha_{02}+\alpha_{20}+\alpha_{22} \geq 0$, $\alpha_{10}+\alpha_{12} \geq 0$, $\alpha_{01}, \alpha_{11}, \alpha_{21} \geq 0$, $\alpha_{20}+\alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$, and $\gamma_0, \gamma_1 \geq 0$. Replacing x_n and y in (4.10), we can derive the desired result. □

5. The existence of attractive points and fixed points

Regarding the existence of $(1,0)$-attractive points of T, we know the following result.

Theorem 5.1 ([13]). *Let C be a nonempty subset of a real Hilbert space H, and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping. Suppose that T satisfies one of the following conditions:*

(1) $\alpha_{0\bullet}+\alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0,\ \gamma_0+\gamma_1 \geq 0,\ \gamma_2 \geq 0;$

(2) $\alpha_{\bullet 0}+\alpha_{\bullet 1} \geq 0,\ \alpha_{\bullet 2} \geq 0,\ \alpha_{\bullet 1} > 0,\ \beta_0+\beta_1 \geq 0,\ \beta_2 \geq 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0,$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3). *If there exists an element $x \in C$ such that the sequence $\{T^n x\}$ in C is bounded, then $A_{10}(T)$ is nonempty.*

By adding the assumption that C is closed and convex, the following fixed point theorem was obtained.

Theorem 5.2 ([13]). *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $T : C \to C$ be $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid. Suppose that T satisfies one of the following conditions:*

(1) $\alpha_{0\bullet}+\alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet}+\beta_0 > 0,\ \beta_1, \beta_2 \geq 0,\ \gamma_0+\gamma_1 \geq 0,\ \gamma_2 \geq 0;$

(2) $\alpha_{\bullet 0}+\alpha_{\bullet 1} \geq 0,\ \alpha_{\bullet 2} \geq 0,\ \alpha_{\bullet 1}+\gamma_0 > 0,\ \beta_0+\beta_1 \geq 0,\ \beta_2 \geq 0,\ \gamma_1, \gamma_2 \geq 0,$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3). *If there exists an element $x \in C$ such that the sequence $\{T^n x\}$ in C is bounded, then $F(T)$ is nonempty.*

Regarding the existence of $(2,0)$-attractive points and fixed points of T^2, we have the following results.

Theorem 5.3. *Let C be a nonempty subset of a real Hilbert space H, and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping. Suppose that T satisfies one of the following conditions:*

(1) $\alpha_{0\bullet}+\alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0,\ \gamma_0+\gamma_1 \geq 0,\ \gamma_2 \geq 0;$

(2) $\alpha_{\bullet 0}+\alpha_{\bullet 2} \geq 0,\ \alpha_{\bullet 1} \geq 0,\ \alpha_{\bullet 2} > 0,\ \beta_0+\beta_1 \geq 0,\ \beta_2 \geq 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0;$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3). *If there exists an element $x \in C$ such that the sequence $\{T^n x\}$ in C is bounded, then $A_{20}(T)$ is nonempty.*

Proof. Let $\mu \in (l^\infty)^*$ be a Banach limit. For the bounded sequence $\{T^n x\}$, it holds from Lemma 3.1 that there exists a unique element $u \in \overline{co}\{T^n x\}\ (\subset H)$ such that

$$\mu_n \langle T^n x,\ v\rangle = \langle u,\ v\rangle \tag{5.1}$$

for all $v \in H$. We show that $u \in A_{20}(T)$.

Case (2). Suppose that $\alpha_{\bullet 0}+\alpha_{\bullet 2}\geq 0$, $\alpha_{\bullet 1}\geq 0$, $\alpha_{\bullet 2}>0$, $\beta_0+\beta_1\geq 0$, $\beta_2\geq 0$, and $\gamma_0,\gamma_1,\gamma_2\geq 0$. Let $y\in C$. From (4.2), it is enough to prove that

$$\left\|T^2y-y\right\|^2+2\left\langle T^2y-y,\ y-u\right\rangle\leq 0.$$

Since T is $(\alpha_{ij},\beta_i,\gamma_i;\,i,j=0,1,2)$-generic 2-generalized hybrid, it holds from (1.1) that

$$\begin{aligned}(5.2)\qquad &\alpha_{00}\|T^nx-y\|^2+\alpha_{01}\|T^nx-Ty\|^2+\alpha_{02}\|T^nx-T^2y\|^2\\ &+\alpha_{10}\|T^{n+1}x-y\|^2+\alpha_{11}\|T^{n+1}x-Ty\|^2+\alpha_{12}\|T^{n+1}x-T^2y\|^2\\ &+\alpha_{20}\|T^{n+2}x-y\|^2+\alpha_{21}\|T^{n+2}x-Ty\|^2+\alpha_{22}\|T^{n+2}x-T^2y\|^2\\ &+\beta_0\|T^nx-T^{n+1}x\|^2+\beta_1\|T^{n+1}x-T^{n+2}x\|^2+\beta_2\|T^{n+2}x-T^nx\|^2\\ &+\gamma_0\|y-Ty\|^2+\gamma_1\|Ty-T^2y\|^2+\gamma_2\|T^2y-y\|^2\leq 0\end{aligned}$$

for all $n\in\mathbb{N}$. Since $\beta_2\geq 0$, $\gamma_0,\gamma_1,\gamma_2\geq 0$, we obtain that

$$\begin{aligned}&\alpha_{00}\left\|T^nx-y\right\|^2+\alpha_{01}\left\|T^nx-Ty\right\|^2+\alpha_{02}\left\|T^nx-T^2y\right\|^2\\ &+\alpha_{10}\left\|T^{n+1}x-y\right\|^2+\alpha_{11}\left\|T^{n+1}x-Ty\right\|^2+\alpha_{12}\left\|T^{n+1}x-T^2y\right\|^2\\ &+\alpha_{20}\left\|T^{n+2}x-y\right\|^2+\alpha_{21}\left\|T^{n+2}x-Ty\right\|^2+\alpha_{22}\left\|T^{n+2}x-T^2y\right\|^2\\ &+\beta_0\left\|T^nx-T^{n+1}x\right\|^2+\beta_1\left\|T^{n+1}x-T^{n+2}x\right\|^2\leq 0.\end{aligned}$$

It holds that

$$\begin{aligned}&\alpha_{\bullet 0}\left\|T^nx-y\right\|^2+\alpha_{\bullet 1}\left\|T^nx-Ty\right\|^2+\alpha_{\bullet 2}\left\|T^nx-T^2y\right\|^2\\ &+\alpha_{10}\left(\left\|T^{n+1}x-y\right\|^2-\left\|T^nx-y\right\|^2\right)+\alpha_{11}\left(\left\|T^{n+1}x-Ty\right\|^2-\left\|T^nx-Ty\right\|^2\right)\\ &+\alpha_{12}\left(\left\|T^{n+1}x-T^2y\right\|^2-\left\|T^nx-T^2y\right\|^2\right)\\ &+\alpha_{20}\left(\left\|T^{n+2}x-y\right\|^2-\left\|T^nx-y\right\|^2\right)+\alpha_{21}\left(\left\|T^{n+2}x-Ty\right\|^2-\left\|T^nx-Ty\right\|^2\right)\\ &+\alpha_{22}\left(\left\|T^{n+2}x-T^2y\right\|^2-\left\|T^nx-T^2y\right\|^2\right)\\ &+\beta_0\left\|T^nx-T^{n+1}x\right\|^2+\beta_1\left\|T^{n+1}x-T^{n+2}x\right\|^2\leq 0.\end{aligned}$$

Applying the Banach limit μ, we obtain that

$$\begin{aligned}&\alpha_{\bullet 0}\mu_n\left\|T^nx-y\right\|^2+\alpha_{\bullet 1}\mu_n\left\|T^nx-Ty\right\|^2+\alpha_{\bullet 2}\mu_n\left\|T^nx-T^2y\right\|^2\\ &\qquad\qquad+\left(\beta_0+\beta_1\right)\mu_n\left\|T^nx-T^{n+1}x\right\|^2\leq 0.\end{aligned}$$

