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THE REPRESENTATION OF n-COLLINEARITY IN n-NORMED SPACE

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Abstract. In a n-normed space X, for the n-colinear elements $u, x_1, \dots, x_n$, we prove the uniqueness of u such that $u, x_1, \dots, x_n$ satisfy some inequalities. This answers two open questions given by W. Shatanawi and M. Postolache [13].

1. Introduction

Let X and Y be metric spaces, a mapping $f : X \to Y$ is called an *isometry* if f satisfies $d_Y(f(x), f(y)) = d_X(x, y)$ for all $x, y \in X$, where $d_X(\cdot, \cdot)$ and $d_Y(\cdot, \cdot)$ denote the metrics in the spaces X and Y, respectively.

The classical Mazur-Ulam Theorem [9] tells us that every surjective isometry between real normed spaces must be a linear mapping up to a translation. In 1970, Aleksandrov [1] posed the following problem:

> For a mapping f between two metric spaces preserving a single conservative distance, whether it is an isometry.

The n-normed spaces was introduced by Misiak [10] in 1989. Since 2004, many people start to study the Aleksandrov problem and Mazur-Ulam theorem in n-normed spaces (see the reference [2–8, 11, 12]). The representation of n-collinearity and 2-collinearity play a major role on the problem of conservative distance and other geometric properties (see [5–8]). Many interesting results on n-collinearity have been proved, for example preserving n-collinearity is equivalent to preserving the n-distances (see [8]).

This paper will study the representation of n-collinearity and prove the uniqueness of representation of n-collinearity, which answers two open problems given by W. Shatanawi and M. Postolache [13] in 2016.

Definition 1.1. Assume that X is a real linear space with $\mathtt{dim}X \geq n$ and $\|\cdot, \dots, \cdot\| : X^n \to \mathbb{R}$ is a non-negative function satisfying:

(1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;
(2) $\|x_1, \dots, x_n\| = \|x_{j_1}, \dots, x_{j_n}\|$ for every permutation $(j_1, \dots, j_n)$ of $(1, \dots, n)$;
(3) $\|\alpha x_1, \dots, x_n\| = |\alpha|\|x_1, \dots, x_n\|$;
(4) $\|x + y, x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$.

for any $\alpha \in \mathbb{R}$ and any $x_1, \dots, x_n \in X$. Then the function $\|\cdot, \dots, \cdot\|$ is called the *n-norm* on X, and $(X, \|\cdot, \dots, \cdot\|)$ is called a *linear n-normed space*.

Definition 1.2. Let X be a real n-normed spaces, we can define the following:

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(a) The points $x, y, z \in X$ are said to be 2-*collinear* if there exists $t \in \mathbb{R}$ such that $z - x = t(y - x)$.
(b) The points $x_0, x_1, \ldots, x_n$ are said to be n-*collinear* if for every $0 \leq i \leq n$, the elements $\{x_j - x_i \,:\, 0 \leq j \neq i \leq n\}$ are linearly dependent.

Remark 1.3. If the points $x_0, x_1, ..., x_n$ in X are n-collinear, then there are n scalars $\lambda_0, \lambda_1, ..., \lambda_n$, where not all λ_i are equal to zero, such that

$$x_0 = \frac{\sum_{i=1}^{n} \lambda_i x_i}{\sum_{i=1}^{n} \lambda_i}.$$

2. Main results

Wasfi Shatanawi and M. Postolache [13] gave the following representation for n-collinear maps.

