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DISTANCES BETWEEN CONVEX COMBINATIONS IN HADAMARD SPACES

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Abstract. The paper aims to research the difference between the convex combinations which have different orders of combining three points in Hadamard spaces. It is interesting to consider ranges of distances between two convex combinations. The main finding is that the difference of shapes between an Hadamard space and the Euclidean plane gives a relation of between two convex combinations which have different orders of combining points from each other in the Hadamard space.

1. Introduction

The study of metric spaces without linear structures has played a vital role in various branches of pure and applied sciences. H. Busemann [7] developed a theory of non-positive curvature for path-metric spaces, based on a simple axiom of convexity of distance functions. By referring to this theory, B. H. Bowditch [4] introduced "Busemann spaces" which is a notion of non-positively curve metric spaces. Busemann spaces satisfy many fundamental metric, geometric, and topological properties [19]. A large subclass of Busemann spaces is constituted by non-positively curved metric spaces introduced by A. D. Alexandrov [1], known also under the name of "CAT(0) spaces". The difference between CAT(0) spaces and Busemann spaces is similar to that between Hilbert spaces and strictly convex Banach spaces. CAT(0) spaces also satisfy many fundamental properties and can be characterized in many natural ways [5]. CAT(0) spaces found numerous applications in geometric group theory and metric geometry. A useful example of a CAT(0) space is an $\mathbb{R}$-tree [22], whose study found applications in mathematics, biology, medicine, and computer science. Against such a background, it is worth making a study of geometry of non-positively curve metric spaces.

In geometry of metric spaces, convex combinations of some points are one of important elements. In fact, it is possible to characterize some kinds of non-positively curve metric spaces by comparing triangles in that space to triangles in the Euclidean plane. Therefore, convex combinations of three points is also concerned with geometry of metric spaces. Convex combinations in non-positively curve metric spaces are difficult elements to deal with. In metric spaces that provide a natural setting for considering triangles (which we will define later), a convex combination of two points with two coefficients which sum to 1 is a unique point. However,

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if we consider three points and three coefficients which sum to 1, then their corresponding convex combination is not clearly defined since it may depend on the order of combining the points. On the other hand, this convex combination is a unique point in Banach spaces since the spaces have linear structures. Based on the above fact, B. Beauzamy [3] studied characterization of uniform convexity of Banach spaces using convex combinations of three points with three coefficients. Since the characterization obtained is an important result, other researchers used it in order to study geometry of Banach spaces [12, 21].

In this paper, we study properties of convex combinations of three points with three coefficients which sum to 1 in uniformly convex Busemann spaces and Hadamard spaces. Namely, we put a pair of three points and consider a triangle consisting of the points. Convex combinations of the points are inside the triangle. In particular, we take up two convex combinations which have different orders of combining same points from each other. We research ranges of distances between the convex combinations using uniform convexity of uniquely geodesic spaces. The main finding is that a distance between two convex combinations is less than or equal to a value given by a comparison of shapes between the triangle in the Euclidean plane and the triangle in an Hadamard space. Moreover, we find results of distances between the convex combinations which are generalizations of a property of uniformly convex Banach spaces proved by Beauzamy.

In Section 2, we present several preliminary definitions and results. In Section 3, we prove ranges of distances between the convex combinations of three points with three coefficients which sum to 1 in uniformly convex Busemann space and Hadamard spaces. In Section 4, we consider $\mathbb{R}$-trees which is specific cases of Hadamard spaces. We prove values of distances between the convex combinations inside equilateral triangle.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers.