It follows from $\beta_0+\beta_1\geq 0$ that

$$(5.3)\qquad \alpha_{\bullet 0}\mu_n\left\|T^nx-y\right\|^2+\alpha_{\bullet 1}\mu_n\left\|T^nx-Ty\right\|^2+\alpha_{\bullet 2}\mu_n\left\|T^nx-T^2y\right\|^2\leq 0.$$

Since $\alpha_{\bullet 1}\geq 0$, we have that

$$\alpha_{\bullet 0}\mu_n\left\|T^nx-y\right\|^2+\alpha_{\bullet 2}\mu_n\left\|T^nx-T^2y\right\|^2\leq 0.$$

Using $\alpha_{\bullet 0}+\alpha_{\bullet 2}\geq 0$, we obtain that

$$\alpha_{\bullet 2}\mu_n\left\|T^nx-T^2y\right\|^2\ \leq\ -\alpha_{\bullet 0}\mu_n\left\|T^nx-y\right\|^2$$

$$\leq \quad \alpha_{\bullet 2}\mu_n \|T^n x - y\|^2 .$$

Since $\alpha_{\bullet 2} > 0$, we have that

$$\mu_n \left\|T^n x - T^2 y\right\|^2 \leq \mu_n \|T^n x - y\|^2 ,$$

and thus,

$$\mu_n \left(\|T^n x - y\|^2 + 2\left\langle T^n x - y,\ y - T^2 x\right\rangle + \left\|y - T^2 x\right\|^2\right) \leq \mu_n \|T^n x - y\|^2 .$$

This means that

$$\mu_n \left(2\left\langle T^n x - y,\ y - T^2 x\right\rangle + \left\|y - T^2 x\right\|^2\right) \leq 0.$$

From (5.1), it holds that

$$2\left\langle u - y,\ y - T^2 x\right\rangle + \left\|y - T^2 x\right\|^2 \leq 0 \tag{5.4}$$

for all $y \in C$. This implies from (4.2) that $u \in A_{20}(T)$.

Case (1). Suppose that $\alpha_{0\bullet} + \alpha_{2\bullet} \geq 0$, $\alpha_{1\bullet} \geq 0$, $\alpha_{2\bullet} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$, $\gamma_0 + \gamma_1 \geq 0$, and $\gamma_2 \geq 0$. We can obtain the desired result by replacing the variables y and $T^n x$ in (5.2). □

Theorem 5.4. *Let C be a nonempty, closed and convex subset of a real Hilbert space H, and let $T : C \to C$ be $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid. Suppose that T satisfies one of the following conditions:*

(1) $\alpha_{0\bullet} + \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} + \beta_2 > 0,\ \beta_0, \beta_1 \geq 0,\ \gamma_0 + \gamma_1 \geq 0,\ \gamma_2 \geq 0;$

(2) $\alpha_{\bullet 0} + \alpha_{\bullet 2} \geq 0,\ \alpha_{\bullet 1} \geq 0,\ \alpha_{\bullet 2} + \gamma_2 > 0,\ \beta_0 + \beta_1 \geq 0,\ \beta_2 \geq 0,\ \gamma_0, \gamma_1 \geq 0;$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3). If there exists an element $x \in C$ such that the sequence $\{T^n x\}$ in C is bounded, then $F\left(T^2\right)$ is nonempty.

Proof. Let $\mu \in (l^\infty)^*$ be a Banach limit. From Lemma 3.1, it holds that for the bounded sequence $\{T^n x\}$, there exists a unique element $u \in \overline{co}\{T^n x\}$ such that

$$\mu_n \langle T^n x,\ v\rangle = \langle u,\ v\rangle \tag{5.5}$$

for all $v \in H$. Note that since C is closed and convex, we have that $\overline{co}\{T^n x\} \subset C$. Thus, $u \in C$. We show that $u \in F\left(T^2\right)$.

Case (2). Suppose that $\alpha_{\bullet 0} + \alpha_{\bullet 2} \geq 0$, $\alpha_{\bullet 1} \geq 0$, $\alpha_{\bullet 2} + \gamma_2 > 0$, $\beta_0 + \beta_1 \geq 0$, $\beta_2 \geq 0$, and $\gamma_0, \gamma_1 \geq 0$. Since T is $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid, we have from (1.1) that

$$\begin{aligned}
&\alpha_{00}\|T^n x - u\|^2 + \alpha_{01}\|T^n x - Tu\|^2 + \alpha_{02}\|T^n x - T^2 u\|^2 \\
&+\alpha_{10}\|T^{n+1} x - u\|^2 + \alpha_{11}\|T^{n+1} x - Tu\|^2 + \alpha_{12}\|T^{n+1} x - T^2 u\|^2 \\
&+\alpha_{20}\|T^{n+2} x - u\|^2 + \alpha_{21}\|T^{n+2} x - Tu\|^2 + \alpha_{22}\|T^{n+2} x - T^2 u\|^2 \\
&+\beta_0\|T^n x - T^{n+1} x\|^2 + \beta_1\|T^{n+1} x - T^{n+2} x\|^2 + \beta_2\|T^{n+2} x - T^n x\|^2 \\
&+\gamma_0\|u - Tu\|^2 + \gamma_1\|Tu - T^2 u\|^2 + \gamma_2\|T^2 u - u\|^2 \leq 0
\end{aligned} \tag{5.6}$$

for all $n \in \mathbb{N}$. Since $\beta_2, \gamma_0, \gamma_1 \geq 0$, we obtain that

$$\alpha_{00}\|T^n x - u\|^2 + \alpha_{01}\|T^n x - Tu\|^2 + \alpha_{02}\left\|T^n x - T^2 u\right\|^2$$

$$+\alpha_{10}\left\|T^{n+1}x-u\right\|^2+\alpha_{11}\left\|T^{n+1}x-Tu\right\|^2+\alpha_{12}\left\|T^{n+1}x-T^2u\right\|^2$$
$$+\alpha_{20}\left\|T^{n+2}x-u\right\|^2+\alpha_{21}\left\|T^{n+2}x-Tu\right\|^2+\alpha_{22}\left\|T^{n+2}x-T^2u\right\|^2$$
$$+\beta_0\left\|T^nx-T^{n+1}x\right\|^2+\beta_1\left\|T^{n+1}x-T^{n+2}x\right\|^2+\gamma_2\left\|T^2u-u\right\|^2\le 0,$$

and hence,

$$\alpha_{\bullet 0}\left\|T^nx-u\right\|^2+\alpha_{\bullet 1}\left\|T^nx-Tu\right\|^2+\alpha_{\bullet 2}\left\|T^nx-T^2u\right\|^2$$
$$+\alpha_{10}\left(\left\|T^{n+1}x-u\right\|^2-\left\|T^nx-u\right\|^2\right)+\alpha_{11}\left(\left\|T^{n+1}x-Tu\right\|^2-\left\|T^nx-Tu\right\|^2\right)$$
$$+\alpha_{12}\left(\left\|T^{n+1}x-T^2u\right\|^2-\left\|T^nx-T^2u\right\|^2\right)$$
$$+\alpha_{20}\left(\left\|T^{n+2}x-u\right\|^2-\left\|T^nx-u\right\|^2\right)+\alpha_{21}\left(\left\|T^{n+2}x-Tu\right\|^2-\left\|T^nx-Tu\right\|^2\right)$$
$$+\alpha_{22}\left(\left\|T^{n+2}x-T^2u\right\|^2-\left\|T^nx-T^2u\right\|^2\right)$$
$$+\beta_0\left\|T^nx-T^{n+1}x\right\|^2+\beta_1\left\|T^{n+1}x-T^{n+2}x\right\|^2+\gamma_2\left\|T^2u-u\right\|^2\le 0.$$

Applying the Banach limit μ, we obtain that

$$\alpha_{\bullet 0}\mu_n\left\|T^nx-u\right\|^2+\alpha_{\bullet 1}\mu_n\left\|T^nx-Tu\right\|^2+\alpha_{\bullet 2}\mu_n\left\|T^nx-T^2u\right\|^2$$
$$+\left(\beta_0+\beta_1\right)\mu_n\left\|T^nx-T^{n+1}x\right\|^2+\gamma_2\left\|T^2u-u\right\|^2\le 0.$$

