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EXISTENCE THEOREMS OF CONE SADDLE-POINTS IN SET OPTIMIZATION APPLYING NONLINEAR SCALARIZATIONS

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Abstract. There are two types of criteria of solutions for the set-valued optimization problem, the vectorial criterion and set optimization criterion. The first criterion consists of looking for efficient points of set-valued map and is called set-valued vector optimization problem. On the other hand, Kuroiwa-Tanaka-Ha and Jahn-Ha started developing a new approach to set-valued optimization which is based on comparison among values of the set-valued map. We treat the second type criterion and call it a set optimization problem.

In this paper, we consider set-valued saddle point problem. First, we investigate several types of cone-convexity and cone continuity of set-valued map. By using scalarizing technique for set-valued map, we give existence theorems of cone saddle-points for set-valued map.

1. Introduction

Let Y be a topological vector space ordered by a closed convex cone $C \subset Y$. Let X be a nonempty set and $F : X \to 2^Y$ a set-valued map with domain X ($F(x) \neq \emptyset$ for each $x \in X$). The set-valued optimization problem is formalized as follows:

$$(\mathrm{P}) \begin{cases} \text{Optimize} & F(x) \\ \text{Subject to} & x \in X \end{cases}$$

The above problem is based on comparison among values of F, that is, whole images $F(x)$ (for details see Kuroiwa-Tanaka-Ha [25] and Jahn-Ha [17]) and seems to be more natural for set-valued optimization problem.

In this paper, we consider set-valued saddle point problem as a natural generalizations of vector-valued saddle point problem. Then we present existence theorems of cone saddle-points in set optimization problem by using nonlinear scalarization technique for sets.

The organization of this paper is as follows. First, we introduce some types of nonlinear scalarizing technique for sets [2] which are generalization of Gerstewitz's scalarizing functions for the vector-valued case [7, 9, 10]. Then we correct some errors of [2]. The revised results contain some improvements of [2, 11, 12]. Next, we introduce several types of cone-convexity, cone-concavity and cone continuity of set-valued map and investigate their properties. Lastly, by using scalarizing technique for set-valued maps, we give existence theorems of cone saddle-points for set-valued maps which are natural extensions of Sion's minimax theorem [33].

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2. Mathematical preliminaries

2.1. **Mathematical terminology and notation.** Throughout of this paper, let Z be a normed vector space and 0_Z the origin of Z. For a set $A \subset Z$, int$A$ and cl$A$ denote the topological interior and the topological closure, respectively. We denote $\mathcal{V}$ by the family of nonempty subsets of Z. The sum of two sets $V_1, V_2 \in \mathcal{V}$ and the product of $\alpha \in \mathbb{R}$ and $V \in \mathcal{V}$ are defined by

$$V_1 + V_2 := \{v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2\} \qquad \alpha V := \{\alpha v \mid v \in V\}.$$

In this paper, we assume that $C \subset Z$ is a solid closed convex cone, that is, $\mathrm{int}C \neq \emptyset$, $\mathrm{cl}C = C$, $C + C \subset C$ and $t \cdot C \subset C$ for all $t \in [0, \infty)$.

Lemma 2.1 ([24]). *For $C \subset Z$ a closed convex cone and $A, B, V \in \mathcal{V}$, the following statements hold:*

(i) $C + C = C$;
(ii) $C + \mathrm{int}C = \mathrm{int}C$;
(iii) $\mathrm{cl}A + \mathrm{cl}B \subset \mathrm{cl}(A + B)$;
(iv) $\mathrm{cl}(V + C) + C = \mathrm{cl}(V + C)$.

Proof. The last property follows from properties (i) and (iii). □

2.2. **Preliminaries in vector optimization.** A cone C is called pointed if $C \cap (-C) = \{0_Z\}$ and solid if $\mathrm{int}C \neq \emptyset$, respectively.

Definition 2.2. For $a, b \in Z$ and a solid convex cone $C \subset Z$, we define

$$a \leq_C b \quad \text{by} \quad b - a \in C \qquad\qquad a \leq_{\mathrm{int}C} b \quad \text{by} \quad b - a \in \mathrm{int}C.$$

Proposition 2.3. *For $x \in Z$ and $y \in Z$, the following statements hold:*

(i) *$x \leq_C y$ implies that $x + z \leq_C y + z$ for all $z \in Z$,*
(ii) *$x \leq_C y$ implies that $\alpha x \leq_C \alpha y$ for all $\alpha \geq 0$,*
(iii) *$\leq_C$ is reflexive and transitive. Moreover, if C is pointed, $\leq_C$ is antisymmetric and hence a partial order.*

We say that a point $a \in A \subset Z$ is a minimal [resp. weak minimal] point of A if there is no $\hat{a} \in A \setminus \{a\}$ such that $\hat{a} \leq_C a$ [resp. $\hat{a} \leq_{\mathrm{int}C} a$]. The above definition is equivalent to

$$A \cap (a - C) = \{a\} \quad [\text{resp. } A \cap (a - \mathrm{int}C) = \emptyset].$$

Similarly, we say that a point $a \in A \subset Z$ is a maximal [resp.weak maximal] point of A if there is no $\hat{a} \in A \setminus \{a\}$ such that $a \leq_C \hat{a}$ [resp. $a \leq_{\mathrm{int}C} \hat{a}$]. The above definition is equivalent to

$$A \cap (a + C) = \{a\} \quad [\text{resp. } A \cap (a + \mathrm{int}C) = \emptyset].$$

We denote by $\mathrm{Min}(A; C)$[resp. $\mathrm{wMin}(A; \mathrm{int}C)$] and $\mathrm{Max}(A; C)$[resp. $\mathrm{wMax}(A; \mathrm{int}C)$] the set of minimal [resp. weak minimal] and maximal [resp. weak maximal] points of A with respect to C [resp. int$C$], respectively. We can easily see that

$$\mathrm{Min}(A; C) \subset \mathrm{wMin}(A; \mathrm{int}C) \subset A \qquad \text{and} \qquad \mathrm{Max}(A; C) \subset \mathrm{wMax}(A; \mathrm{int}C) \subset A.$$

2.3. **Preliminaries in set optimization.** We consider several types of binary relationships on $\mathcal{V}$ by using a solid convex cone $C \subset Y$.

Definition 2.4 (Kuroiwa-Tanaka-Ha [25], Jahn-Ha [17])**.** For A, $B \in \mathcal{V}$ and a solid closed convex cone $C \subset Y$, we define

(**lower type**) $A \leq_C^l B$ by $B \subset A + C$ $(A \leq_{\text{int}C}^l B$ by $B \subset A + \text{int}C)$,

(**upper type**) $A \leq_C^u B$ by $A \subset B - C$ $(A \leq_{\text{int}C}^u B$ by $A \subset B - \text{int}C)$,

(**lower and upper type**) $A \leq_C^{l\&u} B$ by $B \subset A + C$ and $A \subset B - C$

$(A \leq_{\text{int}C}^{l\&u} B$ by $B \subset A + \text{int}C$ and $A \subset B - \text{int}C)$.

Proposition 2.5 ([27])**.** *For A, B, $D \in \mathcal{V}$, the following statements hold.*

(i) $A \leq_C^l B$ *implies* $(A + D) \leq_C^l (B + D)$ *and* $A \leq_C^u B$ *implies* $(A + D) \leq_C^u (B + D)$.

(ii) $A \leq_C^l B$ *implies* $\alpha A \leq_C^l \alpha B$ *for* $\alpha \geq 0$ *and* $A \leq_C^u B$ *implies* $\alpha A \leq_C^u \alpha B$ *for* $\alpha \geq 0$.

(iii) $\leq_C^l$ *and* $\leq_C^u$ *are reflexive and transitive.*

Proposition 2.6 (see also [2])**.** *For A, $B \in \mathcal{V}$, the following statements hold.*

(i) $A \leq_C^{l\&u} B$ *implies* $A \leq_C^l B$ *and* $A \leq_C^{l\&u} B$ *implies* $A \leq_C^u B$.

(ii) $A \leq_C^l B$ *and* $A \leq_C^u B$ *are not comparable, that is,* $A \leq_C^l B$ *does not imply* $A \leq_C^u B$ *and* $A \leq_C^u B$ *does not imply* $A \leq_C^l B$.

Definition 2.7 ([29])**.** It is said that $A \in \mathcal{V}$ is

(i) C-closed $\big[(-C)$-closed$\big]$ if $A + C$ $[A - C]$ is a closed set,

(ii) C-bounded $\big[(-C)$-bounded$\big]$ if for each neighborhood U of zero in Z there is some positive number $t > 0$ such that

$$A \subset tU + C \quad [A \subset tU - C],$$

(iii) C-compact $\big[(-C)$-compact$\big]$ if any cover of A the form $\{U_\alpha + C |\ U_\alpha$ are open$\}$ $[\{U_\alpha - C |\ U_\alpha$ are open$\}]$ admits a finite subcover.

Every C-compact set is C-closed and C-bounded.