Let (X, d) be a metric space. A *path* in X is a continuous map $\gamma : [0, l] \subset \mathbb{R} \to X$. Given a pair of point $x, y \in X$, we say that a path $\gamma : [0, l] \to X$ *joins* x *and* y if $\gamma(0) = x$ and $\gamma(l) = y$. A *geodesic path* in X is an isometry $\gamma : [0, l] \to X$ such that $d(\gamma(s), \gamma(t)) = |s - t|$ for every $s, t \in [0, l]$. A *geodesic segment* $\gamma([0, l]) \subset X$ *from* x *to* y is the image of a geodesic path $\gamma : [0, l] \to X$ joining x and y. Note that a geodesic segment from x to y is not necessarily unique. If no confusion arises, then $[x, y]$ denotes a unique geodesic segment from x to y. A *(uniquely) geodesic space* is a metric space (X, d) if every two points in X can be joined by a (unique) geodesic path. Geodesic spaces provide a natural setting for considering triangles. A point $z \in X$ belongs to the geodesic segment $[x, y]$ if and only if there exists $t \in [0, 1]$ such that $d(x, z) = td(x, y)$ and $d(z, y) = (1 - t)d(x, y)$. For such a point, we write $z := (1 - t)x \oplus ty$, which is a unique point in $[x, y]$ for t.

Given distinct points $x, y \in X$, a *metric midpoint* of x and y is a point $m \in X$ if $d(x, y) = 2d(x, m) = 2d(m, y)$. A complete metric space X is geodesic space if and only if every pair of points in X has a metric midpoint [2, p. 2, Prop. 1.1.3].

Definition 2.1 ([4, pp. 576–577][19, pp. 203–204])**.** A *Busemann space* is a geodesic space (X, d) such that for every two geodesic paths $\gamma_1 : [0, l_1] \to X$ and $\gamma_2 : [0, l_2] \to X$, the map $D_{\gamma_1,\gamma_2} : [0, l_1] \times [0, l_2] \to \mathbb{R}$ defined by

$$D_{\gamma_1,\gamma_2}(t, s) = d\big(\gamma_1(t), \gamma_2(s)\big)$$

is convex.

There are various equivalent ways of expressing the definition of Busemann spaces. Several characterizations of Busemann spaces and other results on Busemann spaces consult the books [14, 19]. This condition gives the following elementary facts:

(1) [2, p. 4, Prop. 1.1.5] Every Busemann space X has a convex metric, that is, for every $x, y, z \in X$ and $t \in [0, 1]$,

$$d(x, (1-t)y \oplus tz) \leq (1-t)d(x, y) + td(x, z). \tag{2.1}$$

(2) [19, p. 210, Prop. 8.1.4] Every Busemann space is a uniquely geodesic space.

(3) [14, p. 40, Def. 6.5] In a Busemann space X, for every $x, y \in X$ and $t, s \in [0, 1]$,

$$d\big((1-t)x \oplus ty, (1-s)x \oplus sy\big) = |t-s|d(x, y). \tag{2.2}$$

Every complete geodesic locally compact, locally convex, and simply connected metric space is a Busemann space. Basic examples of Busemann spaces are the Euclidean space, normed strictly convex vector spaces, $\mathbb{R}$-trees, classical hyperbolic spaces [2], and Riemannian manifolds of global nonpositive sectional curvature [5].

Let (X, d) be a geodesic space. $\triangle(x_1, x_2, x_3)$ denotes a *geodesic triangle* which is a set consisting of geodesic segments $[x_1, x_2], [x_2, x_3], [x_3, x_1] \subset X$. If a point $p \in X$ lies in the union of $[x_i, x_j]$ for some $i, j \in \{1, 2, 3\}$, then we write $p \in\triangle(x_1, x_2, x_3)$. Given a geodesic triangle $\triangle(x_1, x_2, x_3)$, there exists a *comparison triangle* $\triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ which is a geodesic triangle in the Euclidean plane $(\mathbb{E}^2, \|\cdot\|)$ such that $d(x_i, x_j) = \|\bar{x}_i - \bar{x}_j\|$ for $i, j \in \{1, 2, 3\}$. If $p = (1-t)x_i \oplus tx_j \in\triangle(x_1, x_2, x_3)$ for $t \in [0, 1]$ and $i, j \in \{1, 2, 3\}$, then a *comparison point* for p is a point $\bar{p} := (1-t)\bar{x}_i \oplus t\bar{x}_j \in\triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3)$.