Since $\beta_0+\beta_1\ge 0$, we have that

$$\alpha_{\bullet 0}\mu_n\left\|T^nx-u\right\|^2+\alpha_{\bullet 1}\mu_n\left\|T^nx-Tu\right\|^2+\alpha_{\bullet 2}\mu_n\left\|T^nx-T^2u\right\|^2+\gamma_2\left\|T^2u-u\right\|^2\le 0.$$

Since $\alpha_{\bullet 1}\ge 0$, it holds that

$$\alpha_{\bullet 0}\mu_n\left\|T^nx-u\right\|^2+\alpha_{\bullet 2}\mu_n\left\|T^nx-T^2u\right\|^2+\gamma_2\left\|T^2u-u\right\|^2\le 0.$$

This yields that

$$\alpha_{\bullet 0}\mu_n\left\|T^nx-u\right\|^2+\alpha_{\bullet 2}\mu_n\left(\left\|T^nx-u\right\|^2+2\left\langle T^nx-u,\ u-T^2x\right\rangle+\left\|u-T^2u\right\|^2\right)$$
$$+\gamma_2\left\|T^2u-u\right\|^2\le 0.$$

Since $\alpha_{\bullet 0}+\alpha_{\bullet 2}\ge 0$, it holds that

$$\alpha_{\bullet 2}\mu_n\left(2\left\langle T^nx-u,\ u-T^2x\right\rangle+\left\|u-T^2u\right\|^2\right)+\gamma_2\left\|T^2u-u\right\|^2\le 0.$$

We have from (5.5) that

$$\alpha_{\bullet 2}\left(2\left\langle u-u,\ u-T^2x\right\rangle+\left\|u-T^2u\right\|^2\right)+\gamma_2\left\|T^2u-u\right\|^2\le 0,$$

and hence,

$$\left(\alpha_{\bullet 2}+\gamma_2\right)\left\|u-T^2u\right\|^2\le 0.$$

Since $\alpha_{\bullet 2}+\gamma_2>0$, we obtain that $u\in F\left(T^2\right)$.

Case (1). Suppose that $\alpha_{0\bullet}+\alpha_{2\bullet}\ge 0$, $\alpha_{1\bullet}\ge 0$, $\alpha_{2\bullet}+\beta_2>0$, $\beta_0,\beta_1\ge 0$, $\gamma_0+\gamma_1\ge 0$, and $\gamma_2\ge 0$. We can obtain the desired result by replacing the variables u and T^nx in (5.6). □

Regarding the existence of $(2,1)$-attractive points of T, we have the following result.

Theorem 5.5. *Let C be a nonempty subset of a real Hilbert space H, and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i,j = 0,1,2)$-generic 2-generalized hybrid mapping. Suppose that T satisfies one of the following conditions:*

(1) $\alpha_{1\bullet} + \alpha_{2\bullet} \geq 0,\ \alpha_{0\bullet} \geq 0,\ \alpha_{2\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0,\ \gamma_0 + \gamma_1 \geq 0,\ \gamma_2 \geq 0;$

(2) $\alpha_{\bullet 1} + \alpha_{\bullet 2} \geq 0,\ \alpha_{\bullet 0} \geq 0,\ \alpha_{\bullet 2} > 0,\ \beta_0 + \beta_1 \geq 0,\ \beta_2 \geq 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0;$

where the notations $\alpha_{i\bullet}$ and $\alpha_{\bullet i}$ are defined in (1.3). *If there exists an element $x \in C$ such that the sequence $\{T^n x\}$ in C is bounded, then $A_{21}(T)$ is nonempty.*

Proof. Let $\mu \in (l^\infty)^*$ be a Banach limit. For the bounded sequence $\{T^n x\}$, we obtain from Lemma 3.1 that there exists a unique element $u \in \overline{co}\,\{T^n x\}\,(\subset H)$ such that

$$\mu_n \langle T^n x,\ v\rangle = \langle u,\ v\rangle \tag{5.7}$$

for all $v \in H$. We show that $u \in A_{21}(T)$.

Case (2). Suppose that $\alpha_{\bullet 1} + \alpha_{\bullet 2} \geq 0$, $\alpha_{\bullet 0} \geq 0$, $\alpha_{\bullet 2} > 0$, $\beta_0 + \beta_1 \geq 0$, $\beta_2 \geq 0$, and $\gamma_0, \gamma_1, \gamma_2 \geq 0$. Let $y \in C$. From (4.3), it suffices to prove that

$$\left\|T^2 y - Ty\right\|^2 + 2\left\langle T^2 y - Ty,\ Ty - u\right\rangle \leq 0.$$

As the proof of Theorem 5.3, we can obtain (5.3):

$$\alpha_{\bullet 0}\mu_n \left\|T^n x - y\right\|^2 + \alpha_{\bullet 1}\mu_n \left\|T^n x - Ty\right\|^2 + \alpha_{\bullet 2}\mu_n \left\|T^n x - T^2 y\right\|^2 \leq 0$$

since $\beta_2 \geq 0$, $\gamma_0, \gamma_1, \gamma_2 \geq 0$, and $\beta_0 + \beta_1 \geq 0$. Since $\alpha_{\bullet 0} \geq 0$, the following holds:

$$\alpha_{\bullet 1}\mu_n \left\|T^n x - Ty\right\|^2 + \alpha_{\bullet 2}\mu_n \left\|T^n x - T^2 y\right\|^2 \leq 0.$$

From $\alpha_{\bullet 1} + \alpha_{\bullet 2} \geq 0$,

$$\begin{aligned}\alpha_{\bullet 2}\mu_n \left\|T^n x - T^2 y\right\|^2 &\leq -\alpha_{\bullet 1}\mu_n \left\|T^n x - Ty\right\|^2 \\ &\leq \alpha_{\bullet 2}\mu_n \left\|T^n x - Ty\right\|^2.\end{aligned}$$

Therefore, since $\alpha_{\bullet 2} > 0$, we have that

$$\mu_n \left\|T^n x - T^2 y\right\|^2 \leq \mu_n \left\|T^n x - Ty\right\|^2.$$

Thus, it holds that

$$\mu_n \left(\left\|T^n x - Ty\right\|^2 + 2\left\langle T^n x - Ty,\ Ty - T^2 y\right\rangle + \left\|Ty - T^2 y\right\|^2\right) \leq \mu_n \left\|T^n x - Ty\right\|^2,$$

and thus,

$$\mu_n \left(2\left\langle T^n x - Ty,\ Ty - T^2 y\right\rangle + \left\|Ty - T^2 y\right\|^2\right) \leq 0.$$

By using (5.7), we have

$$2\left\langle u - Ty,\ Ty - T^2 y\right\rangle + \left\|Ty - T^2 y\right\|^2 \leq 0$$

for all $y \in C$. This implies that $u \in A_{21}(T)$.

Case (1). Suppose that $\alpha_{1\bullet} + \alpha_{2\bullet} \geq 0$, $\alpha_{0\bullet} \geq 0$, $\alpha_{2\bullet} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$, $\gamma_0 + \gamma_1 \geq 0$, and $\gamma_2 \geq 0$. As in the proof of Theorem 5.3, we can derive the desired result. □

6. Weak convergence theorems

In this section, we present weak convergence theorems for finding attractive points of generic 2-generalized hybrid mappings without assuming that the domains of the mappings are closed. For finding fixed points of the mappings, we additionally assume that the domains are closed. The fundamentals of the proofs were improved by many authors; see, for example, [8], [26], [17], [11], and [4].

Theorem 6.1. *Let C be a nonempty and convex subset of a real Hilbert space H. Let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping that satisfies one of the following conditions:*

$$(1)\ \alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0;$$

$$(2)\ \alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0,\ \alpha_{\bullet 2} \geq 0,\ \alpha_{\bullet 1} > 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0.$$

Suppose that $A_{10}(T)$ is nonempty. Let $P_{A_{10}(T)}$ be the metric projection from H onto $A_{10}(T)$. Let $a, b \in (0, 1)$ such that $a \leq b$, and let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers such that $a_n + b_n + c_n = 1$ and $0 < a \leq a_n, b_n, c_n \leq b < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as

$$x_{n+1} = a_n x_n + b_n T x_n + c_n T^2 x_n \ (\in C)$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges weakly to an element $\overline{x}$ of $A_{10}(T)$, where $\overline{x} \equiv \lim_{n\to\infty} P_{A_{10}(T)} x_n$.