Definition 2.8 ([15])**.** It is said that $A \in \mathcal{V}$ is C-proper $\big[(-C)$-proper$\big]$ if

$$A + C \neq Z \qquad [A - C \neq Z].$$

We denote by $\mathcal{V}_C$ the family of C-proper subsets of Z, $\mathcal{V}_{-C}$ the family of $(-C)$-proper subsets of Z and $\mathcal{V}_{\pm C}$ the family of C-proper and $(-C)$-proper subsets of Z, respectively.

Remark 2.9. It sometimes happens that $\leq_C^l$ is equivalent to $\leq_{\text{int}C}^l$. Thus when we need to distinguish between $A \leq_C^l B$ and $A \leq_{\text{int}C}^l B$ for $A, B \in \mathcal{V}$, we assume C-closeness of A. Similarly, when we need to distinguish between $A \leq_C^u B$ and $A \leq_{\text{int}C}^u B$, we assume $(-C)$-closeness of B. Furthermore, when we need to distinguish between $A \leq_C^{l\&u} B$ and $A \leq_{\text{int}C}^{l\&u} B$, we assume C-closeness of A and $(-C)$-closeness of B. (see example [2]).

Introducing the equivalence relations

$$A \sim_l B \Longleftrightarrow A \leq_C^l B \quad \text{and} \quad B \leq_C^l A,$$

$$A \sim_u B \Longleftrightarrow A \leq_C^u B \quad \text{and} \quad B \leq_C^u A,$$

$$A \sim_{l\&u} B \Longleftrightarrow A \leq_C^{l\&u} B \quad \text{and} \quad B \leq_C^{l\&u} A,$$

we can generate a partial ordering on the set of equivalence classes which are denoted by $[\cdot]^l$, $[\cdot]^u$ and $[\cdot]^{l\&u}$, respectively. We can easily see that

$$A \in [B]^l \Longleftrightarrow A + C = B + C,$$

$$A \in [B]^u \Longleftrightarrow A - C = B - C,$$

$$A \in [B]^{l\&u} \Longleftrightarrow A + C = B + C \quad \text{and} \quad A - C = B - C.$$

Definition 2.10. ($l[u, l\&u]$-minimal and $l[u, l\&u]$-weak minimal element [15, 17, 26]) Let $\mathcal{S} \subset \mathcal{V}$. We say that $\bar{A} \in \mathcal{S}$ is a $l[u, l\&u]$-minimal element if for any $A \in \mathcal{S}$,

$$A \leq_C^{l[u,l\&u]} \bar{A} \quad \text{implies} \quad \bar{A} \leq_C^{l[u,l\&u]} A.$$

Moreover, $\bar{A} \in \mathcal{S}$ is a $l[u, l\&u]$-weak minimal element if for any $A \in \mathcal{S}$,

$$A \leq_{\text{int}C}^{l[u,l\&u]} \bar{A} \quad \text{implies} \quad \bar{A} \leq_{\text{int}C}^{l[u,l\&u]} A.$$

We denote the family of $l[u, l\&u]$-minimal elements of $\mathcal{S}$ by $l[u, l\&u]$-$\text{Min}(\mathcal{S}; C)$ and the family of $l[u, l\&u]$-weak minimal elements of $\mathcal{S}$ by $l[u, l\&u]$-$\text{wMin}(\mathcal{S}; \text{int}C)$.

Definition 2.11. ($l[u, l\&u]$-maximal and $l[u, l\&u]$-weak maximal element [17, 26]) Let $\mathcal{S} \subset \mathcal{V}$. We say that $\bar{A} \in \mathcal{S}$ is a $l[u, l\&u]$-maximal element if for any $A \in \mathcal{S}$,

$$\bar{A} \leq_C^{l[u,l\&u]} A \quad \text{implies} \quad A \leq_C^{l[u,l\&u]} \bar{A}.$$

Moreover, $\bar{A} \in \mathcal{S}$ is a $l[u, l\&u]$-weak maximal element if for any $A \in \mathcal{S}$,

$$\bar{A} \leq_{\text{int}C}^{l[u,l\&u]} A \quad \text{implies} \quad A \leq_{\text{int}C}^{l[u,l\&u]} \bar{A}.$$

We denote the family of $l[u, l\&u]$-maximal elements of $\mathcal{S}$ by $l[u, l\&u]$-$\text{Max}(\mathcal{S}; C)$ and the family of $l[u, l\&u]$-weak maximal elements of $\mathcal{S}$ by $l[u, l\&u]$-$\text{wMax}(\mathcal{S}; \text{int}C)$.

We can easily see that

$$l[u, l\&u]\text{-Min}(\mathcal{S}; C) \subset l[u, l\&u]\text{-wMin}(\mathcal{S}; \text{int}C) \subset \mathcal{S},$$

$$l[u, l\&u]\text{-Max}(\mathcal{S}; C) \subset l[u, l\&u]\text{-wMax}(\mathcal{S}; \text{int}C) \subset \mathcal{S}.$$

3. Nonlinear scalarizations

3.1. **Nonlinear scalarizing functions for sets.** In this subsection, we assume that $k^0 \in C \setminus (-C)$. In 1980s, Gerstewitz [7] introduced a nonlinear scalarizing function in vector optimization problem. The nonlinear scalarizing function is known as the Gerstewitz's function.

$$\varphi_{C,k^0} : Y \to (-\infty, \infty], \quad \varphi_{C,k^0}(y) = \inf\{t \in \mathbb{R} \,|\, y \leq_C tk^0\} = \inf\{t \in \mathbb{R} \,|\, y \in tk^0 - C\}$$

The above scalarization method, which is also found in a similar form [31], contains the linear scalarization as a special case. After in [8, 9], they derived the essential properties of the Gerstewitz's function in vector optimization problem, for instance, monotonicity properties, sublinear properties. Also, the above scalarizing function has a dual form as follows:

$$\psi_{C,k^0} : Y \to [-\infty, \infty),\ \psi_{C,k^0}(y) = \sup\{t \in \mathbb{R} \,|\, tk^0 \leq_C y\} = \sup\{t \in \mathbb{R} \,|\, y \in tk^0 + C\}.$$

$$\varphi_{C,k^0}(y) = -\psi_{C,k^0}(-y).$$

These functions have wide applications in vector optimization (see also Luc [29], Göpfert-Riahi-Tammer-Zălinescu [10]).

After that, we investigated the properties of the following two-variable infimum type $h_{\inf} : Z \times Z \to (-\infty, \infty]$ and supremum type $h_{\sup} : Z \times Z \to [-\infty, \infty)$ of nonlinear scalarizing function for vector optimization problem, which are generalizations of the above scalarizing function (see [1]):

$$h_{\inf}(y, a) = \inf\{t \in \mathbb{R} \,|\, y \leq_C tk^0 + a\} = \inf\{t \in \mathbb{R} \,|\, y \in tk^0 + a - C\},$$

$$h_{\sup}(y, a) = \sup\{t \in \mathbb{R} \,|\, tk^0 + a \leq_C y\} = \sup\{t \in \mathbb{R} \,|\, y \in tk^0 + a + C\},$$

($h_{\inf}(y, a) := \varphi_{C,k^0}(y - a)$ for $a, y \in Z$). We can easily show

$$h_{\sup}(y, a) = -h_{\inf}(-y, -a).$$

The investigation of scalarizing functions for sets begun at around 2000. In the 2000s decade there were four important papers [5, 6, 13, 15]. In the last decade, many authors have been investigated sublinear scalarizing technique for set optimization problem (see [2, 3, 4, 11, 12, 14, 18, 21, 22, 23, 30, 32, 36] and their references therein).

In this section, we investigate detailed properties of the following nonlinear scalarizing functions for sets, which are natural extension of $h_{\inf}$ and $h_{\sup}$. Agreeing $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$, we define

$$h^l_{\inf}, h^u_{\inf}, h^{l\&u}_{\inf} : \mathcal{V} \times \mathcal{V} \to [-\infty, \infty] \quad \text{and}$$

$$h^l_{\sup}, h^u_{\sup}, h^{l\&u}_{\sup} : \mathcal{V} \times \mathcal{V} \to [-\infty, \infty]$$

as follows. The functions $h^l_{\inf}, h^u_{\inf}, h^{l\&u}_{\inf}, h^l_{\sup}, h^u_{\sup}, h^{l\&u}_{\sup}$ play the role of utility functions.