Definition 2.2 ([16, p. 119])**.** A *CAT(0) space* is a geodesic space (X, d) such that for every $\triangle(x, y, z) \subset X$, $p \in [x, y]$ and $q \in [x, z]$,

$$d(p, q) \leq \|\bar{p} - \bar{q}\|, \tag{2.3}$$

where $\bar{p}$ and $\bar{q}$ are comparison points respectively for p and q. An *Hadamard space* is a complete CAT(0) space.

The condition (2.3) is equivalent to the *CN inequality* [6, p. 64] [2, pp. 5–6], that is,

$$d\left(x, \frac{1}{2}y \oplus \frac{1}{2}z\right)^2 \leq \frac{1}{2}d(x, y)^2 + \frac{1}{2}d(x, z)^2 - \frac{1}{4}d(y, z)^2$$

for every $x, y, z \in X$. The following lemma is a generalization of the CN inequality.

Lemma 2.3 ([9, p. 2574, Lem. 2.5])**.** Let (X,d) be a CAT(0) space. Then

$$d(x,(1-t)y\oplus tz)^2\le(1-t)d(x,y)^2+td(x,z)^2-t(1-t)d(y,z)^2$$

for every $x,y,z\in X$ and $t\in[0,1]$.

A CAT(0) space X is geodesically connected and if every geodesic triangle in X is at least as "thin" as its comparison triangle in the Euclidean plane $\mathbb{E}^2$. Obviously, all CAT(0) spaces are Busemann spaces and thus uniquely geodesic spaces, but not vice-versa. The CAT(0) space has been investigated in various fields in mathematics, and a great deal of results have been obtained [5]. Basic examples of Hadamard spaces are Euclidean spaces, Hilbert spaces, $\mathbb{R}$-trees, classical hyperbolic spaces, complete simply connected Riemannian manifolds of nonpositive sectional curvature, Euclidean Bruhat-Tits buildings [6], the Hilbert ball [15], and other important spaces included in none of the above classes [5].

The notion of uniform convexity in Banach spaces appeared in Clarkson [8] whereas a modulus of convexity in hyperbolic spaces was coined by Goebel and Reich [15]. Similarly, there exists the notion of uniform convexity in metric spaces [11] and a modulus of convexity can be defined in geodesic spaces [18]. This modulus has been a very useful tool in geometry of geodesic spaces and it basically gives information about how square or rotund balls are.

Definition 2.4 ([18, pp. 468–469])**.** A uniquely geodesic space (X,d) is said to be *uniformly convex* if for every $r>0$ and $\epsilon\in(0,2]$ there exists $\delta\in(0,1]$ such that for every $a,x,y\in X$,

$$\left.\begin{array}{l} d(x,a)\le r\\ d(y,a)\le r\\ d(x,y)\ge\epsilon r\end{array}\right\}\quad\Rightarrow\quad d\left(\frac{1}{2}x\oplus\frac{1}{2}y,a\right)\le(1-\delta)r.$$

A *modulus of convexity* is a mapping $\delta:X\times(0,\infty)\times(0,2]\to(0,1]$ providing such

$$\delta=\delta(a,r,\epsilon):=\inf\left\{1-\frac{1}{r}d\left(\frac{1}{2}x\oplus\frac{1}{2}y,a\right)\ :\ d(x,a)\le r,d(y,a)\le r,d(x,y)\ge\epsilon r\right\}$$

for given $a\in X,r>0$ and $\epsilon\in(0,2]$.

While in Banach spaces there exists a natural modulus of convexity for each space which only depends on ϵ, in geodesic spaces we need to assume that the modulus depends on three variables: the center of the ball given by a, the radius of the ball given by r, and the separation condition given by ϵ. By the CN inequality, the modulus of convexity of a Hilbert space is a modulus of convexity for every CAT(0) space, so that every CAT(0) space is uniformly convex [17, p. 391, Prop. 8].

3. Convex combinations inside geodesic triangles

In this section, we study some properties of metrics between convex combinations inside geodesic triangles of uniformly convex Busemann spaces and Hadamard spaces.