Additionally, if C is closed in H, then $\{x_n\}$ converges weakly to a fixed point $\widehat{x} \equiv \lim_{n\to\infty} P_{F(T)} x_n$ of T, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Proof. **Case (2)**. Suppose that $\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0$, $\alpha_{\bullet 2} \geq 0$, $\alpha_{\bullet 1} > 0$, and $\gamma_0, \gamma_1, \gamma_2 \geq 0$. It follows from [24] that $A_{10}(T)$ is closed and convex. Since it is assumed that $A_{10}(T)$ is nonempty, there exists the metric projection $P_{A_{10}(T)}$ from H onto $A_{10}(T)$. First, we show that

$$\|x_{n+1} - q\| \leq \|x_n - q\| \tag{6.1}$$

for all $q \in A_{10}(T)$ and $n \in \mathbb{N}$. Indeed, since $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are sequences of positive real numbers such that $a_n + b_n + c_n = 1$, we have that

$$\begin{aligned}
\|x_{n+1} - q\| &= \left\|a_n x_n + b_n T x_n + c_n T^2 x_n - q\right\| \\
&= \left\|a_n (x_n - q) + b_n (T x_n - q) + c_n \left(T^2 x_n - q\right)\right\| \\
&\leq a_n \|x_n - q\| + b_n \|T x_n - q\| + c_n \left\|T^2 x_n - q\right\| \\
&\leq a_n \|x_n - q\| + b_n \|x_n - q\| + c_n \|x_n - q\| \\
&= \|x_n - q\|.
\end{aligned}$$

Thus, $\{\|x_n - q\|\}$ is convergent in $\mathbb{R}$, and then, $\{x_n\}$ is bounded. Furthermore, we obtain from (6.1) and Lemma 3.2 that $\left\{P_{A_{10}(T)} x_n\right\}$ is convergent in $A_{10}(T)$.

Next, we demonstrate that

$$a_nb_n \|x_n - Tx_n\|^2 + b_nc_n \left\|Tx_n - T^2x_n\right\|^2 + c_na_n \left\|T^2x_n - x_n\right\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \tag{6.2}$$

for all $q \in A_{10}(T)$ and $n \in \mathbb{N}$. Indeed, using Lemma 3.3, we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \left\|a_n(x_n - q) + b_n(Tx_n - q) + c_n\left(T^2x_n - q\right)\right\|^2 \\
&= a_n\|x_n - q\|^2 + b_n\|Tx_n - q\|^2 + c_n\left\|T^2x_n - q\right\|^2 \\
&- a_nb_n\|x_n - Tx_n\|^2 - b_nc_n\left\|Tx_n - T^2x_n\right\|^2 - c_na_n\left\|T^2x_n - x_n\right\|^2 \\
&\leq a_n\|x_n - q\|^2 + b_n\|x_n - q\|^2 + c_n\|x_n - q\|^2 \\
&- a_nb_n\|x_n - Tx_n\|^2 - b_nc_n\left\|Tx_n - T^2x_n\right\|^2 - c_na_n\left\|T^2x_n - x_n\right\|^2 \\
&= \|x_n - q\|^2 - a_nb_n\|x_n - Tx_n\|^2 - b_nc_n\left\|Tx_n - T^2x_n\right\|^2 - c_na_n\left\|T^2x_n - x_n\right\|^2,
\end{aligned}$$

which implies that (6.2) holds. Since the sequence $\{\|x_n - q\|\}$ is convergent, we have from (6.2) that

$$x_n - Tx_n \to 0, \quad Tx_n - T^2x_n, \quad T^2x_n - x_n \to 0 \tag{6.3}$$

as $n \to \infty$.

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ for some $u \in H$. From (6.3) and Lemma 4.11, we have that $u \in A_{10}(T)$. We prove that $x_n \rightharpoonup u$. Assume that $x_{n_j} \rightharpoonup u_1$ and $x_{n_k} \rightharpoonup u_2$, where $\{x_{n_j}\}$ and $\{x_{n_k}\}$ are subsequences of $\{x_n\}$. From (6.3) and Lemma 4.11, it holds that $u_1, u_2 \in A_{10}(T)$. Thus, two sequences $\{\|x_n - u_1\|\}$ and $\{\|x_n - u_2\|\}$ are convergent. Define $a \equiv \lim_{n\to\infty}\left(\|x_n - u_1\|^2 - \|x_n - u_2\|^2\right) (\in \mathbb{R})$. It holds that

$$\|x_n - u_1\|^2 - \|x_n - u_2\|^2 = -2\langle x_n,\ u_1 - u_2\rangle + \|u_1\|^2 - \|u_2\|^2.$$

Since $x_{n_j} \rightharpoonup u_1$, we have that $a = -2\langle u_1,\ u_1 - u_2\rangle + \|u_1\|^2 - \|u_2\|^2$. Similarly, since $x_{n_k} \rightharpoonup u_2$, we have that $a = -2\langle u_2,\ u_1 - u_2\rangle + \|u_1\|^2 - \|u_2\|^2$. As a result, we obtain $\langle u_1 - u_2,\ u_1 - u_2\rangle = 0$. This means that $u_1 = u_2$, and thus, $x_n \rightharpoonup u$.

We demonstrate that $u = \overline{x}\left(\equiv \lim_{n\to\infty} P_{A_{10}(T)}x_n\right)$. Since $u \in A_{10}(T)$, it follows that

$$\left\langle x_n - P_{A_{10}(T)}x_n,\ P_{A_{10}(T)}x_n - u\right\rangle \geq 0$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we have that $\langle u - \overline{x},\ \overline{x} - u\rangle \geq 0$. This means that $u = \overline{x}$. Hence, we obtain $x_n \rightharpoonup \overline{x}$.

Suppose, in addition to the other assumptions, that C is closed in H. Since C is weakly closed and $x_n \rightharpoonup \overline{x}\,(\in A_{10}(T))$, we have that $\overline{x} \in C \cap A_{10}(T)$. From Lemma 4.6, $\overline{x} \in F(T)$. Thus, $F(T)$ is nonempty. From Lemma 3.4, T is quasi-nonexpansive. Hence, $F(T)$ is closed and convex, and there exists the metric projection $P_{F(T)}$ from H onto $F(T)$. As in the proof of (6.1), we can obtain

$$\|x_{n+1} - q\| \leq \|x_n - q\|$$

for all $q \in F(T)$ and $n \in \mathbb{N}$. Thus, we have from Lemma 3.2 that $\{P_{F(T)}x_n\}$ converges strongly to an element $\widehat{x}$ of $F(T)$, that is, $\widehat{x} = \lim_{n\to\infty} P_{F(T)}x_n$. We prove that

$$\overline{x} \left(= \lim_{n\to\infty} P_{A_{10}(T)}x_n\right) = \widehat{x} \left(= \lim_{n\to\infty} P_{F(T)}x_n\right).$$

Since $\overline{x} \in F(T)$, we have from the property of the metric projection that

$$\langle x_n - P_{F(T)}x_n,\ P_{F(T)}x_n - \overline{x}\rangle \geq 0$$

for all $n \in \mathbb{N}$. Since $x_n \rightharpoonup \overline{x}$ and $P_{F(T)}x_n \to \widehat{x}$, we have that $\langle \overline{x} - \widehat{x}, \widehat{x} - \overline{x}\rangle \geq 0$, and then, $\widehat{x} = \overline{x}$. Thus, $\{x_n\}$ converges weakly to $\widehat{x} = \lim_{n\to\infty} P_{F(T)}x_n$.