$$h^l_{\inf}(V_1, V_2) = \inf\{t \in \mathbb{R} \,\big|\, V_1 \leq^l_C tk^0 + V_2\} = \inf\{t \in \mathbb{R} \,|\, tk^0 + V_2 \subset V_1 + C\},$$

$$h^u_{\inf}(V_1, V_2) = \inf\{t \in \mathbb{R} \,|\, V_1 \leq^u_C tk^0 + V_2\} = \inf\{t \in \mathbb{R} \,|\, V_1 \subset tk^0 + V_2 - C\},$$

$$h^{l\&u}_{\inf}(V_1,V_2)=\inf\{t\in\mathbb{R}\,\Big|\,V_1\leq^{l\&u}_C tk^0+V_2\}$$
$$=\inf\{t\in\mathbb{R}\,\big|\,tk^0+V_2\subset V_1+C\quad\text{and}\quad V_1\subset tk^0+V_2-C\},$$
$$h^{l}_{\sup}(V_1,V_2)=\sup\{t\in\mathbb{R}\,\Big|\,tk^0+V_2\leq^{l}_C V_1\}=\sup\{t\in\mathbb{R}\,\big|\,V_1\subset tk^0+V_2+C\},$$
$$h^{u}_{\sup}(V_1,V_2)=\sup\{t\in\mathbb{R}\,\big|\,tk^0+V_2\leq^{u}_C V_1\}=\sup\{t\in\mathbb{R}\,\big|\,tk^0+V_2\subset V_1-C\},$$
$$h^{l\&u}_{\sup}(V_1,V_2)=\sup\{t\in\mathbb{R}\,\Big|\,tk^0+V_2\leq^{l\&u}_C V_1\}$$
$$=\sup\{t\in\mathbb{R}\,\big|\,V_1\subset tk^0+V_2+C\quad\text{and}\quad tk^0+V_2\subset V_1-C\}.$$

Proposition 3.1 (see also [2]). *The following statements hold:*

(i) $h^l_{\sup}(V_1,V_2)=-h^u_{\inf}(-V_1,-V_2)$;
(ii) $h^u_{\sup}(V_1,V_2)=-h^l_{\inf}(-V_1,-V_2)$;
(iii) $h^{l\&u}_{\sup}(V_1,V_2)=-h^{l\&u}_{\inf}(-V_1,-V_2)$;
(iv) $h^l_{\inf}(V_1,V_2)\leq h^{l\&u}_{\inf}(V_1,V_2)$ *and* $h^u_{\inf}(V_1,V_2)\leq h^{l\&u}_{\inf}(V_1,V_2)$;
(v) $h^{l\&u}_{\sup}(V_1,V_2)\leq h^l_{\sup}(V_1,V_2)$ *and* $h^{l\&u}_{\sup}(V_1,V_2)\leq h^u_{\sup}(V_1,V_2)$.

Proof. Conclusion (iv) and (v) are straightforward from Proposition 2.6. □

In this section, we correct some errors of the proofs of [2], which are mainly based on [11, 12]. The revised results contain some improvements of [2, 11, 12].

Definition 3.2. We say that the function $f:\mathcal{V}\to[-\infty,\infty]$ is

(i) $\leq^l_C$-increasing if $V_1\leq^l_C V_2$ implies $f(V_1)\leq f(V_2)$,
(ii) strictly $\leq^l_{\mathrm{int}C}$-increasing if $V_1\leq^l_{\mathrm{int}C} V_2$ $(V_1\neq V_2)$ implies $f(V_1)<f(V_2)$.

The definitions of $\leq^u_C$-increasing, $\leq^{l\&u}_C$-increasing, strictly $\leq^u_{\mathrm{int}C}$-increasing and strictly $\leq^{l\&u}_{\mathrm{int}C}$-increasing are similar to the above ones, respectively.

Theorem 3.3 (see also [2]). *The functions* $h^l_{\inf}, h^u_{\inf}, h^{l\&u}_{\inf}, h^l_{\sup}, h^u_{\sup}, h^{l\&u}_{\sup}$ *have the following properties:*

(i) $h^l_{\inf}(\cdot,V)$ *and* $h^l_{\sup}(\cdot,V)$ *are* $\leq^l_C$*-increasing for every* $V\in\mathcal{V}$;
(ii) $h^u_{\inf}(\cdot,V)$ *and* $h^u_{\sup}(\cdot,V)$ *are* $\leq^u_C$*-increasing for every* $V\in\mathcal{V}$;
(iii) $h^{l\&u}_{\inf}(\cdot,V)$ *and* $h^{l\&u}_{\sup}(\cdot,V)$ *are* $\leq^{l\&u}_C$*-increasing for every* $V\in\mathcal{V}$.

Proof. Condition (iii) is straightforward from the monotonicity of $h^l_{\inf}(\cdot,V)$, $h^u_{\inf}(\cdot,V)$, $h^l_{\sup}(\cdot,V)$ and $h^u_{\sup}(\cdot,V)$, respectively. □

3.2. Non-convex separation type theorems.

Theorem 3.4 (l-inf type, revised version of [2]). *Let* $C\subset Z$ *be a solid closed convex cone and* $k^0\in\mathrm{int}C$.

(i) *If* $V_1\in\mathcal{V}_C$ *is* $(-C)$*-bounded and* $V_2\in\mathcal{V}$ *is* C*-bounded, then* $h^l_{\inf}(\cdot,\cdot)$ *is a real-valued function.*
(ii) *Moreover, if* $V_1\in\mathcal{V}_C$ *is* C*-closed and* $V_2\in\mathcal{V}$*, then we have*

$$V_2\not\subset V_1+C\Longleftrightarrow h^l_{\inf}(V_1,V_2)>0.$$

(iii) *Furthermore, if* $V_1 \in \mathcal{V}_C$ *and* $V_2 \in \mathcal{V}$ *is* C*-compact, then we have*

$$V_2 \not\subset V_1 + \mathrm{int}C \Longleftrightarrow h^l_{\inf}(V_1, V_2) \geq 0.$$

Proof. (i) Firstly, we show

$$V_1 \in \mathcal{V} : C\text{−proper} \iff h^l_{\inf}(V_1, V_2) > -\infty.$$

If $V_1 + C = Z$ for $V_1 \in \mathcal{V}$, then we have $tk^0 + V_2 \subset V_1 + C$ for all $t \in \mathbb{R}$ and $V_2 \in \mathcal{V}$, which is equivalent to $h^l_{\inf}(V_1, V_2) = -\infty$. Conversely, let $tk^0 + V_2 \subset V_1 + C$ for all $t \in \mathbb{R}$ and $V_1, V_2 \in \mathcal{V}$. Then we have

$$tk^0 + V_2 + C \subset V_1 + C + C \subset V_1 + C.$$

For $k^0 \in \mathrm{int}C$, it is known that

$$\bigcup_{t\in\mathbb{R}} (tk^0 + C) = Z$$

and hence $V_1 + C = Z$.

Next, we prove $h^l_{\inf}(V_1, V_2) < \infty$. Since $V_2 \in \mathcal{V}$ is C-bounded, for the neighborhood of zero

$$U = -k^0 + \mathrm{int}C$$

there exists $t_2 > 0$ such that $V_2 \subset t_2(-k^0 + \mathrm{int}C) + C$ and hence

$$t_2k^0 + V_2 \subset C.$$

Moreover, since $V_1 \in \mathcal{V}$ is $(-C)$-bounded, for the neighborhood of zero

$$U = k^0 - \mathrm{int}C$$

there exists $t_1 > 0$ such that $V_1 \subset t_1(k^0 - \mathrm{int}C) - C$. Then we have that

$$0_Z \in V_1 - V_1 \subset t_1k^0 - V_1 - \mathrm{int}C$$

and hence

$$0_Z \in -t_1k^0 + V_1 + \mathrm{int}C.$$

Thus, we obtain $C \subset -t_1k^0 + V_1 + C$. Therefore, we have

$$t_2k^0 + V_2 \subset C \subset -t_1k^0 + V_1 + C$$

and hence

$$(t_1 + t_2)k^0 + V_2 \subset V_1 + C,$$

that is, $h^l_{\inf}(V_1, V_2) \leq t_1 + t_2$.

(ii) We show the following relationship:

$$(\star) \quad h^l_{\inf}(V_1, V_2) \leq t \Longleftrightarrow tk^0 + V_2 \subset \mathrm{cl}(V_1 + C).$$

We define

$$\Lambda^l_-(V_1, V_2) := \{t \in \mathbb{R} \,|\, tk^0 + V_2 \subset \mathrm{int}(V_1 + C)\},$$
$$\Lambda^l(V_1, V_2) := \{t \in \mathbb{R} \,|\, tk^0 + V_2 \subset V_1 + C\},$$
$$\Lambda^l_+(V_1, V_2) := \{t \in \mathbb{R} \,|\, tk^0 + V_2 \subset \mathrm{cl}(V_1 + C)\}.$$

Then we have obviously that $\Lambda^l_-(V_1,V_2)\subset\Lambda^l(V_1,V_2)\subset\Lambda^l_+(V_1,V_2)$ and hence

$$\inf\Lambda^l_+(V_1,V_2)\le\inf\Lambda^l(V_1;V_2)(=h^l_{\inf}(V_1,V_2))\le\inf\Lambda^l_-(V_1,V_2).$$