Proposition 3.1. Let (X,d) be a complete Busemann space. If $p,q \in X$ and $\mu \in [0,1]$, then

$$\frac{1+\mu}{2}p \oplus \frac{1-\mu}{2}q = \mu p \oplus (1-\mu)\left(\frac{1}{2}p \oplus \frac{1}{2}q\right).$$

Proof. Let $u = \mu p \oplus (1-\mu)(p/2 \oplus q/2)$. Since $u \in [p,q]$, there exists $\alpha \in [0,1]$ such that $\alpha p \oplus (1-\alpha)q = u$. We have

$$d(p,u) = (1-\mu)d\left(p, \frac{1}{2}p \oplus \frac{1}{2}q\right) = \frac{1-\mu}{2}d(p,q) = \frac{1-\mu}{2} \cdot \frac{1}{1-\alpha}d(p,u).$$

Therefore $1-\alpha = (1-\mu)/2$ and $\alpha = (1+\mu)/2$. □

Theorem 3.2. Let (X,d) be a complete Busemann space. Let $x_1, x_2, x_3 \in X$ be three distinct points such that $d(x_1,x_2) = d(x_1,x_3) = r$. Set $\epsilon r = d(x_2,x_3)$. If X is uniformly convex, then for every $t \in [0,1]$,

$$d\left(x_1, \frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\left(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right)\right) \le r\left(\frac{1+t}{2} - t\delta(x_1,r,\epsilon)\right).$$

Proof. By uniform convexity of X, we have $d(x_1, x_2/2 \oplus x_3/2) \le \big(1-\delta(x_1,r,\epsilon)\big)r$. Using Proposition 3.1, applied to $p = x_3, q = x_2, \mu = (1-t)/(1+t)$, we obtain

$$\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3 = \frac{1-t}{1+t}x_3 \oplus \frac{2t}{1+t}\left(\frac{1}{2}x_2 \oplus \frac{1}{2}x_3\right).$$

Therefore

$$\begin{aligned}
&d\left(x_1, \frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\left(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right)\right)\\
&= \frac{1+t}{2}d\left(x_1, \frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right) = \frac{1+t}{2}d\left(x_1, \frac{1-t}{1+t}x_3 \oplus \frac{2t}{1+t}\left(\frac{1}{2}x_2 \oplus \frac{1}{2}x_3\right)\right)\\
&\le \frac{1-t}{2}d(x_1,x_3) + td\left(x_1, \frac{1}{2}x_2 \oplus \frac{1}{2}x_3\right)\\
&\le \frac{(1-t)r}{2} + t\big(1-\delta(x_1,r,\epsilon)\big)r = r\left(\frac{1+t}{2} - t\delta(x_1,r,\epsilon)\right).
\end{aligned}$$

□

Theorem 3.3. Let (X,d) be an Hadamard space. Let $x_1,x_2,x_3 \in X$ be three distinct points. Let

$$p_t = \frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\left(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right)$$

for $t \in [0,1]$. Then

$$\begin{aligned}
&\frac{1}{2}d(p_t,x_3)^2 + \frac{1}{2}d(p_t,(1-t)x_1 \oplus tx_2)^2\\
&\le (1-t)\left(\frac{d(x_1,x_3)}{2}\right)^2 + t\left(\frac{d(x_2,x_3)}{2}\right)^2 - t(1-t)\left(\frac{d(x_1,x_2)}{2}\right)^2.
\end{aligned} \tag{3.1}$$

Proof. Using Lemma 2.3, we obtain

$$\begin{aligned} d(p_t,x_3)^2 &\le \frac{1-t}{2}d(x_1,x_3)^2+\frac{1+t}{2}d\biggl(\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3,x_3\biggr)^2\\ &\quad-\frac{1-t}{2}\cdot\frac{1+t}{2}d\biggl(x_1,\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3\biggr)^2\\ &=\frac{1-t}{2}d(x_1,x_3)^2+\frac{1+t}{2}\biggl(\frac{t}{1+t}d(x_2,x_3)\biggr)^2\\ &\quad-\frac{(1-t)(1+t)}{4}d\biggl(x_1,\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3\biggr)^2\\ &=\frac{1-t}{2}d(x_1,x_3)^2+\frac{t^2}{2(1+t)}d(x_2,x_3)^2\\ &\quad-\frac{(1-t)(1+t)}{4}d\biggl(x_1,\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3\biggr)^2. \end{aligned}\tag{3.2}$$