Theorem 2.1 (see [13, Proposition 3.1])**.** *Let X be a real n-normed spaces and $x_1, x_2, \ldots, x_n \in X$, then $u = \frac{t_1x_1+t_2x_2+\ldots+t_nx_n}{t_1+t_2+\ldots+t_n}$ for some scalars $t_1, t_2, ..., t_n$, which are not all equal to 0, satisfies the property: for all $j \in \{2, 3, ..., n-1\}$ and some $c \in X$ with $||x_1 - c, x_2 - c, ..., x_n - c|| \neq 0$, we have that*

$$\begin{aligned} &||x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_j - u, x_{j+1} - c, \ldots, x_n - c|| \\ = &\frac{|\sum_{i=1, i\neq j}^{n} t_i|}{|\sum_{i=1}^{n} t_i|} ||x_1 - c, x_2 - c, \ldots, x_n - c||, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &||x_1 - u, x_2 - c, \ldots, x_{j-1} - c, x_j - c, x_{j+1} - c, \ldots, x_n - c|| \\ = &\frac{|\sum_{i=2}^{n} t_j|}{|\sum_{i=1}^{n} t_i|} ||x_1 - c, x_2 - c, \ldots, x_n - c||, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &||x_1 - c, x_2 - c, \ldots, x_{n-1} - c, x_n - u|| \\ = &\frac{|\sum_{i=1}^{n-1} t_j|}{|\sum_{i=1}^{n} t_i|} ||x_1 - c, x_2 - c, \ldots, x_n - c||, \end{aligned} \tag{2.3}$$

Open Problems: Whether u is the unique element satisfying the above relations (2.1)-(2.3)?

In the following, we show the answers are positive.

Lemma 2.2. *Let λ_1, λ_2 and t_1, t_2 are real numbers with*

$$\frac{|\lambda_i|}{|\lambda_1 + \lambda_2|} = \frac{|t_i|}{|t_1 + t_2|}, \quad \forall\, i = 1, 2.$$

Then there exists a positive number $\kappa > 0$ such that either

(i) $\lambda_i = \kappa t_i$ *for all* $i = 1, 2$
or
(ii) $\lambda_i = -\kappa t_i$ *for* $i = 1, 2$.

Proof. For some $i = 1, 2$, it is obviously that $\lambda_i = 0$ if and only if $t_i = 0$. So the result is true when some $\lambda_i = 0$. We next assume that $\lambda_i, t_i \neq 0$ for all $i = 1, 2$.

Let $\kappa = \frac{|\lambda_1+\lambda_2|}{|t_1+t_2|}$, then $\kappa|t_1 + t_2| = |\lambda_1 + \lambda_2|$ and $|\lambda_i| = \kappa|t_i|$, $\lambda_i = \kappa\xi_i t_i$ for $i = 1, 2$, where $\xi_i = 1$ or -1. So we have that

$$|t_1 + t_2| = |t_1 + \xi_1\xi_2 t_2|.$$

If $\xi_1\xi_2 = -1$, we have that $t_2 = 0$, which is impossible. This shows that $\xi_1\xi_2 = 1$. This implies that $\lambda_i = \kappa t_i$ for all $i = 1, 2$ or $\lambda_i = -\kappa t_i$ for $i = 1, 2$.

□

Remark 2.3. The above Lemma is not true for $n \geq 3$.

Example 2.4. Let $t_1, t_2, \ldots, t_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are real numbers with $t_2 = -t_1, \lambda_1 = -t_1, \lambda_2 = t_1, \lambda_i = t_i$ for each $i = 3, 4, \ldots, n$. It is obviously

$$\frac{|\lambda_j|}{|\lambda_1 + \lambda_2 + \ldots + \lambda_n|} = \frac{|t_j|}{|t_1 + t_2 + \ldots + t_n|}$$

for any $j = 1, 2, \ldots, n$.