We assume $h^l_{\inf}(V_1,V_2)\le t$. Then by the definitions of $h^l_{\inf}$ and Λ^l being of epigraphical type (that is, $t\in\Lambda^l$ and $\hat{t}>t$ implies $\hat{t}\in\Lambda^l$, see [2]), we have

$$a+\left(t+\frac{1}{n}\right)k^0\in V_1+C$$

for all $n\in\mathbb{N}$ and $a\in V_2$. Taking the limit when $n\to\infty$, we obtain $tk^0+V_2\subset\mathrm{cl}(V_1+C)$. Conversely, by the definitions of $h^l_{\inf}$, we show

$$\inf\Lambda^l_+(V_1,V_2)=\inf\Lambda^l(V_1,V_2)=\inf\Lambda^l_-(V_1,V_2).$$

We assume contrary that $\inf\Lambda^l_+(V_1,V_2)<\inf\Lambda^l_-(V_1,V_2)$. Then there exists $t_1,t_2,t_3\in\mathbb{R}$ such that $\inf\Lambda^l_+(V_1,V_2)<t_1<t_2<t_3<\inf\Lambda^l_-(V_1,V_2)$. By $\inf\Lambda^l_+(V_1,V_2)<t_1$ $[t_1k^0+V_2\subset\mathrm{cl}(V_1+C)]$ and using (iv) of Lemma 2.1, we have

$$(*)\quad t_1k^0+V_2+C\subset\mathrm{cl}(V_1+C)+C=\mathrm{cl}(V_1+C).$$

On the other hand, using (ii) of Lemma 2.1, we have

$$\begin{aligned}(**)\quad t_3k^0+V_2\subset t_3k^0+V_2+C&=t_2k^0+V_2+C+(t_3-t_2)k^0\\&\subset t_2k^0+V_2+\mathrm{int}C\subset\mathrm{int}(t_1k^0+V_2+C).\end{aligned}$$

By $(*)$, we have the following inclusion (the last equality follows from [24])

$$(***)\quad\mathrm{int}(t_1k^0+V_2+C)\subset\mathrm{int}(\mathrm{cl}(V_1+C))=\mathrm{int}(V_1+C).$$

By $(**)$ and $(***)$, we obtain $t_3k^0+V_2\subset\mathrm{int}(V_1+C)$, which contradicts $t_3<\inf\Lambda^l_-(V_1,V_2)$.

(iii) We show the following relationship:

$$h^l_{\inf}(V_1,V_2)<0\iff V_2\subset V_1+\mathrm{int}C.$$

Let $h^l_{\inf}(V_1,V_2)<0$. Then there exists $t_1\in\mathbb{R}$ such that $h^l_{\inf}(V_1,V_2)\le t_1<0$. By using $(\star)$, we have

$$V_2=t_1k^0-t_1k^0+V_2\subset\mathrm{cl}(V_1+C)-t_1k^0\subset V_1+\mathrm{int}C.$$

Conversely, let $V_2\subset V_1+\mathrm{int}C$. For $k^0\in\mathrm{int}C$, it is known that

$$\mathrm{int}C=\bigcup_{\varepsilon>0}((\varepsilon k^0+\mathrm{int}C)+C).$$

Therefore, we have

$$V_2\subset V_1+\mathrm{int}C=\bigcup_{\varepsilon>0}\big((V_1+\varepsilon k^0+\mathrm{int}C)+C\big)$$

and $\{(V_1+\varepsilon k^0+\mathrm{int}C)+C\}_{\varepsilon>0}$ is a cover of V_2. Since V_2 is C-compact, we can find $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_m>0$ such that

$$V_2\subset\bigcup_{i=1}^m\big((V_1+\varepsilon_ik^0+\mathrm{int}C)+C\big)=V_1+\varepsilon_0k^0+\mathrm{int}C$$

where $\varepsilon_0 := \min\{\varepsilon_i | i = 1, 2, \dots, m\} > 0$. Then we have $-\varepsilon_0 k^0 + V_2 \subset \mathrm{cl}(V_1 + C)$ and therefore $h^l_{\inf}(V_1, V_2) \leq -\varepsilon_0 < 0$. □

Theorem 3.5 (u-inf type, revised version of [2]). *Let $C \subset Z$ be a solid closed convex cone and $k^0 \in \mathrm{int}C$.*

(i) *If $V_1 \in \mathcal{V}$ is $(-C)$-bounded and $V_2 \in \mathcal{V}_{-C}$ is C-bounded, then $h^u_{\inf}(\cdot, \cdot)$ is a real-valued function.*

(ii) *Moreover, if $V_1 \in \mathcal{V}$ and $V_2 \in \mathcal{V}_{-C}$ is $(-C)$-closed, then we have*

$$V_1 \not\subset V_2 - C \Longleftrightarrow h^u_{\inf}(V_1, V_2) > 0.$$

(iii) *Furthermore, if $V_1 \in \mathcal{V}$ is $(-C)$-compact and $V_2 \in \mathcal{V}_{-C}$, then we have*

$$V_1 \not\subset V_2 - \mathrm{int}C \Longleftrightarrow h^u_{\inf}(V_1, V_2) \geq 0.$$

Proof. (i) Firstly, we show

$$V_2 \in \mathcal{V} : (-C)\text{−proper} \iff h^u_{\inf}(V_1, V_2) > -\infty.$$

If $V_2 - C = Z$ for $V_2 \in \mathcal{V}$, then we have $V_1 \subset tk^0 + V_2 - C$ for all $t \in \mathbb{R}$ and $V_1 \in \mathcal{V}$, which is equivalent to $h^u_{\inf}(V_1, V_2) = -\infty$. Conversely, let $V_1 \subset tk^0 + V_2 - C$ for all $t \in \mathbb{R}$ and $V_1, V_2 \in \mathcal{V}$. Then we have

$$V_1 - tk^0 - C \subset V_2 - C - C \subset V_2 - C.$$

For $k^0 \in \mathrm{int}C$, it is known that

$$\bigcup_{t \in \mathbb{R}} (-tk^0 - C) = Z$$

and hence $V_2 - C = Z$.

Next, we prove $h^u_{\inf}(V_1, V_2) < \infty$. Since $V_1 \in \mathcal{V}$ is $(-C)$-bounded, for the neighborhood of zero

$$U = k^0 - \mathrm{int}C$$

there exists $t_1 > 0$ such that $V_1 \subset t_1(k^0 - \mathrm{int}C) - C$ and hence

$$V_1 \subset t_1 k^0 - C.$$

Moreover, Since $V_2 \in \mathcal{V}$ is C-bounded, for the neighborhood of zero

$$U = -k^0 + \mathrm{int}C$$

there exists $t_2 > 0$ such that $V_2 \subset t_2(-k^0 + \mathrm{int}C) + C$. Then we have that

$$0_Z \in V_2 - V_2 \subset -t_2 k^0 - V_2 + \mathrm{int}C$$

and hence

$$0_Z \in t_2 k^0 + V_2 - \mathrm{int}C.$$

Thus, we obtain $-C \subset t_2 k^0 + V_2 - C$. Therefore, we have

$$V_1 \subset t_1 k^0 - C \subset t_1 k^0 + t_2 k^0 + V_2 - C = (t_1 + t_2)k^0 + V_2 - C$$

that is, $h^u_{\inf}(V_1, V_2) \leq t_1 + t_2$.

(ii) We show the following relationship:

$$h^u_{\rm inf}(V_1, V_2) \le t \Longleftrightarrow V_1 \subset {\rm cl}(tk^0 + V_2 - C).$$

We define

$$\Lambda^u_-(V_1, V_2) := \{t \in \mathbb{R} \,|\, V_1 \subset {\rm int}(tk^0 + V_2 - C)\},$$
$$\Lambda^u(V_1, V_2) := \{t \in \mathbb{R} \,|\, V_1 \subset tk^0 + V_2 - C\},$$
$$\Lambda^u_+(V_1, V_2) := \{t \in \mathbb{R} \,|\, V_1 \subset {\rm cl}(tk^0 + V_2 - C)\}.$$

Then we have obviously that

$$\inf \Lambda^u_+(V_1, V_2) \le \inf \Lambda^u(V_1, V_2) (= h^u_{\rm inf}(V_1, V_2)) \le \inf \Lambda^u_-(V_1, V_2).$$

We assume $h^u_{\rm inf}(V) \le t$. Then by the definitions of $h^u_{\rm inf}$ and Λ^u being of epigraphical type (that is, $t \in \Lambda^u$ and $\hat{t} > t$ implies $\hat{t} \in \Lambda^u$, see [2]), we have

$$y - \left(t + \frac{1}{n}\right)k^0 \in V_2 - C$$

for all $n \in \mathbb{N}$ and $y \in V_1$. Taking the limit when $n \to \infty$, we obtain $V_1 \subset {\rm cl}(tk^0 + V_2 - C)$. Conversely, by the definitions of $h^u_{\rm inf}$, we show

$$\inf \Lambda^u_+(V_1, V_2) = \inf \Lambda^u(V_1, V_2) = \inf \Lambda^u_-(V_1, V_2).$$

We assume contrary that $\inf \Lambda^u_+(V_1, V_2) < \inf \Lambda^u_-(V_1, V_2)$. Then there exists $t_1, t_2, t_3, t_4 \in \mathbb{R}$ such that $\inf \Lambda^u_+(V_1, V_2) < t_1 < t_2 < t_3 < t_4 < \inf \Lambda^u_-(V_1, V_2)$. By $\inf \Lambda^u_+(V_1, V_2) < t_1$, we have

$$(*) \quad V_1 \subset {\rm cl}(t_1k^0 + V_2 - C) \subset t_2k^0 + V_2 - C.$$

On the other hand, using (ii) of Lemma 2.1, we have

$$\begin{aligned}(**) \quad t_2k^0 + V_2 - C &= t_3k^0 + (t_2 - t_3)k^0 + V_2 - C \\ &\subset t_3k^0 + V_2 - {\rm int}C \subset {\rm int}(t_4k^0 + V_2 - C).\end{aligned}$$

By $(*)$ and $(**)$, we obtain $V_1 \subset {\rm int}(t_4k^0 + V_2 - C)$, which contradicts $t_4 < \inf \Lambda^u_-(V_1, V_2)$.