In the same way, we have

$$\begin{aligned} &d(p_t,(1-t)x_1\oplus tx_2)^2\\ &\le(1-t)d(p_t,x_1)^2+td(p_t,x_2)^2-(1-t)td(x_1,x_2)^2\\ &=(1-t)\biggl(\frac{1+t}{2}d\biggl(\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3,x_1\biggr)\biggr)^2-(1-t)td(x_1,x_2)^2+td(p_t,x_2)^2\\ &=\frac{(1-t)(1+t)^2}{4}d\biggl(x_1,\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3\biggr)^2-(1-t)td(x_1,x_2)^2+td(p_t,x_2)^2. \end{aligned}\tag{3.3}$$

Moreover,

$$\begin{aligned} d(p_t,x_2)^2 &\le \frac{1-t}{2}d(x_1,x_2)^2+\frac{1+t}{2}d\biggl(\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3,x_2\biggr)^2\\ &\quad-\frac{1-t}{2}\cdot\frac{1+t}{2}d\biggl(x_1,\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3\biggr)^2\\ &=\frac{1-t}{2}d(x_1,x_2)^2+\frac{1+t}{2}\biggl(\frac{1}{1+t}d(x_3,x_2)\biggr)^2\\ &\quad-\frac{(1-t)(1+t)}{4}d\biggl(x_1,\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3\biggr)^2\\ &=\frac{1-t}{2}d(x_1,x_2)^2+\frac{1}{2(1+t)}d(x_3,x_2)^2\\ &\quad-\frac{(1-t)(1+t)}{4}d\biggl(x_1,\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3\biggr)^2. \end{aligned}\tag{3.4}$$

Combining (3.4) with (3.3), we get

$$\begin{aligned} &d(p_t,(1-t)x_1\oplus tx_2)^2\\ &\le\frac{(1-t)(1+t)^2}{4}d\biggl(x_1,\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3\biggr)^2-(1-t)td(x_1,x_2)^2\\ &\quad+t\biggl\{\frac{1-t}{2}d(x_1,x_2)^2+\frac{1}{2(1+t)}d(x_3,x_2)^2-\frac{(1-t)(1+t)}{4}d\biggl(x_1,\frac{t}{1+t}x_2\oplus\frac{1}{1+t}x_3\biggr)^2\biggr\} \end{aligned}$$

$$= \frac{(1-t)(1+t)}{4} d\Big(x_1, \frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\Big)^2 - \frac{t(1-t)}{2} d(x_1,x_2)^2 + \frac{t}{2(1+t)} d(x_3,x_2)^2. \tag{3.5}$$

We conclude from (3.2) and (3.5) that

$$\begin{aligned}
&\frac{1}{2}d(p_t,x_3)^2 + \frac{1}{2}d(p_t,(1-t)x_1 \oplus tx_2)^2 \\
&\le \frac{1}{2}\left\{\frac{1-t}{2}d(x_1,x_3)^2 + \frac{t^2}{2(1+t)}d(x_2,x_3)^2 - \frac{(1-t)(1+t)}{4}d\Big(x_1, \frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\Big)^2\right\} \\
&\quad + \frac{1}{2}\left\{\frac{(1-t)(1+t)}{4}d\Big(x_1, \frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\Big)^2 - \frac{t(1-t)}{2}d(x_1,x_2)^2 + \frac{t}{2(1+t)}d(x_2,x_3)^2\right\} \\
&= \frac{1-t}{4}d(x_1,x_3)^2 + \frac{t(1+t)}{4(1+t)}d(x_2,x_3)^2 - \frac{t(1-t)}{4}d(x_1,x_2)^2 \\
&= (1-t)\Big(\frac{d(x_1,x_3)}{2}\Big)^2 + t\Big(\frac{d(x_2,x_3)}{2}\Big)^2 - t(1-t)\Big(\frac{d(x_1,x_2)}{2}\Big)^2.
\end{aligned}$$