Case (1). Suppose that $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0$. As in the proof of Case (2), we can derive the desired result. □

Theorem 6.2. *Let C be a nonempty and convex subset of a real Hilbert space H. Let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping that satisfies one of the following conditions:*

$$(1)\ \alpha_{00} + \alpha_{02} + \alpha_{20} + \alpha_{22} \geq 0,\ \alpha_{10} + \alpha_{12} \geq 0,\ \alpha_{01}, \alpha_{11}, \alpha_{21} \geq 0,$$
$$\alpha_{20} + \alpha_{22} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0,\ \gamma_0, \gamma_1 \geq 0;$$

$$(2)\ \alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0,\ \alpha_{01} + \alpha_{21} \geq 0,\ \alpha_{10}, \alpha_{11}, \alpha_{12} \geq 0,$$
$$\alpha_{02} + \alpha_{22} > 0,\ \beta_0, \beta_1 \geq 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0.$$

Suppose that $A_{20}(T)$ is nonempty. Let $P_{A_{20}(T)}$ be the metric projection from H onto $A_{20}(T)$. Let $a, b \in (0, 1)$ such that $a \leq b$, and let $\{a_n\}$ and $\{c_n\}$ be sequences of real numbers such that $a_n + c_n = 1$ and $0 < a \leq a_n, c_n \leq b < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as

$$x_{n+1} = a_n x_n + c_n T^2 x_n\ (\in C)$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges weakly to an element $\overline{x}$ of $A_{20}(T)$, where $\overline{x} \equiv \lim_{n\to\infty} P_{A_{20}(T)}x_n$.

Additionally, if C is closed in H, then the sequence $\{x_n\}$ converges weakly to a fixed point $\widehat{x} \equiv \lim_{n\to\infty} P_{F(T^2)}x_n$ of T^2, where $P_{F(T^2)}$ is the metric projection from H onto $F(T^2)$.

Proof. **Case (2)**. Suppose that $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0,\ \alpha_{01} + \alpha_{21} \geq 0,\ \alpha_{10}, \alpha_{11}, \alpha_{12} \geq 0,\ \alpha_{02} + \alpha_{22} > 0,\ \beta_0, \beta_1 \geq 0$, and $\gamma_0, \gamma_1, \gamma_2 \geq 0$. Since $A_{20}(T) = A_{10}(T^2)$, it follows from [24] that $A_{20}(T)$ is closed and convex. Furthermore, we assume that $A_{20}(T)$ is nonempty. Thus, there exists the metric projection $P_{A_{20}(T)}$ from H onto $A_{20}(T)$. We show that

$$\|x_{n+1} - q\| \leq \|x_n - q\| \tag{6.4}$$

for all $q \in A_{20}(T)$ and $n \in \mathbb{N}$. Indeed, since $\{a_n\}$ and $\{c_n\}$ are sequence of positive real numbers such that $a_n + c_n = 1$, we have that

$$\begin{aligned}\|x_{n+1} - q\| &= \left\|\left(a_n x_n + c_n T^2 x_n\right) - q\right\| \\ &= \left\|a_n (x_n - q) + c_n \left(T^2 x_n - q\right)\right\|\end{aligned}$$

$$\begin{aligned}&\leq a_n \|x_n - q\| + c_n \left\|T^2 x_n - q\right\| \\ &\leq a_n \|x_n - q\| + c_n \|x_n - q\| \\ &= \|x_n - q\| .\end{aligned}$$

Thus, $\{\|x_n - q\|\}$ is convergent in $\mathbb{R}$, and then, $\{x_n\}$ is bounded. Furthermore, we obtain from Lemma 3.2 that $\left\{P_{A_{20}(T)} x_n\right\}$ is convergent in $A_{20}(T)$.

Next, we demonstrate that

$$a_n c_n \left\|x_n - T^2 x_n\right\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \tag{6.5}$$

for all $q \in A_{20}(T)$ and $n \in \mathbb{N}$. Indeed, using Lemma 3.3, we have that

$$\begin{aligned}&\|x_{n+1} - q\|^2 \\ &= \left\|a_n (x_n - q) + c_n \left(T^2 x_n - q\right)\right\|^2 \\ &= a_n \|x_n - q\|^2 + c_n \left\|T^2 x_n - q\right\|^2 - a_n c_n \left\|x_n - T^2 x_n\right\|^2 \\ &\leq a_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 - a_n c_n \left\|x_n - T^2 x_n\right\|^2 \\ &= \|x_n - q\|^2 - a_n c_n \left\|x_n - T^2 x_n\right\|^2 ,\end{aligned}$$

which means that (6.5) holds. Since the sequence $\{\|x_n - q\|\}$ is convergent, we have from (6.5) that

$$x_n - T^2 x_n \to 0 \tag{6.6}$$

as $n \to \infty$.

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ for some $u \in H$. From (6.6) and Lemma 4.12, we have that $u \in A_{20}(T)$. We prove that $x_n \rightharpoonup u$. Assume that $x_{n_j} \rightharpoonup u_1$ and $x_{n_k} \rightharpoonup u_2$, where $\left\{x_{n_j}\right\}$ and $\{x_{n_k}\}$ are subsequences of $\{x_n\}$. From (6.6) and Lemma 4.12, $u_1, u_2 \in A_{20}(T)$. Thus, two sequences $\{\|x_n - u_1\|\}$ and $\{\|x_n - u_2\|\}$ are convergent. Define $a \equiv \lim_{n\to\infty} \left(\|x_n - u_1\|^2 - \|x_n - u_2\|^2\right) (\in \mathbb{R})$. It holds that

$$\|x_n - u_1\|^2 - \|x_n - u_2\|^2 = -2 \langle x_n,\ u_1 - u_2\rangle + \|u_1\|^2 - \|u_2\|^2 .$$

Since $x_{n_j} \rightharpoonup u_1$, we have that $a = -2\langle u_1,\ u_1 - u_2\rangle + \|u_1\|^2 - \|u_2\|^2$. Similarly, since $x_{n_k} \rightharpoonup u_2$, we have that $a = -2\langle u_2,\ u_1 - u_2\rangle + \|u_1\|^2 - \|u_2\|^2$. Consequently, we obtain $\langle u_1 - u_2,\ u_1 - u_2\rangle = 0$. This means that $u_1 = u_2$, and thus, $x_n \rightharpoonup u$.

We show that $u = \overline{x} \left(\equiv \lim_{n\to\infty} P_{A_{20}(T)} x_n\right)$. Since $u \in A_{20}(T)$, it follows that

$$\left\langle x_n - P_{A_{20}(T)} x_n,\ P_{A_{20}(T)} x_n - u\right\rangle \geq 0$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we have that $\langle u - \overline{x},\ \overline{x} - u\rangle \geq 0$. This means that $u = \overline{x}$. Hence, we obtain $x_n \rightharpoonup \overline{x} \left(\equiv \lim_{k\to\infty} P_{A_{20}(T)} x_k\right)$.

Suppose, in addition to the other assumptions, that C is closed in H. Since C is weakly closed and $x_n \rightharpoonup \overline{x} (\in A_{20}(T))$, we have that $\overline{x} \in C \cap A_{20}(T)$. From Lemma 4.7, it holds that $\overline{x} \in F\left(T^2\right)$. Thus, $F\left(T^2\right)$ is nonempty. Furthermore, from Lemma 3.5, T^2 is quasi-nonexpansive. Therefore, $F\left(T^2\right)$ is closed and convex.

Consequently, there exists the metric projection $P_{F(T^2)}$ from H onto $F(T^2)$. In much the same way as for the proof of (6.4), we can obtain

$$\|x_{n+1} - q\| \leq \|x_n - q\|$$

for all $q \in F(T^2)$ and $n \in \mathbb{N}$. Thus, we have from Lemma 3.2 that $\{P_{F(T^2)}x_n\}$ converges strongly to an element $\widehat{x}$ of $F(T^2)$, that is, $\widehat{x} \equiv \lim_{n\to\infty} P_{F(T^2)}x_n$. We prove that

$$\overline{x}\left(\equiv \lim_{n\to\infty} P_{A_{20}(T)}x_n\right) = \widehat{x}\left(\equiv \lim_{n\to\infty} P_{F(T^2)}x_n\right).$$

Since $\overline{x} \in F\left(T^2\right)$, we have from the property of the metric projection that

$$\langle x_n - P_{F(T^2)}x_n,\ P_{F(T^2)}x_n - \overline{x}\rangle \geq 0$$

for all $n \in \mathbb{N}$. Since $x_n \rightharpoonup \overline{x}$ and $P_{F(T^2)}x_n \to \widehat{x}$, we have that $\langle \overline{x}-\widehat{x}, \widehat{x}-\overline{x}\rangle \geq 0$, which means that $\widehat{x} = \overline{x}$. Thus, $\{x_n\}$ converges weakly to $\widehat{x} = \lim_{n\to\infty} P_{F(T^2)}x_n \in F\left(T^2\right)$.