(iii) We show the following relationship:

$$h^u_{\rm inf}(V_1, V_2) < 0 \iff V_1 \subset V_2 - {\rm int}C.$$

Let $h^u_{\rm inf}(V_1, V_2) < 0$. Then there exists $t_1 \in \mathbb{R}$ such that $h^u_{\rm inf}(V_1, V_2) \le t_1 < 0$. By the definition of $h^u_{\rm inf}$, we have

$$V_1 \subset {\rm cl}(t_1k^0 + V_2 - C) \subset V_2 - {\rm int}C.$$

Conversely, let $V_1 \subset V_2 - {\rm int}C$. For $k^0 \in {\rm int}C$, it is known that

$${\rm int}C = \bigcup_{\varepsilon>0} \big((\varepsilon k^0 + {\rm int}C) + C\big).$$

Therefore, we have

$$V_1 \subset V_2 - {\rm int}C = \bigcup_{\varepsilon>0} \big((V_2 - \varepsilon k^0 - {\rm int}C) - C\big)$$

and $\{(V_2 - \varepsilon k^0 - \mathrm{int}C) - C\}_{\varepsilon>0}$ is a cover of V_1. Since $V_1 \in \mathcal{V}$ is $(-C)$-compact, we can find $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m > 0$ such that

$$V_1 \subset \bigcup_{i=1}^{m} \big((V_2 - \varepsilon_0 k^0 - \mathrm{int}C) - C\big) = V_2 - \varepsilon_0 k^0 - \mathrm{int}C \subset V_2 - \varepsilon_0 k^0 - C$$

where $\varepsilon_0 := \min\{\varepsilon_i | i = 1, 2, \ldots, m\} > 0$. Then we have $V_1 \subset \mathrm{cl}(V_2 - \varepsilon_0 k^0 - C)$ and therefore $h^u_{\inf}(V_1, V_2) \leq -\varepsilon_0 < 0$. □

Combining the above results, we obtain the following non-convex separation type theorem with respect to lower and upper type set relation.

Theorem 3.6 ($l\&u$-inf type). *Let $C \subset Z$ be a solid closed convex cone and $k^0 \in \mathrm{int}C$.*

(i) *If $V_1 \in \mathcal{V}_C$ is $(-C)$-bounded and $V_2 \in \mathcal{V}_{-C}$ is C-bounded, then $h^{l\&u}_{\inf}(\cdot,\cdot)$ is a real-valued function.*

(ii) *Moreover, if $V_1 \in \mathcal{V}_C$ is C-closed and $V_2 \in \mathcal{V}_{-C}$ is $(-C)$-closed, then we have*

$$V_2 \not\subset V_1 + C \quad \text{and} \quad V_1 \not\subset V_2 - C \Longleftrightarrow h^{l\&u}_{\inf}(V_1, V_2) > 0.$$

(iii) *Furthermore, if $V_1 \in \mathcal{V}_C$ is $(-C)$-compact and $V_2 \in \mathcal{V}_{-C}$ is C-compact, then we have*

$$V_2 \not\subset V_1 + \mathrm{int}C \quad \text{and} \quad V_1 \not\subset V_2 - \mathrm{int}C \Longleftrightarrow h^{l\&u}_{\inf}(V_1, V_2) \geq 0.$$

By using proposition 3.1, we obtain the following supremum type separation type theorems.

Theorem 3.7 (l-sup type, revised version of [2]). *Let $C \subset Z$ be a solid closed convex cone and $k^0 \in \mathrm{int}C$.*

(i) *If $V_1 \in \mathcal{V}$ is C-bounded and $V_2 \in \mathcal{V}_C$ is $(-C)$-bounded, then $h^l_{\sup}(\cdot,\cdot)$ is a real-valued function.*

(ii) *Moreover, if $V_1 \in \mathcal{V}$ and $V_2 \in \mathcal{V}_C$ is C-closed, then we have*

$$V_1 \not\subset V_2 + C \Longleftrightarrow h^l_{\sup}(V_1, V_2) < 0.$$

(iii) *Furthermore, if $V_1 \in \mathcal{V}$ is C-compact and $V_2 \in \mathcal{V}_C$, then we have*

$$V_1 \not\subset V_2 + \mathrm{int}C \Longleftrightarrow h^l_{\sup}(V_1, V_2) \leq 0.$$

Theorem 3.8 (u-sup type, revised version of [2]). *Let $C \subset Z$ be a solid closed convex cone and $k^0 \in \mathrm{int}C$.*

(i) *If $V_1 \in \mathcal{V}_{-C}$ is C-bounded and $V_2 \in \mathcal{V}$ is $(-C)$-bounded, then $h^u_{\sup}(\cdot,\cdot)$ is a real-valued function.*

(ii) *Moreover, if $V_1 \in \mathcal{V}_{-C}$ is $(-C)$-closed and $V_2 \in \mathcal{V}$, then we have*

$$V_2 \not\subset V_1 - C \Longleftrightarrow h^u_{\sup}(V_1, V_2) < 0.$$

(iii) *Furthermore, if $V_1 \in \mathcal{V}_{-C}$ and $V_2 \in \mathcal{V}$ is $(-C)$-compact, then we have*

$$V_2 \not\subset V_1 - \mathrm{int}C \Longleftrightarrow h^u_{\sup}(V_1, V_2) \leq 0.$$

Theorem 3.9 ($l\&u$-sup type)**.** *Let $C \subset Z$ be a solid closed convex cone and $k^0 \in \mathrm{int}C$.*

(i) *If $V_1 \in \mathcal{V}_{-C}$ is C-bounded and $V_2 \in \mathcal{V}_C$ is $(-C)$-bounded, then $h^{l\&u}_{\sup}(\cdot,\cdot)$ is a real-valued function.*

(ii) *Moreover, if $V_1 \in \mathcal{V}_{-C}$ is $(-C)$-closed and $V_2 \in \mathcal{V}_C$ is C-closed, then we have*

$$V_1 \not\subset V_2 + C \quad \text{and} \quad V_2 \not\subset V_1 - C \Longleftrightarrow h^{l\&u}_{\sup}(V_1, V_2) < 0.$$

(iii) *Furthermore, if $V_1 \in \mathcal{V}_{-C}$ is C-compact and $V_2 \in \mathcal{V}_C$ is $(-C)$-compact, then we have*

$$V_1 \not\subset V_2 + \mathrm{int}C \quad \text{and} \quad V_2 \not\subset V_1 - \mathrm{int}C \Longleftrightarrow h^{l\&u}_{\sup}(V_1, V_2) \le 0.$$

4. Existence theorems of cone-saddle points

Let X, Y be nonempty sets and $f : X \times Y \to Z$ be a vector-valued function. The vector-valued saddle-point problem is to find a pair $\bar{x} \in X$ and $\bar{y} \in Y$ such that

$$\text{(P)} \qquad f(\bar{x}, \bar{y}) \in \mathrm{wMax}(f(\bar{x}, Y); \mathrm{int}C) \cap \mathrm{wMin}(f(X, \bar{y}); \mathrm{int}C).$$

A point $(x, y) \in X \times Y$ is said to be a weak C-saddle point of function f on $X \times Y$, if it is a solution of the problem (see [35] and their references therein).

Definition 4.1. Let K be a convex set in a real vector space X, Z a normed space with the partial ordering by a solid pointed convex cone $C \subset Z$. A vector-valued function $f : X \to Z$ is said to be

(i) C-quasi-convex on K if for each $x_1, x_2 \in K$, $\lambda \in [0, 1]$ and $z \in Z$, we have that

$$f(x_1), f(x_2) \in z - C \quad \text{implies} \quad f(\lambda x_1 + (1 - \lambda)x_2) \in z - C,$$

(ii) C-properly quasi-convex on K if either

$$f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_1) - C \qquad \text{or} \qquad f(\lambda x_1 + (1 - \lambda)x_2) \in f(x_2) - C,$$

for every $x_1, x_2 \in K$ and $\lambda \in [0, 1]$.