□

Theorem 3.4. Let (X,d) be an Hadamard space. Let $x_1,x_2,x_3 \in X$ be three distinct points such that $d(x_1,x_2) = d(x_1,x_3) = r$. Set $\epsilon r = d(x_2,x_3)$. Then for every $t \in [0,1]$,

$$\begin{aligned}
&d\left(\frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\Big(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\Big), \frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big)\right) \\
&\le \frac{1}{2}\sqrt{(1-t)^2r^2 + t\epsilon^2r^2 - d(x_3,(1-t)x_1 \oplus tx_2)^2}.
\end{aligned}$$

Proof. Let

$$p_t = \frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\Big(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\Big) \quad \text{and} \quad q_t = \frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big)$$

for $t \in [0,1]$. Using Lemma 2.3 and Theorem 3.3, we obtain

$$\begin{aligned}
d(p_t,q_t)^2 &\le \frac{1}{2}d(p_t,x_3)^2 + \frac{1}{2}d(p_t,(1-t)x_1 \oplus tx_2)^2 - \frac{1}{4}d(x_3,(1-t)x_1 \oplus tx_2)^2 \\
&\le (1-t)\Big(\frac{r}{2}\Big)^2 + t\Big(\frac{\epsilon r}{2}\Big)^2 - t(1-t)\Big(\frac{r}{2}\Big)^2 - \frac{1}{4}d(x_3,(1-t)x_1 \oplus tx_2)^2 \\
&= \frac{1}{4}\big\{(1-t)^2r^2 + t\epsilon^2r^2 - d(x_3,(1-t)x_1 \oplus tx_2)^2\big\}.
\end{aligned}$$

Therefore

$$d(p_t,q_t) \le \frac{1}{2}\sqrt{(1-t)^2r^2 + t\epsilon^2r^2 - d(x_3,(1-t)x_1 \oplus tx_2)^2}.$$

□

Remark 1. In a Hilbert space $(H,\|\cdot\|)$, Lemma 2.3 is Stewart's theorem, that is,

$$\big\|x - \big((1-t)y + tz\big)\big\|^2 = (1-t)\|x-y\|^2 + t\|x-z\|^2 - t(1-t)\|y-z\|^2$$

for every $x,y,z \in H$ and $t \in [0,1]$. If $\|x-y\| = \|y-z\| = r$ and $\|x-z\| = \epsilon r$, we have

$$\big\|x - \big((1-t)y + tz\big)\big\|^2 = (1-t)r^2 + t\epsilon^2r^2 - t(1-t)r^2 = (1-t)^2r^2 + t\epsilon^2r^2. \tag{3.6}$$

Convex combinations p_t and q_t in the proof of Theorem 3.4 have different orders of combining three points. Let $\bar{x}_1, \bar{x}_2, \bar{x}_3 \in \mathbb{E}^2$ be comparison points respectively for x_1, x_2, x_3 in an Hadamard space X. Combining (3.6) with Theorem 3.4, $d(p_t, q_t)$ is less than or equal to a value given by the difference between $\|\bar{x}_3 - ((1-t)\bar{x}_1 + t\bar{x}_2)\|^2$ and $d(x_3, (1-t)x_1 \oplus tx_2)^2$. This implies that the difference of shapes between a geodesic triangle $\triangle(x_1, x_2, x_3) \subset X$ and a comparison triangle $\triangle(\bar{x}_1, \bar{x}_2, \bar{x}_3) \subset \mathbb{E}^2$ gives a relation of between two convex combinations which have different orders of combining same points from each other in X.