Case (1). Suppose that $\alpha_{00}+\alpha_{02}+\alpha_{20}+\alpha_{22} \geq 0$, $\alpha_{10}+\alpha_{12} \geq 0$, $\alpha_{01}, \alpha_{11}, \alpha_{21} \geq 0$, $\alpha_{20}+\alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$, and $\gamma_0, \gamma_1 \geq 0$. As in the proof of Case (2), we can derive the desired result. □

7. Strong convergence theorems

In this section, we present strong convergence theorems for finding attractive points of generic 2-generalized hybrid mappings without assuming that the domains of the mappings are closed. Additionally, for finding fixed points of the mappings, we assume that the domains are closed. The fundamentals of the proof were developed in [27], [12], [5], and [19].

Theorem 7.1. *Let C be a nonempty and convex subset of H. Let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping that satisfies one of the following conditions:*

$$(1)\ \alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0;$$
$$(2)\ \alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0,\ \alpha_{\bullet 2} \geq 0,\ \alpha_{\bullet 1} > 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0.$$

Suppose that $A_{10}(T)$ is nonempty. Let $P_{A_{10}(T)}$ be the metric projection from H onto $A_{10}(T)$. Let $a, b \in (0, 1)$ such that $a \leq b$, and let $\{\lambda_n\}$, $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $(0, 1)$ such that

$$\lambda_n \to 0, \qquad \sum_{n=1}^{\infty} \lambda_n = \infty,$$
$$a_n + b_n + c_n = 1, \quad 0 < a \leq a_n, b_n, c_n \leq b < 1, \quad \forall n \in \mathbb{N}.$$

Let $\{z_n\}$ be a sequence in C such that $z_n \to z$. Define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n z_n + (1 - \lambda_n)\left(a_n x_n + b_n T x_n + c_n T^2 x_n\right)$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges strongly to an attractive point $\overline{z} \in A_{10}(T)$, where $\overline{z} \equiv P_{A_{10}(T)}z$.

Additionally, if C is closed in H, then $\{x_n\}$ converges strongly to a fixed point $\widehat{z} = P_{F(T)}z \in F(T)$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Proof. **Case (2)**. Suppose that $\alpha_{\bullet 0}+\alpha_{\bullet 1}\geq 0$, $\alpha_{\bullet 2}\geq 0$, $\alpha_{\bullet 1}>0$, and $\gamma_0,\gamma_1,\gamma_2\geq 0$. Define $y_n\equiv a_n x_n+b_n T x_n+c_n T^2 x_n\in C$ for all $n\in\mathbb{N}$. Then, we have that $x_{n+1}=\lambda_n z_n+(1-\lambda_n)\,y_n\in C$.

First, we show that $x_n\to\overline{z}\equiv P_{A_{10}(T)}z$. It is easy to verify that

$$\|y_n-q\|\leq\|x_n-q\| \tag{7.1}$$

for all $q\in A_{10}(T)$ and $n\in\mathbb{N}$. Indeed, since $q\in A_{10}(T)$, $a_n+b_n+c_n=1$, and $a_n,b_n,c_n\geq 0$, we have that

$$\begin{aligned}\|y_n-q\| &\equiv \left\|a_n x_n+b_n T x_n+c_n T^2 x_n-q\right\|\\ &\leq a_n\|x_n-q\|+b_n\|Tx_n-q\|+c_n\left\|T^2x_n-q\right\|\\ &\leq a_n\|x_n-q\|+b_n\|x_n-q\|+c_n\|x_n-q\|\\ &= \|x_n-q\|.\end{aligned}$$

We can demonstrate that the sequence $\{x_n\}$ is bounded by using mathematical induction. Indeed, let $q\in A_{10}(T)$, and define

$$M\equiv\max\left\{\sup_{k\in\mathbb{N}}\|z_k-q\|\,,\ \ \|x_1-q\|\right\}.$$

Since $\{z_n\}$ is bounded, M is a real number. We prove that $\|x_n-q\|\leq M$ for all $n\in\mathbb{N}$. (i) It is obvious for the case of $n=1$. (ii) Assume that $\|x_k-q\|\leq M$ for some $k\in\mathbb{N}$. We have from (7.1) that

$$\begin{aligned}\|x_{k+1}-q\| &\leq \|\lambda_k z_k+(1-\lambda_k)\,y_k-q\|\\ &\leq \lambda_k\|z_k-q\|+(1-\lambda_k)\|y_k-q\|\\ &\leq \lambda_k\|z_k-q\|+(1-\lambda_k)\|x_k-q\|\\ &\leq \lambda_k M+(1-\lambda_k)\,M=M.\end{aligned}$$

Hence, $\{x_n\}$ is bounded.

Let us show that the following inequality holds:

$$\begin{aligned}a_nb_n\|x_n-Tx_n\|^2+b_nc_n\|Tx_n-T^2x_n\|^2+c_na_n\|T^2x_n-x_n|^2\\ \leq\lambda_n\|z_n-q\|^2+\|x_n-q\|^2-\|x_{n+1}-q\|^2\end{aligned} \tag{7.2}$$

for all $q\in A_{10}(T)$ and $n\in\mathbb{N}$. Indeed, using Lemma 3.3, we obtain

$$\begin{aligned}\|x_{n+1}-q\|^2 &= \|\lambda_n(z_n-q)+(1-\lambda_n)(y_n-q)\|^2\\ &\leq \lambda_n\|z_n-q\|^2+(1-\lambda_n)\|y_n-q\|^2\\ &\leq \lambda_n\|z_n-q\|^2+\left\|a_n(x_n-q)+b_n(Tx_n-q)+c_n\left(T^2x_n-q\right)\right\|^2\\ &= \lambda_n\|z_n-q\|^2+a_n\|x_n-q\|^2+b_n\|Tx_n-q\|^2+c_n\left\|T^2x_n-q\right\|^2\\ &\quad -a_nb_n\|x_n-Tx_n\|^2-b_nc_n\left\|Tx_n-T^2x_n\right\|^2-c_na_n\left\|T^2x_n-x_n\right\|^2\\ &\leq \lambda_n\|z_n-q\|^2+a_n\|x_n-q\|^2+b_n\|x_n-q\|^2+c_n\|x_n-q\|^2\\ &\quad -a_nb_n\|x_n-Tx_n\|^2-b_nc_n\left\|Tx_n-T^2x_n\right\|^2-c_na_n\left\|T^2x_n-x_n\right\|^2\\ &= \lambda_n\|z_n-q\|^2+\|x_n-q\|^2\end{aligned}$$

$$- a_n b_n \|x_n - Tx_n\|^2 - b_n c_n \left\|Tx_n - T^2 x_n\right\|^2 - c_n a_n \left\|T^2 x_n - x_n\right\|^2 .$$

Thus, the relationship (7.2) follows. Furthermore, it holds that

$$\|x_{n+1} - x_n\| \leq \lambda_n \|z_n - x_n\| + \|Tx_n - x_n\| + \left\|T^2 x_n - x_n\right\| \tag{7.3}$$

for all $n \in \mathbb{N}$. This inequality can be ascertained as follows:

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\lambda_n z_n + (1 - \lambda_n)\, y_n - x_n\| \\ &\leq \lambda_n \|z_n - x_n\| + (1 - \lambda_n)\, \|y_n - x_n\| \\ &\leq \lambda_n \|z_n - x_n\| + \|a_n x_n + b_n T x_n + c_n T^2 x_n - (a_n + b_n + c_n)\, x_n\| \\ &\leq \lambda_n \|z_n - x_n\| + b_n \|Tx_n - x_n\| + c_n \|T^2 x_n - x_n\| \\ &\leq \lambda_n \|z_n - x_n\| + \|Tx_n - x_n\| + \|T^2 x_n - x_n\|. \end{aligned}$$

Define $X_n = \|x_n - \overline{z}\|^2 \, (\geq 0)$, where $\overline{z} = P_{A_{10}(T)} z$. Our goal is to show that $X_n \to 0$ as $n \to \infty$. Let us divide the rest of the proof into two cases.