Definition 4.2. Let X be a topological space and Z a normed space. A vector-valued function $f : X \to Z$ is said to be C-continuous at X if the set $\{x \in X | f(x) \le_C z\}$ is closed for all $z \in Z$.

Many researchers have investigated existence theorem of cone saddle-point by using scalarizing technique (see for instance, [35] and references therein). Afterwards, Kimura and Tanaka presented an existence theorem of cone saddle-point by using scalarizing function $h_{\inf}(\cdot; 0_Y)$.

Theorem 4.3 (Kimura-Tanaka [20])**.** *Let X and Y be nonempty compact convex sets in two normed spaces, respectively, and Z a normed space with a partial ordering induced by a solid pointed convex cone $C \subset Z$. If a vector-valued function $f : X \times Y \to Z$ satisfies that*

(i) *$x \mapsto f(x, y)$ is C-continuous and C-quasi-convex on X for every $y \in Y$,*

(ii) *$y \mapsto f(x,y)$ is $(-C)$-continuous and $(-C)$-properly quasi-convex on Y for every $x \in X$,*

then f has at least one weak C-saddle point.

We obtain another type of existence theorem of cone saddle-point by using scalarizing function $h_{\sup}(\cdot; 0_Y)$.

Theorem 4.4 (Araya [1])**.** *Let X and Y be nonempty compact convex sets in two normed spaces, respectively, and Z a normed space with a partial ordering induced by a solid pointed convex cone $C \subset Z$. If a vector-valued function $f : X \times Y \to Z$ satisfies that*

(i) *$x \mapsto f(x,y)$ is C-continuous and C-properly quasi-convex on X for every $y \in Y$,*
(ii) *$y \mapsto f(x,y)$ is $(-C)$-continuous and $(-C)$ quasi-convex on Y for every $x \in X$,*

then f has at least one weak C-saddle point.

The aim of this section is to generalize the above existence theorems to set-valued map as an application of the scalarizations for sets.

4.1. **Set-valued saddle-point problem and some definitions.** We first give some definitions of set-valued saddle point problem as a natural generalizations of vector-valued saddle point problem.

Definition 4.5. Let X, Y be nonempty sets and $F : X \times Y \to \mathcal{V}$ be a set-valued map. The set-valued saddle-point problem is to find a pair $\bar{x} \in X$ and $\bar{y} \in Y$ such that

(l-P): $F(\bar{x},\bar{y}) \in l\text{-wMax}(F(\bar{x},Y);\mathrm{int}C) \cap l\text{-wMin}(F(X,\bar{y});\mathrm{int}C)$,
(u-P): $F(\bar{x},\bar{y}) \in u\text{-wMax}(F(\bar{x},Y);\mathrm{int}C) \cap u\text{-wMin}(F(X,\bar{y});\mathrm{int}C)$,
($l\&u$-P): $F(\bar{x},\bar{y}) \in (l\&u)\text{-wMax}(F(\bar{x},Y);\mathrm{int}C) \cap (l\&u)\text{-wMin}(F(X,\bar{y});\mathrm{int}C)$.

A point $(x,y) \in X \times Y$ is said to be a weak l-C-saddle point [resp. weak u-C-saddle point, weak $(l\&u)$-C-saddle point] of F on $X \times Y$, if it is a solution of the problem.

We give some definitions of cone-convexity and cone-continuity of set-valued map in a similar way as [25, 27, 29, 34].

Definition 4.6 (see also [25, 27, 29, 34])**.** Let K be a convex set in a real vector space X. A set-valued map $F : X \to \mathcal{V}$ is said to be

(i) l-properly quasi C-convex [u-properly quasi C-convex, $(l\&u)$-properly quasi C-convex] on K if either

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{l[u,l\&u]} F(x_1) \quad \text{or} \quad F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{l[u,l\&u]} F(x_2)$$

for every $x_1, x_2 \in K$ and $\lambda \in [0,1]$,

(ii) l-properly quasi C-concave [u-properly quasi C-concave, ($l\&u$)-properly quasi C-concave] on K if either

$$F(x_1) \leq_C^{l[u,l\&u]} F(\lambda x_1 + (1-\lambda)x_2) \quad \text{or} \quad F(x_2) \leq_C^{l[u,l\&u]} F(\lambda x_1 + (1-\lambda)x_2)$$

for every $x_1, x_2 \in K$ and $\lambda \in [0,1]$,

(iii) l-naturally quasi C-convex [u-naturally quasi C-convex, ($l\&u$)-naturally quasi C-convex] on K if for each $x_1, x_2 \in K$ and $\lambda \in [0,1]$, there exists $\mu \in [0,1]$ such that

$$F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{l[u,l\&u]} \mu F(x_1) + (1-\mu)F(x_2),$$

(iv) l-naturally quasi C-concave [u-naturally quasi C-concave, ($l\&u$)-naturally quasi C-concave] on K if if for each $x_1, x_2 \in K$ and $\lambda \in [0,1]$, there exists $\mu \in [0,1]$ such that

$$\mu F(x_1) + (1-\mu)F(x_2) \leq_C^{l[u,l\&u]} F(\lambda x_1 + (1-\lambda)x_2),$$

(v) l-quasi C-convex [u-quasi C-convex, ($l\&u$)-quasi C-convex] on K if for each $x_1, x_2 \in K$, $\lambda \in [0,1]$ and $V \in \mathcal{V}$, we have that

$$F(x_1) \leq_C^{l[u,l\&u]} V, \quad F(x_2) \leq_C^{l[u,l\&u]} V \quad \text{implies} \quad F(\lambda x_1 + (1-\lambda)x_2) \leq_C^{l[u,l\&u]} V,$$

(vi) l-quasi C-concave [u-quasi C-concave, ($l\&u$)-quasi C-concave] on K if for each $x_1, x_2 \in K$, $\lambda \in [0,1]$ and $V \in \mathcal{V}$, we have that

$$V \leq_C^{l[u,l\&u]} F(x_1), \quad V \leq_C^{l[u,l\&u]} F(x_2) \quad \text{implies} \quad V \leq_C^{l[u,l\&u]} F(\lambda x_1 + (1-\lambda)x_2).$$

Proposition 4.7 (see also [25, 27, 29, 34])**.** *We can confirm the following facts:*

- $\big[l[u,l\&u]\textit{-properly quasi } C\textit{-convex}\big] \subset \big[l[u,l\&u]\textit{-naturally quasi } C\textit{-convex}\big]$,
- $\big[l[u,l\&u]\textit{-naturally quasi } C\textit{-convex}\big] \subset \big[l[u,l\&u]\textit{-quasi } C\textit{-convex}\big]$,
- $\big[l\textit{-properly quasi } (-C)\textit{-convex}\big] = \big[u\textit{-properly quasi } C\textit{-concave}\big]$,
- $\big[u\textit{-properly quasi } (-C)\textit{-convex}\big] = \big[l\textit{-properly quasi } C\textit{-concave}\big]$,
- $\big[(l\&u)\textit{-properly quasi } (-C)\textit{-convex}\big] = \big[(l\&u)\textit{-properly quasi } C\textit{-concave}\big]$,
- $\big[l\textit{-naturally quasi } (-C)\textit{-convex}\big] = \big[u\textit{-naturally quasi } C\textit{-concave}\big]$,
- $\big[u\textit{-naturally quasi } (-C)\textit{-convex}\big] = \big[l\textit{-naturally quasi } C\textit{-concave}\big]$,
- $\big[(l\&u)\textit{-naturally quasi } (-C)\textit{-convex}\big] = \big[(l\&u)\textit{-naturally quasi } C\textit{-concave}\big]$,
- $\big[l\textit{-quasi } (-C)\textit{-convex}\big] = \big[u\textit{-quasi } C\textit{-concave}\big]$,
- $\big[u\textit{-quasi } (-C)\textit{-convex}\big] = \big[l\textit{-quasi } C\textit{-concave}\big]$,
- $\big[(l\&u)\textit{-quasi } (-C)\textit{-convex}\big] = \big[(l\&u)\textit{-quasi } C\textit{-concave}\big]$.