Theorem 2.5. *Let X and Y be two real n-normed spaces and $x, y \in X$ with $x \neq y$, then $u = \frac{\lambda_1 x+\lambda_2 y}{\lambda_1+\lambda_2}$ is unique element such that u, x, y are 2-collinear and it satisfies the following relations:*

$$|x - c, y - u, x_3 - c, \ldots, x_n - c|| = \frac{|\lambda_1|}{|\lambda_1 + \lambda_2|}||x - c, y - c, x_3 - c, \ldots, x_n - c|| \quad (2.4)$$

and

$$||x - u, y - c, x_3 - c, \ldots, x_n - c|| = \frac{|\lambda_2|}{|\lambda_1 + \lambda_2|}||x - c, y - c, x_3 - c, \ldots, x_n - c|| \quad (2.5)$$

for some $c, x_3, \ldots, x_n \in X$ with $||x - c, y - c, x_3 - c, ..., x_n - c|| \neq 0$.

Proof. Choose $c, x_3, \ldots, x_n \in X$ with $||x - c, y - c, x_3 - c, \ldots, x_n - c|| \neq 0$, it follows from Theorem 2.1 that $u = \frac{\lambda_1 x+\lambda_2 y}{\lambda_1+\lambda_2}$ satisfies equations (2.4) and (2.5). So it suffices to show the uniqueness.

Assume that v is an element in X with v, x, y are 2-collinear with

$$||x - c, y - v, x_3 - c, \ldots, x_n - c|| = \frac{|\lambda_1|}{|\lambda_1 + \lambda_2|}||x - c, y - c, x_3 - c, \ldots, x_n - c||.$$

When write $v = \frac{t_1 x+t_2 y}{t_1+t_2}$, we can have that

$$||x - c, y - v, x_3 - c, \ldots, x_n - c|| = \frac{|t_1|}{|t_1 + t_2|}||x - c, x_2 - c, x_3 - c, ..., x_n - c||,$$

and hence $\frac{|\lambda_1|}{|\lambda_1+\lambda_2|} = \frac{|t_1|}{|t_1+t_2|}$. Similarly, we can derive that $\frac{|\lambda_2|}{|\lambda_1+\lambda_2|} = \frac{|t_2|}{|t_1+t_2|}$. Therefore, by Lemma 2.2, there exists $\kappa > 0$ such that $\lambda_i = \kappa t_i$ (for each $i = 1, 2$) or all $\lambda_i = -\kappa t_i$ (for each $i = 1, 2$). This implies that $v = u$. □

Theorem 2.6. *Let X be a real n-normed spaces, if $u, x_1, x_2, \ldots, x_n$ are n-collinear elements, then $u = \frac{x_1+x_2+\ldots+x_n}{n}$ is the unique element satisfying the following relations:*

$$||x_1-c, x_2-c, \ldots, x_{j-1}-c, x_j-u, x_{j+1}-c, x_n-c|| = \frac{n-1}{n}||x_1-c, x_2-c, \ldots, x_n-c||$$

for all $j = 2, 3, \ldots, n-1$,

$$||x_1 - u, x_2 - c, \ldots, x_{j-1} - c, x_j - c, x_{j+1} - c, \ldots, x_n - c|| \\ = \frac{n-1}{n}||x_1 - c, x_2 - c, x_3 - c, \ldots, x_n - c||,$$

and

$$||x_1 - c, x_2 - c, \ldots, x_{n-1} - c, x_n - u|| = \frac{n-1}{n}||x_1 - c, x_2 - c, x_3 - c, \ldots, x_n - c||$$

for some $c \in X$ with $||x_1 - c, x_2 - c, x_3 - c, \ldots, x_n - c|| \neq 0$.

Proof. Suppose that $||x - c, x_2 - c, x_3 - c, \ldots, x_n - c|| \neq 0$. To prove the uniqueness, assume that $v, x_1, x_2, \ldots, x_n$ are n-collinear satisfying the equations in the theorem. Since $x_1 - v, x_2 - v, \ldots, x_n - v$ are linearly dependent, there are n scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$v = \frac{\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n}{\lambda_1 + \lambda_2 + ...\lambda_n}.$$