Theorem 3.5. Let (X, d) be an Hadamard space. Let $x_1, x_2, x_3 \in X$ be three distinct points such that $d(x_1, x_2) = d(x_1, x_3) = r$. Set $\epsilon r = d(x_2, x_3)$. Then for every $t \in [0, 1]$,

$$\begin{aligned} &d\left(x_1, \frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big)\right) \\ &\le r\left(\frac{1+t}{2} - t\delta(x_1, r, \epsilon)\right) + \frac{1}{2}\sqrt{(1-t)^2r^2 + t\epsilon^2r^2 - d(x_3, (1-t)x_1 \oplus tx_2)^2}. \end{aligned}$$

Proof. Using the triangle inequality and Theorems 3.2 and 3.4, we obtain

$$\begin{aligned} &d\Big(x_1, \frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big)\Big) \\ &\le d\left(x_1, \frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\left(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right)\right) \\ &\quad + d\left(\frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\left(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right), \frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big)\right) \\ &\le r\left(\frac{1+t}{2} - t\delta(x_1, r, \epsilon)\right) + \frac{1}{2}\sqrt{(1-t)^2r^2 + t\epsilon^2r^2 - d(x_3, (1-t)x_1 \oplus tx_2)^2}. \end{aligned}$$

□

If (X, d) is a Banach space with norm $\|\cdot\|$, convex combinations of a finite number of points do not depend on the order of combining the points. Since the modulus of convexity of a Banach space is

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+ty}{2}\right\| \ : \ \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\right\},$$

the following lemma is included in Theorems 3.2 and 3.5 with $r = 1$.

Lemma 3.6 ([3, p. 191, Lem. 2])**.** Let $(X, \|\cdot\|)$ be a Banach space. Let $x, y \in X$ be two distinct points of norm 1. Set $\epsilon = \|x - y\|$. If X is uniformly convex, then for every $t \in [0, 1]$,

$$\left\|\frac{x+ty}{2}\right\| \le \frac{1+t}{2} - t\delta(\epsilon).$$

This implies that Theorems 3.2 and 3.5 are generalizations of a property of uniformly convex Banach spaces.

4. Convex combinations of $\mathbb{R}$-trees

In this section, we study some properties of metrics between convex combinations inside geodesic triangles of $\mathbb{R}$-trees. In particular, we focus on triangles in which all three sides are equal.

Definition 4.1 ([22, p. 379])**.** An $\mathbb{R}$-*tree* is a uniquely geodesic metric space X such that if $x, y, z \in X$ are such that $[x, z] \cap [z, y] = \{z\}$, then $[x, z] \cup [z, y] = [x, y]$.

The following is an immediate consequence of the definition of an $\mathbb{R}$-tree:

(R1) For $x, y, z \in X$ there exists a point $w \in X$ such that $[x, y] \cap [x, z] = [x, w]$.

An $\mathbb{R}$-tree may be defined as an Hadamard space. The largest modulus of convexity can be easily calculated for $\mathbb{R}$-trees.

Lemma 4.2 ([10, p. 652, Thm. 5.9])**.** The modulus of convexity of an $\mathbb{R}$-tree giving the largest δ for each a, r, and ϵ coincides with the modulus of the real line, that is, it can be written as $\delta(\epsilon) = \epsilon/2$.

Let (X, d) be an $\mathbb{R}$-tree and $\triangle(x_1, x_2, x_3)$ a geodesic triangle in X. By (R1), there exists $w \in \triangle(x_1, x_2, x_3)$ such that $[x_1, x_2] \cap [x_1, x_3] = [x_1, w]$. An *equilateral triangle* is a geodesic triangle $\triangle(x_1, x_2, x_3)$ such that $d(x_1, x_2) = d(x_2, x_3) = d(x_3, x_1)$. An equilateral triangle $\triangle(x_1, x_2, x_3)$ satisfies

$$d(x_1, w) = d(x_2, w) = d(x_3, w) = \frac{1}{2} d(x_i, x_j), \tag{4.1}$$

where $i, j \in \{1, 2, 3\}$ with $i \neq j$. This implies that w is a midpoint of every $[x_i, x_j]$.