Proof. Using the several definitions of convexity and following the same line as Theorem 1 and 2 of [34], we obtain the first inclusions. The latter part is clear from the definitions. $\square$

Lemma 4.8. *Let K be a convex set in a real vector space X and $k^0 \in \mathrm{int}C$. If a set-valued map $F : X \to \mathcal{V}$ is $l[u,l\&u]$-quasi C-convex, then $h_{\inf}^{l[u,l\&u]}(F(\cdot), V)$ and $h_{\sup}^{l[u,l\&u]}(F(\cdot), V)$ are quasi-convex on K for every $V \in \mathcal{V}$.*

Proof. Let for all $\alpha \in \mathbb{R}$
$$\mathrm{Lev}(h^l_{\inf}(F(\cdot), V); \alpha) := \{x \in K | h^l_{\inf}(F(x), V) \leq \alpha\}.$$
Let $\lambda \in [0, 1]$ and $x_1, x_2 \in \mathrm{Lev}(h^l_{\inf}(F(\cdot); V); \alpha)$. Then we have
$$h^l_{\inf}(F(x_1), V) \leq \alpha, \qquad h^l_{\inf}(F(x_2), V) \leq \alpha.$$
By the l-quasi C-convexity of F, we have
$$F(x_1) \leq^l_C V_1, \quad F(x_2) \leq^l_C V_1 \quad \text{implies} \quad F(\lambda x_1 + (1-\lambda)x_2) \leq^l_C V_1.$$
We take $V_1 \in \mathcal{V}$ such that $h^l_{\inf}(V_1, V) = \alpha$. By using Theorem 3.3, we have
$$h^l_{\inf}(F(x_1), V) \leq h^l_{\inf}(V_1, V) = \alpha, \quad h^l_{\inf}(F(x_2), V) \leq h^l_{\inf}(V_1, V) = \alpha$$
$$\Longrightarrow h^l_{\inf}(F(\lambda x_1 + (1-\lambda)x_2), V) \leq h^l_{\inf}(V_1, V) = \alpha$$
which implies that $\lambda x_1 + (1-\lambda)x_2 \in \mathrm{Lev}(h^l_{\inf}(F(\cdot), V); \alpha)$. Similarly, we can prove the rest part of the proofs. □

Lemma 4.9. *Let K be a convex set in a real vector space X and $k^0 \in \mathrm{int}C$. If a set-valued map $F : X \to \mathcal{V}$ is $l[u, l\&u]$-quasi C-concave, then $h^{l[u,l\&u]}_{\inf}(F(\cdot), V)$ and $h^{l[u,l\&u]}_{\sup}(F(\cdot), V)$ are quasi-concave on K for every $V \in \mathcal{V}$.*

Proof. In a similar way as the above lemma, we obtain the conclusions. □

Definition 4.10. Let X be a topological space. A set-valued function $F : X \to \mathcal{V}$ is said to be

(i) l-C-lower semi-continuous (u-$(-C)$-upper semi-continuous) at X if the set $\{x \in X | F(x) \leq^l_C V\} = \{x \in X | V \leq^u_{(-C)} F(x)\}$ is closed for all $V \in \mathcal{V}$,

(ii) u-C-lower semi-continuous (l-$(-C)$-upper semi-continuous) at X if the set $\{x \in X | F(x) \leq^u_C V\} = \{x \in X | V \leq^l_{(-C)} F(x)\}$ is closed for all $V \in \mathcal{V}$,

(iii) $(l\&u)$-C-lower semi-continuous ($(l\&u)$-$(-C)$-upper semi-continuous) at X if the set $\{x \in X | F(x) \leq^{l\&u}_C V\} = \{x \in X | V \leq^{l\&u}_{(-C)} F(x)\}$ is closed for all $V \in \mathcal{V}$,

(iv) l-C-upper semi-continuous (u-$(-C)$-lower semi-continuous) at X if the set $\{x \in X | V \leq^l_C F(x)\} = \{x \in X | F(x) \leq^u_{(-C)} V\}$ is closed for all $V \in \mathcal{V}$,

(v) u-C-upper semi-continuous (l-$(-C)$-lower semi-continuous) at X if the set $\{x \in X | V \leq^u_C F(x)\} = \{x \in X | F(x) \leq^l_{(-C)} V\}$ is closed for all $V \in \mathcal{V}$,

(vi) $(l\&u)$-C-upper semi-continuous ($(l\&u)$-$(-C)$-lower semi-continuous) at X if the set $\{x \in X | V \leq^{l\&u}_C F(x)\} = \{x \in X | F(x) \leq^{l\&u}_{(-C)} V\}$ is closed for all $V \in \mathcal{V}$.

Lemma 4.11. *Let X be a topological space and $k^0 \in \mathrm{int}C$. If a set-valued map $F : X \to \mathcal{V}$ is*

(i) *$l[u, l\&u]$-C-lower semi-continuous, then $h^{l[u,l\&u]}_{\inf}(F(\cdot), V)$ and $h^{l[u,l\&u]}_{\sup}(F(\cdot), V)$ are lower semi-continuous for every $V \in \mathcal{V}$,*

(ii) *$l[u, l\&u]$-C-upper semi-continuous, then $h^{l[u,l\&u]}_{\inf}(F(\cdot), V)$ and $h^{l[u,l\&u]}_{\sup}(F(\cdot), V)$ are upper semi-continuous for every $V \in \mathcal{V}$.*

Proof. By using Theorem 3.3, we obtain the conclusions. □

4.2. **Existence theorems of cone-saddle point.**

Theorem 4.12 (Existence of weak l-C-saddle point)**.** *Let X and Y be nonempty compact convex sets in two normed spaces, respectively. If a C-proper and C-compact-valued map $F : X \times Y \to \mathcal{V}_C$ satisfies that*

(i) *$F(\cdot, y)$ is l-quasi C-convex on X for every $y \in Y$,*
(ii) *$F(\cdot, y)$ is l-C-lower semi-continuous on X for every $y \in Y$,*
(iii) *$F(x, \cdot)$ is l-quasi C-concave on Y for every $x \in X$,*
(iv) *$F(x, \cdot)$ is l-C-upper semi-continuous on Y for every $x \in X$,*

then F has at least one weak l-C-saddle point.

Proof. **(I: by using scalarizing function $h^l_{\inf}$)**
We see that by Lemma 4.8 and Lemma 4.11 the map $x \mapsto h^l_{\inf}(F(\cdot, y), V)$ is lower semi-continuous and quasi-convex on X for every $V \in \mathcal{V}_C$. Moreover, we see that by Lemma 4.9 and Lemma 4.11, the map $y \mapsto h^l_{\inf}(F(x, \cdot), V)$ is upper semi-continuous and quasi-concave on Y for every $V \in \mathcal{V}_C$. By Sion's minimax theorem [33], $h^l_{\inf}(F, V)$ has a saddle point for every $V \in \mathcal{V}_C$, that is, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$0 = h^l_{\inf}(F(\bar{x}, y), F(\bar{x}, y)) \leq h^l_{\inf}(F(\bar{x}, \bar{y}), F(\bar{x}, y)) \leq h^l_{\inf}(F(x, \bar{y}), F(\bar{x}, y)) \quad \text{and}$$
$$h^l_{\inf}(F(\bar{x}, y), F(\bar{x}, \bar{y})) \leq h^l_{\inf}(F(\bar{x}, \bar{y}), F(\bar{x}, \bar{y})) = 0 \leq h^l_{\inf}(F(x, \bar{y}), F(\bar{x}, \bar{y})).$$

By (iii) of Theorem 3.4, F has at least one weak l-C-saddle point.

(II: by using scalarizing function $h^l_{\sup}$)
Similarly, we see that by Lemma 4.8 and Lemma 4.11 the map $x \mapsto h^l_{\sup}(F(\cdot, y), V)$ is lower semi-continuous and quasi-convex on X for every $V \in \mathcal{V}_C$. Moreover, we see that by Lemma 4.9 and Lemma 4.11, the map $y \mapsto h^l_{\sup}(F(x, \cdot), V)$ is upper semi-continuous and quasi-concave on Y for every $V \in \mathcal{V}_C$. By Sion's minimax theorem [33], $h^l_{\sup}(F, V)$ has a saddle point for every $V \in \mathcal{V}_C$, that is, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$h^l_{\sup}(F(\bar{x}, y), F(x, \bar{y})) \leq h^l_{\sup}(F(\bar{x}, \bar{y}), F(x, \bar{y})) \leq h^l_{\sup}(F(x, \bar{y}), F(x, \bar{y})) = 0 \quad \text{and}$$
$$h^l_{\sup}(F(\bar{x}, y), F(\bar{x}, \bar{y})) \leq h^l_{\sup}(F(\bar{x}, \bar{y}), F(\bar{x}, \bar{y})) = 0 \leq h^l_{\sup}(F(x, \bar{y}), F(\bar{x}, \bar{y})).$$

By (iii) of Theorem 3.7, F has at least one weak l-C-saddle point. □

By using $h^u_{\inf}$, $h^u_{\sup}$, $h^{l\&u}_{\inf}$, $h^{l\&u}_{\sup}$, we obtain existence theorems of u-C-saddle point and weak $(l\&u)$-C-saddle point in a similar way as Theorem 4.12.

Theorem 4.13 (Existence of weak u-C-saddle point)**.** *Let X and Y be nonempty compact convex sets in two normed spaces, respectively. If a $(-C)$-proper and $(-C)$-compact-valued map $F : X \times Y \to \mathcal{V}_{-C}$ satisfies that*

(i) *$F(\cdot, y)$ is u-quasi C-convex on X for every $y \in Y$,*
(ii) *$F(\cdot, y)$ is u-C-lower semi-continuous on X for every $y \in Y$,*
(iii) *$F(x, \cdot)$ is u-quasi C-concave on Y for every $x \in X$,*
(iv) *$F(x, \cdot)$ is u-C-upper semi-continuous on Y for every $x \in X$,*

then F has at least one weak u-C-saddle point.