It follows from the equations (2.1)-(2.3) that

$$\begin{aligned} &||x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_j - v, x_{j+1} - c, x_n - c|| \\ (2.6) \qquad = &\frac{|\sum_{i=1,i\neq j}^{n} \lambda_i|}{|\sum_{i=1}^{n} \lambda_i|}||x_1 - c, x_2 - c, \ldots, x_n - c|| \end{aligned}$$

for all $j = 2, 3, \ldots, n-1$,

$$\begin{aligned} &||x_1 - v, x_2 - c, \ldots, x_{j-1} - c, x_j - c, x_{j+1} - c, \ldots, x_n - c|| \\ (2.7) \qquad = &\frac{|\sum_{i=2}^{n} \lambda_j|}{|\sum_{i=1}^{n} \lambda_i|}||x_1 - c, x_2 - c, x_3 - c, \ldots, x_n - c||, \end{aligned}$$

and

$$\begin{aligned} &||x_1 - c, x_2 - c, \ldots, x_{n-1} - c, x_n - v|| \\ (2.8) \qquad = &\frac{|\sum_{i=1}^{n-1} \lambda_j|}{|\sum_{i=1}^{n} \lambda_i|}||x_1 - c, x_2 - c, x_3 - c, \ldots, x_n - c||. \end{aligned}$$

Therefore, for any $j = 1, 2, \ldots, n$, we have that

$$\begin{aligned} n|\sum_{i=1,i\neq j}^{n} \lambda_i| = (n-1)|\sum_{i=1}^{n} \lambda_i| &= |(n-1)\lambda_1 + (n-1)\lambda_2 + ... + (n-1)\lambda_n| \\ &= |\sum_{j=1}^{n}\sum_{i=1,i\neq j}^{n} \lambda_i|. \end{aligned}$$

This implies that $|\sum_{i=1,i\neq j}^{n}\lambda_i| = |\sum_{i=1,i\neq k}^{n}\lambda_i|$ for any $k, j = 1, 2, \ldots, n$ and

$$n|\sum_{i=1,i\neq j}^{n}\lambda_i| \leq \sum_{j=1}^{n}|\sum_{i=1,i\neq j}^{n}\lambda_i| = n|\sum_{i=1,i\neq j}^{n},\lambda_i|.$$

This implies that

$$|\sum_{j=1}^{n}\sum_{i=1,i\neq j}^{n}\lambda_i| = \sum_{j=1}^{n}|\sum_{i=1,i\neq j}^{n}\lambda_i|.$$

So we get that $\sum_{i=1,i\neq j}^{n}\lambda_i$, $j = 1, 2, \ldots, n$ are all positive or all negative, which together with $|\sum_{i=1,i\neq j}^{n}\lambda_i| = |\sum_{i=1,i\neq k}^{n}\lambda_i|$ for all $j, k = 1, 2, \ldots, n$, implies that

$$\sum_{i=1,i\neq j}^{n}\lambda_i = \sum_{i=1,i\neq k}^{n}\lambda_i \quad \forall\, k, j = 1, 2, \ldots, n.$$

Therefore, for any $j = 1, 2, \ldots, n$, it follows from $(n-1)\sum_{i=1}^{n}\lambda_i = n\sum_{i=1,i\neq j}^{n}\lambda_i$ and $(n-1)\lambda_j = \sum_{i=1,i\neq j}^{n}\lambda_i$ that

$$\lambda_1 = \lambda_2 = \ldots = \lambda_n.$$

This shows that $v = \frac{x_1+x_2+\ldots+x_n}{n}$. □

Theorem 2.7. *Let X be a real n-normed spaces, if $u, x_1, x_2, \ldots, x_n$ are n-collinear elements, then $u = \frac{x_1+x_2+\ldots+x_n}{n}$ is the unique element satisfying the following relations:*

(1). $||x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_j - u, x_{j+1} - c, \ldots, x_n - c|| = (n-1)||x_1 - c, x_3 - c, \ldots, x_{j-1} - c, x_1 - u, x_{j+1} - c, \ldots, x_n - c||$ *for all* $j = 2, 3, \ldots, n$*;*