Case (A). Suppose that there exists a natural number n' such that $X_{n+1} \leq X_n$ for all $n \geq n'$. In this case, the sequence $\{X_n\}$ is convergent. Since $\overline{z} \in A_{10}(T)$, it holds from (7.2) that

$$\begin{aligned} a_n b_n \|x_n - Tx_n\|^2 - b_n c_n \|Tx_n - T^2 x_n\|^2 - c_n a_n \|T^2 x_n - x_n\|^2 &\leq \lambda_n \|z_n - \overline{z}\|^2 + \|x_n - \overline{z}\|^2 - \|x_{n+1} - \overline{z}\|^2 \\ &= \lambda_n \|z_n - \overline{z}\|^2 + X_n - X_{n+1} \end{aligned} \tag{7.4}$$

for all $n \in \mathbb{N}$. Since $\{z_n\}$ is bounded and $\lambda_n \to 0$, we have that

$$x_n - Tx_n \to 0, \qquad Tx_n - T^2 x_n \to 0, \qquad T^2 x_n - x_n \to 0. \tag{7.5}$$

Then, it holds from (7.3) that

$$x_{n+1} - x_n \to 0. \tag{7.6}$$

Using (2.1) and (7.1), we obtain

$$\begin{aligned} X_{n+1} &= \|x_{n+1} - \overline{z}\|^2 \\ &= \|\lambda_n (z_n - \overline{z}) + (1 - \lambda_n)(y_n - \overline{z})\|^2 \\ &\leq (1 - \lambda_n)^2 \|y_n - \overline{z}\|^2 + 2\lambda_n \langle x_{n+1} - \overline{z},\ z_n - \overline{z} \rangle \\ &\leq (1 - \lambda_n) \|x_n - \overline{z}\|^2 + 2\lambda_n \left(\langle x_{n+1} - x_n,\ z_n - \overline{z} \rangle + \langle x_n - \overline{z},\ z_n - \overline{z} \rangle \right) \\ &\equiv (1 - \lambda_n) X_n + 2\lambda_n \left(\langle x_{n+1} - x_n,\ z_n - \overline{z} \rangle + \langle x_n - \overline{z},\ z_n - \overline{z} \rangle \right) \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\{z_n\}$ is bounded, it holds from (7.6) that $\langle x_{n+1} - x_n, z_n - \overline{z} \rangle \to 0$. Since $\sum_{n=1}^{\infty} \lambda_n = \infty$ is assumed, from Lemma 3.6, it suffices to prove that

$$\limsup_{n \to \infty} \langle x_n - \overline{z},\ z_n - \overline{z} \rangle \leq 0.$$

Since the sequences $\{x_n\}$ is bounded and $z_n \to z$, we can assume, without loss of generality, that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n \to \infty} \langle x_n - \overline{z},\ z_n - \overline{z} \rangle = \lim_{i \to \infty} \langle x_{n_i} - \overline{z},\ z_{n_i} - \overline{z} \rangle$$

and $x_{n_i} \rightharpoonup u$ for some $u \in H$. Lemma 4.11 and (7.5) imply that $u \in A_{10}(T)$. Then, we have from $z_n \to z$ and $\overline{z} = P_{A_{10}(T)} z$ that

$$\begin{aligned}\limsup_{n\to\infty} \langle x_n - \overline{z},\ z_n - \overline{z}\rangle &= \lim_{i\to\infty} \langle x_{n_i} - \overline{z},\ z_{n_i} - \overline{z}\rangle \\ &= \langle u - \overline{z},\ z - \overline{z}\rangle \leq 0.\end{aligned}$$

This completes the proof for Case (A).

Case (B). Suppose that there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let n_0 be a natural number such that

$$\{k \in \mathbb{N} : k \leq n_0,\ X_k < X_{k+1}\}$$

is nonempty. Define

$$\tau(n) = \max\{k \in \mathbb{N} : k \leq n,\ X_k < X_{k+1}\}, \quad \forall n \geq n_0.$$

From Lemma 3.7, we have that

$$\tau(n) \to \infty \text{ as } n \to \infty; \tag{7.7}$$

$$X_n \leq X_{\tau(n)+1}, \quad \forall n \geq n_0; \tag{7.8}$$

$$X_{\tau(n)} < X_{\tau(n)+1}, \quad \forall n \geq n_0. \tag{7.9}$$

From (7.8), it suffices to show that $X_{\tau(n)+1} \to 0$. From the assumptions of this theorem, we have that

$$\lambda_{\tau(n)} \to 0 \text{ as } n \to \infty, \tag{7.10}$$

$$a_{\tau(n)} + b_{\tau(n)} + c_{\tau(n)} = 1 \text{ and} \tag{7.11}$$

$$0 < a \leq a_{\tau(n)},\ b_{\tau(n)},\ c_{\tau(n)} \leq b < 1 \tag{7.12}$$

for all $n \geq n_0$. Since $\overline{z} \in A_{10}(T)$, inequalities (7.1)–(7.3) yields

$$\left\|y_{\tau(n)} - \overline{z}\right\| \leq \left\|x_{\tau(n)} - \overline{z}\right\|, \tag{7.13}$$

$$\begin{aligned} &a_{\tau(n)} b_{\tau(n)} \|x_{\tau(n)} - Tx_{\tau(n)}\|^2 + b_{\tau(n)} c_{\tau(n)} \left\|Tx_{\tau(n)} - T^2 x_{\tau(n)}\right\|^2 \\ &\quad + c_{\tau(n)} a_{\tau(n)} \left\|T^2 x_{\tau(n)} - x_{\tau(n)}\right\|^2 \\ &\leq \lambda_{\tau(n)} \left\|z_{\tau(n)} - \overline{z}\right\|^2 + \left\|x_{\tau(n)} - \overline{z}\right\|^2 - \left\|x_{\tau(n)+1} - \overline{z}\right\|^2 \\ &= \lambda_{\tau(n)} \left\|z_{\tau(n)} - \overline{z}\right\|^2 + X_{\tau(n)} - X_{\tau(n)+1} \end{aligned} \tag{7.14}$$

and

$$\begin{aligned} &\|x_{\tau(n)+1} - x_{\tau(n)}\| \\ &\leq \lambda_{\tau(n)} \|z_{\tau(n)} - x_{\tau(n)}\| + \|Tx_{\tau(n)} - x_{\tau(n)}\| + \|T^2 x_{\tau(n)} - x_{\tau(n)}\|. \end{aligned} \tag{7.15}$$

From (7.9) and (7.14), it holds that

$$\begin{aligned} &a_{\tau(n)} b_{\tau(n)} \left\|x_{\tau(n)} - Tx_{\tau(n)}\right\|^2 + b_{\tau(n)} c_{\tau(n)} \left\|Tx_{\tau(n)} - T^2 x_{\tau(n)}\right\|^2 \\ &\quad + c_{\tau(n)} a_{\tau(n)} \left\|T^2 x_{\tau(n)} - x_{\tau(n)}\right\|^2 \leq \lambda_{\tau(n)} \left\|z_{\tau(n)} - \overline{z}\right\|^2. \end{aligned}$$