Theorem 4.14 (Existence of weak $l\&u$-C-saddle point)**.** *Let X and Y be nonempty compact convex sets in two normed spaces, respectively. If a C-proper, $(-C)$-proper, C-compact and $(-C)$-compact-valued map $F : X \times Y \to \mathcal{V}_{\pm C}$ satisfies that*

(i) *$F(\cdot, y)$ is $(l\&u)$-quasi C-convex on X for every $y \in Y$,*
(ii) *$F(\cdot, y)$ is $(l\&u)$-C-lower semi-continuous on X for every $y \in Y$,*
(iii) *$F(x, \cdot)$ is $(l\&u)$-quasi C-concave on Y for every $x \in X$,*
(iv) *$F(x, \cdot)$ is $(l\&u)$-C-upper semi-continuous on Y for every $x \in X$,*

then F has at least one weak $(l\&u)$-C-saddle point.

Example 4.15 (l-type)**.** We set

$$X = Y = [0, 2], \qquad Z = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+ := \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\},$$

$$F : X \times Y \to \mathcal{V}, \qquad F(x, y) = [x, \infty) \times [y, \infty), \qquad k^0 = (1, 1).$$

We can confirm that F is C-proper and C-compact valued function and F satisfies assumptions of Theorem 4.12. Then we see that $(\bar{x}, \bar{y}) = (0, 2)$ is a weak l-C-saddle point. Moreover, for any $x, y \in [0, 2]$ we have

- $h^l_{\inf}(F(0, y), F(0, y)) = 0 \leq h^l_{\inf}(F(0, 2), F(0, y)) = 2 - y$
  $\leq h^l_{\inf}(F(x, 2), F(0, y)) =\max\{2 - y, x\}$,
- $h^l_{\inf}(F(0, y), F(0, 2)) = 0 \leq h^l_{\inf}(F(0, 2), F(0, 2)) = 0 \leq h^l_{\inf}(F(x, 2), F(0, 2)) = x$,
- $h^l_{\sup}(F(0, y), F(x, 2)) = -\max\{2 - y, x\}$
  $\leq h^l_{\sup}(F(0, 2), F(x, 2)) = -x \leq h^l_{\sup}(F(x, 2), F(x, 2)) = 0$,
- $h^l_{\sup}(F(0, y), F(0, 2)) = y - 2$
  $\leq h^l_{\sup}(F(0, 2), F(0, 2)) = 0 \leq h^l_{\sup}(F(x, 2), F(0, 2)) = 0$.

Example 4.16 (u-type)**.** We set

$$X = Y = [0, 2], \qquad Z = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+,$$

$$F : X \times Y \to \mathcal{V}, \qquad F(x, y) = (-\infty, x] \times (-\infty, y], \qquad k^0 = (1, 1).$$

We can confirm that F is $(-C)$-proper and $(-C)$-compact valued function and F satisfies assumptions of Theorem 4.13. Then we see that $(\bar{x}, \bar{y}) = (0, 2)$ is a weak u-C-saddle point. Moreover, for any $x, y \in [0, 2]$ we have

- $h^u_{\inf}(F(0, y), F(0, y)) = 0 \leq h^u_{\inf}(F(0, 2), F(0, y)) = 2 - y$
  $\leq h^u_{\inf}(F(x, 2), F(0, y)) =\max\{2 - y, x\}$,
- $h^u_{\inf}(F(0, y), F(0, 2)) = 0 \leq h^u_{\inf}(F(0, 2), F(0, 2)) = 0 \leq h^u_{\inf}(F(x, 2), F(0, 2)) = x$,
- $h^u_{\sup}(F(0, y), F(x, 2)) = -\max\{2 - y, x\}$
  $\leq h^u_{\sup}(F(0, 2), F(x, 2)) = -x \leq h^u_{\sup}(F(x, 2), F(x, 2)) = 0$,
- $h^u_{\sup}(F(0, y), F(0, 2)) = y - 2$
  $\leq h^u_{\sup}(F(0, 2), F(0, 2)) = 0 \leq h^u_{\sup}(F(x, 2), F(0, 2)) = 0$.

Example 4.17 ($l\&u$-type)**.** We set

$$X = Y = [0,2], \qquad Z = \mathbb{R}^2, \qquad C = \{(x,y) \mid y \geq 0\},$$

$$F : X \times Y \to \mathcal{V}_{\pm C}, \qquad F(x,y) = [x,\infty) \times [y, y+1], \qquad k^0 = (1,1).$$

We can confirm that F is C-proper, $(-C)$-proper, C-compact and $(-C)$-compact valued function and F satisfies assumptions of Theorem 4.14. Then we see that $(\bar{x},\bar{y}) = (0,2)$ is a weak $(l\&u)$-C-saddle point. Moreover, for any $x, y \in [0,2]$ we have the following inequalities.

- $0 = h^{l\&u}_{\inf}(F(0,y),F(0,y)) \leq h^{l\&u}_{\inf}(F(0,2),F(0,y)) = h^{l\&u}_{\inf}(F(x,2),F(0,y))$
  $= 2 - y,$
- $y - 2 = h^{l\&u}_{\inf}(F(0,y),F(0,2)) \leq h^{l\&u}_{\inf}(F(0,2),F(0,2)) = h^{l\&u}_{\inf}(F(x,2),F(0,2)) = 0,$
- $y - 2 = h^{l\&u}_{\sup}(F(0,y),F(x,2)) \leq h^{l\&u}_{\sup}(F(0,2),F(x,2))$
  $= h^{l\&u}_{\sup}(F(x,2),F(x,2)) = 0,$
- $y - 2 = h^{l\&u}_{\sup}(F(0,y),F(0,2)) \leq h^{l\&u}_{\sup}(F(0,2),F(0,2))$
  $= h^{l\&u}_{\sup}(F(x,2),F(0,2)) = 0.$

Let $\mathcal{S} \subset \mathcal{V}$. By proposition 2.6, we have the following inclusions:

$(l\&u)$-wMin$(\mathcal{S};\mathrm{int}C) \subset l$-wMin$(\mathcal{S};\mathrm{int}C)$,

$(l\&u)$-wMin$(\mathcal{S};\mathrm{int}C) \subset u$-wMin$(\mathcal{S};\mathrm{int}C)$,

$(l\&u)$-wMax$(\mathcal{S};\mathrm{int}C) \subset l$-wMax$(\mathcal{S};\mathrm{int}C)$,

$(l\&u)$-wMax$(\mathcal{S};\mathrm{int}C) \subset u$-wMax$(\mathcal{S};\mathrm{int}C)$.

We consider the following lu-C-saddle point and ul-C-saddle point problem:

(lu-P): $F(\bar{x},\bar{y}) \in l$-wMax$(F(\bar{x},Y);\mathrm{int}C) \cap u$-wMin$(F(X,\bar{y});\mathrm{int}C)$,
(ul-P): $F(\bar{x},\bar{y}) \in u$-wMax$(F(\bar{x},Y);\mathrm{int}C) \cap l$-wMin$(F(X,\bar{y});\mathrm{int}C)$.

By using the above facts, we obtain the following corollary.

Corollary 4.18 (Existence of weak lu and ul-C-saddle point)**.** *We assume the assumptions of Theorem 4.14. Then F has at least one weak lu-C-saddle point and weak ul-C-saddle point (of course, weak l-C-saddle point and weak u-C-saddle point).*

5. Conclusions

In this paper, first we corrected some errors of the properties of the six types of scalarizing functions for sets [2]. Then we relaxed the compactness condition of the result in [12].

Next, we presented three types of existence theorems of cone saddle-point for set-valued map which are natural extensions of Sion's minimax theorem [33] and vector-valued C-saddle point theorem.

(I) In proposition 4.7 and definition 4.10, we also investigated relationships among several concepts of cone-quasi convexity and cone continuity. Especially, we have found that l-type set relation is strong connection with u-type one.

(II) In vector-valued saddle point problem, compactness assumptions of image of f is not needed to obtain the existence of cone saddle-points (see also [20] and [35]).

On the other hand, in set-valued saddle point problem, we need additional conditions on F to show sufficiency of existence of cone saddle-points.

- F: C-proper & C-compact valued map $\Longrightarrow$ existence of weak l-C-saddle point
- F: $(-C)$-proper & $(-C)$-compact valued map
  $\Longrightarrow$ existence of weak u-C-saddle point
- F: C-proper & $(-C)$-proper & C-compact & $(-C)$-compact valued map
  $\Longrightarrow$ existence of weak $l\&u$-C-saddle point

(III) In the proof of existence theorems for set-valued map (Theorem 4.12, 4.13, 4.14), we used two types of nonlinear scalarizing techniques. The previous part of the proof is an inf-type scalarizing technique and the latter part is a sup-type one.

Recently in [16], they revealed strong connections between set optimization problem and uncertain multi-objective optimization problem. Moreover, they clarified that finding robust solutions to uncertain multi-objective optimization problem can be interpreted as an application to set optimization problem. Thus, we expect our results to apply robust game theory and it will be one of the most important subject of game theory.

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