(2). $\|x_2 - u, x_2 - c, \ldots, x_n - c\| = (n-1)\|x_1 - u, x_2 - c, \ldots, x_n - c\|$*;*

(3). $\|x_1-c, x_2-c, \cdots, x_{n-1}-c, x_n-u\| = (n-1)\|x_1-u, x_1-c, x_2-c, \ldots, x_{n-1}-c\|$

for some $c \in X$ *with* $||x_1 - c, x_2 - c, x_3 - c, \ldots, x_n - c|| \neq 0$.

Proof. Suppose that $||x_1-c, x_2-c, x_3-c, \ldots, x_n-c|| \neq 0$. To prove the uniqueness, assume that $v, x_1, x_2, \ldots, x_n$ are n-collinear. Since $x_1 - v, x_2 - v, \ldots, x_n - v$ are linearly dependent, there are n scalars $\lambda_1, \lambda_2, ..., \lambda_n$ such that

$$v = \frac{\lambda_1 x_1 + \lambda_2 x_2 + \ldots + \lambda_n x_n}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}.$$

It follows from [13, Corollary 3.5] that the elements $v, x_1, x_2, \ldots, x_n$ satisfy the following

(i) $|\lambda_j| \cdot ||x_1 - c, x_2 - c, \ldots, x_{j-1} - c, x_j - v, x_{j+1} - c, x_n - c|| = |\sum_{i=1,i\neq j}^{n}\lambda_i| \cdot ||x_1 - v, x_2 - c, \ldots, x_n - c||$ for all $j = 2, 3, ..., n-1$;

(ii) $|\lambda_1|\|x_2 - v, x_2 - c, \ldots, x_n - c\| = |\sum_{i=2}^{n}\lambda_i|\|x_1 - v, x_2 - c, \ldots, x_n - c\|$;

(iii) $|\lambda_n|\|x_1-c, x_2-c, \ldots, x_{n-1}-c, x_n-v\| = |\sum_{i=1}^{n-1}\lambda_i|\|x_1-v, x_1-c, \ldots, x_{n-1}-c\|$.

Then we can derive that

$$|\lambda_j|(n-1) = |\sum_{i=1,i\neq j}^{n}\lambda_i|$$

for all $j = 1, 2, 3, \ldots, n$. Thus we have that

$$\begin{aligned}(n-1)\sum_{j=1}^{n}|\lambda_j| &= \sum_{j=1}^{n}|\sum_{i=1,i\neq j}^{n}\lambda_i| \leq \sum_{j=1}^{n}\sum_{i=1,i\neq j}^{n}|\lambda_i| \\ &= \sum_{i=2}^{n}|\lambda_i| + \sum_{i=1,i\neq 2}^{n}|\lambda_i| + \ldots + \sum_{i=1,i\neq j}^{n}|\lambda_i| + \ldots + \sum_{i=1}^{n-1}|\lambda_i| \\ &= (n-1)(|\lambda_1| + |\lambda_2| + \ldots + |\lambda_n| \\ &= (n-1)\sum_{i=1}^{n}|\lambda_i|.\end{aligned}$$

This implies that $\sum_{i=1,i\neq j}^{n}|\lambda_i| = |\sum_{i=1,i\neq j}^{n}\lambda_i|$ for any $j = 1, 2, \ldots, n$, and thus $\lambda_i, i = 1, 2, ..., n, i \neq j$ are all positive or all negative. This shows that

$$(n-1)\lambda_j = \sum_{i=1,i\neq j}^{n}\lambda_i$$

and

$$n\lambda_j = \sum_{i=1}^{n}\lambda_i = n\lambda_1 = \ldots = n\lambda_n,$$

which implies that $v = u$. Therefore, we showed that u is the unique element satisfying equations (1)-(3) and complete the proof. □

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