Since $\{z_{\tau(n)}\}$ is bounded, we obtain from (7.10) and (7.12) that

$$x_{\tau(n)} - Tx_{\tau(n)} \to 0, \quad Tx_{\tau(n)} - T^2x_{\tau(n)} \to 0, \quad T^2x_{\tau(n)} - x_{\tau(n)} \to 0 \tag{7.16}$$

as $n \to \infty$. Furthermore, (7.10), (7.16) and (7.15) imply that

$$x_{\tau(n)+1} - x_{\tau(n)} \to 0. \tag{7.17}$$

Since $\{x_{\tau(n)}\}$ and $\{x_{\tau(n)+1}\}$ are bounded, we have that

$$X_{\tau(n)+1} - X_{\tau(n)} \to 0. \tag{7.18}$$

Thus, our aim is to prove that $X_{\tau(n)} \to 0$. We have from (2.1) and (7.13) that

$$\begin{aligned}
X_{\tau(n)+1} &\equiv \left\|x_{\tau(n)+1} - \overline{z}\right\|^2 \\
&= \left\|\lambda_{\tau(n)}\left(z_{\tau(n)} - \overline{z}\right) + \left(1 - \lambda_{\tau(n)}\right)\left(y_{\tau(n)} - \overline{z}\right)\right\|^2 \\
&\leq \left(1 - \lambda_{\tau(n)}\right)^2 \left\|y_{\tau(n)} - \overline{z}\right\|^2 + 2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1} - \overline{z},\ z_{\tau(n)} - \overline{z}\right\rangle \\
&\leq \left(1 - \lambda_{\tau(n)}\right) \left\|x_{\tau(n)} - \overline{z}\right\|^2 + 2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1} - \overline{z},\ z_{\tau(n)} - \overline{z}\right\rangle \\
&\equiv \left(1 - \lambda_{\tau(n)}\right) X_{\tau(n)} + 2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1} - \overline{z},\ z_{\tau(n)} - \overline{z}\right\rangle,
\end{aligned}$$

and hence,

$$\lambda_{\tau(n)}X_{\tau(n)} \leq X_{\tau(n)} - X_{\tau(n)+1} + 2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1} - \overline{z},\ z_{\tau(n)} - \overline{z}\right\rangle.$$

From (7.9),

$$\lambda_{\tau(n)}X_{\tau(n)} \leq 2\lambda_{\tau(n)}\left\langle x_{\tau(n)+1} - \overline{z},\ z_{\tau(n)} - \overline{z}\right\rangle.$$

From $\lambda_{\tau(n)} > 0$, we have that

$$\begin{aligned}
X_{\tau(n)} &\leq 2\left\langle x_{\tau(n)+1} - \overline{z},\ z_{\tau(n)} - \overline{z}\right\rangle \\
&= 2\left\langle x_{\tau(n)+1} - x_{\tau(n)},\ z_{\tau(n)} - \overline{z}\right\rangle + 2\left\langle x_{\tau(n)} - \overline{z},\ z_{\tau(n)} - \overline{z}\right\rangle \\
&= 2\left\langle x_{\tau(n)+1} - x_{\tau(n)},\ z_{\tau(n)} - \overline{z}\right\rangle \\
&\quad + 2\left\langle x_{\tau(n)} - \overline{z},\ z_{\tau(n)} - z\right\rangle + 2\left\langle x_{\tau(n)} - \overline{z},\ z - \overline{z}\right\rangle
\end{aligned}$$

Since $\{x_{\tau(n)}\}$ is bounded and $z_{\tau(n)} \to z$, we have from (7.17) that

$$2\left\langle x_{\tau(n)+1} - x_{\tau(n)},\ z_{\tau(n)} - \overline{z}\right\rangle + 2\left\langle x_{\tau(n)} - \overline{z},\ z_{\tau(n)} - z\right\rangle \to 0$$

as $n \to \infty$. Hence, it suffices to prove that

$$\limsup_{n\to\infty}\left\langle x_{\tau(n)} - \overline{z},\ z - \overline{z}\right\rangle.$$

Since $\{x_{\tau(n)}\}$ is bounded, we can assume, without loss of generality, that there is a subsequence $\{x_{\tau(n_i)}\}$ of $\{x_{\tau(n)}\}$ such that

$$\limsup_{n\to\infty}\left\langle x_{\tau(n)} - \overline{z},\ z - \overline{z}\right\rangle = \lim_{i\to\infty}\left\langle x_{\tau(n_i)} - \overline{z},\ z - \overline{z}\right\rangle$$

and $x_{\tau(n_i)} \rightharpoonup u$ for some $u \in H$. From Lemma 4.11 and (7.16), it holds that $u \in A_{10}(T)$. Since $\overline{z} \equiv P_{A_{10}(T)}z$, we obtain

$$\begin{aligned}
\limsup_{n\to\infty}\left\langle x_{\tau(n)} - \overline{z},\ z - \overline{z}\right\rangle &= \lim_{i\to\infty}\left\langle x_{\tau(n_i)} - \overline{z},\ z - \overline{z}\right\rangle \\
&= \left\langle u - \overline{z},\ z - \overline{z}\right\rangle \leq 0.
\end{aligned}$$

This completes the proof for Case (B), and we have shown that $x_n \to \overline{z} \equiv P_{A_{10}(T)}z$.

Next, suppose, in addition to the other assumptions, that C is closed in H. We show that $x_n \to \widehat{z} \equiv P_{F(T)}z$. Since $x_n \to \overline{z} \equiv P_{A_{10}(T)}z$ and C is closed, we have that $\overline{z} \in C \cap A_{10}(T)$. From Lemma 4.6, it holds that $\overline{z} \in F(T)$. Thus, $F(T)$ is nonempty. Since T is quasi-nonexpansive, $F(T)$ is closed and convex. Hence, there exists the metric projection $P_{F(T)}$ from H onto $F(T)$. We demonstrate that

$$(\widehat{z} \equiv)\, P_{F(T)}z = \overline{z} \left(\equiv P_{A_{10}(T)}z\right).$$

Since $\overline{z} \in F(T)$, it suffices to prove that $\|z - \overline{z}\| \leq \|z - v\|$ for all $v \in F(T)$. Let $v \in F(T)$. Since $F(T) \subset A_{10}(T)$, we have that

$$\begin{aligned} \|z - \overline{z}\| &= \inf\{\|z - q\| : q \in A_{10}(T)\} \\ &\leq \inf\{\|z - q\| : q \in F(T)\} \\ &\leq \|z - v\|. \end{aligned}$$

This means that $\overline{z} = P_{F(T)}z\,(\equiv \widehat{z})$. This completes the proof for Case (2).

Case (1). Suppose that $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} > 0$, and $\beta_0, \beta_1, \beta_2 \geq 0$. As in the proof of Case (2), we can derive the desired result. □

As in the proofs of Theorems 6.2 and 7.1, we also have the following strong convergence theorem in a Hilbert space.

Theorem 7.2. *Let C be a nonempty and convex subset of H. Let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping that satisfies one of the following conditions:*

$$\begin{aligned} (1)\ &\alpha_{00} + \alpha_{02} + \alpha_{20} + \alpha_{22} \geq 0,\ \alpha_{10} + \alpha_{12} \geq 0,\ \alpha_{01}, \alpha_{11}, \alpha_{21} \geq 0, \\ &\alpha_{20} + \alpha_{22} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0,\ \gamma_0, \gamma_1 \geq 0; \end{aligned}$$

$$\begin{aligned} (2)\ &\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0,\ \alpha_{01} + \alpha_{21} \geq 0,\ \alpha_{10}, \alpha_{11}, \alpha_{12} \geq 0, \\ &\alpha_{02} + \alpha_{22} > 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0,\ \beta_0, \beta_1 \geq 0. \end{aligned}$$

Suppose that $A_{20}(T)$ is nonempty. Let $P_{A_{20}(T)}$ be the metric projection from H onto $A_{20}(T)$. Let $a, b \in (0,1)$ such that $a \leq b$, and let $\{\lambda_n\}$, $\{a_n\}$ and $\{c_n\}$ be sequences of real numbers in the interval $(0,1)$ such that

$$\lambda_n \to 0, \qquad \sum_{n=1}^{\infty} \lambda_n = \infty,$$

$$a_n + c_n = 1, \quad 0 < a \leq a_n, c_n \leq b < 1 \quad \text{for all } n \in \mathbb{N}.$$

Let $\{z_n\}$ be a sequence in C such that $z_n \to z$. Define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n z_n + (1 - \lambda_n)\left(a_n x_n + c_n T^2 x_n\right)$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges strongly to a $(2,0)$-attractive point $\overline{z} \in A_{20}(T)$, where $\overline{z} \equiv P_{A_{20}(T)}z$.

Additionally, if C is closed in H, then $\{x_n\}$ converges strongly to a fixed point $\widehat{z} \equiv P_{F(T^2)}z \in F\left(T^2\right)$, where $P_{F(T^2)}$ is the metric projection from H onto $F\left(T^2\right)$.

Problem. We do not know whether weak and strong convergence theorems which relate to $(2,1)$-attractive points hold or not.

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