Theorem 4.3. Let (X, d) be an $\mathbb{R}$-tree and $\triangle(x_1, x_2, x_3)$ an equilateral triangle in X. Then

$$\frac{1-t}{2} x_1 \oplus \frac{1+t}{2} \left(\frac{t}{1+t} x_2 \oplus \frac{1}{1+t} x_3 \right) = \frac{1}{2} x_i \oplus \frac{1}{2} x_j$$

for every $t \in [0, 1]$, where $i, j \in \{1, 2, 3\}$ with $i \neq j$.

Proof. Let $m = (1/2)x_i \oplus (1/2)x_j$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$. Clearly for all $t \in [0, 1]$,

$$\frac{t}{1+t} x_2 \oplus \frac{1}{1+t} x_3 \in [m, x_3] \subset [x_1, x_3].$$

By (4.1), we have

$$\begin{aligned} d\left(x_1, \frac{t}{1+t} x_2 \oplus \frac{1}{1+t} x_3 \right) &= d(x_1, m) + d\left(m, \frac{t}{1+t} x_2 \oplus \frac{1}{1+t} x_3 \right) \\ &= d(x_2, m) + d\left(m, \frac{t}{1+t} x_2 \oplus \frac{1}{1+t} x_3 \right) \\ &= d\left(x_2, \frac{t}{1+t} x_2 \oplus \frac{1}{1+t} x_3 \right) = \frac{1}{1+t} d(x_2, x_3). \end{aligned}$$

From this equality and (4.1), we obtain

$$\begin{aligned} d\left(x_1, \frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\left(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right)\right) &= \frac{1+t}{2} d\left(x_1, \frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right) \\ &= \frac{1+t}{2} \cdot \frac{1}{1+t} d(x_2, x_3) \\ &= \frac{1}{2} d(x_1, x_3) = d(x_1, m). \end{aligned}$$

We reaffirm that $m \in [x_1, x_3]$ and

$$\frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\left(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right) \in [x_1, x_3].$$

By a uniqueness of a point m belonging to a unique geodesic segment $[x_1, x_3]$ such that $d(x_1, z) = (1/2)d(x_1, x_3)$, we conclude that

$$\frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\left(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right) = m.$$

□

Remark 2. In an $\mathbb{R}$-tree, let $\triangle(x_1, x_2, x_3)$ be an equilateral triangle such that the length r of its one side. Combining Theorems 3.2 and 4.3 and Lemma 4.2, notice that $\epsilon = 1$, we obtain

$$\begin{aligned} d\left(x_1, \frac{1-t}{2}x_1 \oplus \frac{1+t}{2}\left(\frac{t}{1+t}x_2 \oplus \frac{1}{1+t}x_3\right)\right) &= d\left(x_1, \frac{1}{2}x_1 \oplus \frac{1}{2}x_3\right) = \frac{1}{2}r \\ &\le r\left(\frac{1+t}{2} - t \cdot \frac{\epsilon}{2}\right) = \frac{1}{2}r. \end{aligned}$$

This implies that the inequality in Theorem 3.2 is an equality when $\triangle(x_1, x_2, x_3)$ is an equilateral triangle in an $\mathbb{R}$-tree.

Proposition 4.4. Let (X, d) be an $\mathbb{R}$-tree and $\triangle(x_1, x_2, x_3)$ an equilateral triangle in X. Then

$$\frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big) = \begin{cases} \dfrac{1-t}{2}x_2 \oplus \dfrac{1+t}{2}x_3, & 0 \le t \le \frac{1}{2}, \\ \dfrac{t}{2}x_2 \oplus \dfrac{2-t}{2}x_3, & \frac{1}{2} \le t \le 1. \end{cases} \tag{4.2}$$

Proof. Let $z = (1/4)x_2 \oplus (3/4)x_3$ and $m = (1/2)x_i \oplus (1/2)x_j$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$. Clearly, $z = (1/2)m \oplus (1/2)x_3$ and for every $t \in [0, 1]$,

$$\frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big) \in [m, z] \subset [m, x_3].$$

We now prove the theorem for $0 \le t \le 1/2$. Obviously,

$$\frac{1-t}{2}x_2 \oplus \frac{1+t}{2}x_3 \in [m, z] \subset [m, x_3].$$

By (4.1), we get

$$\begin{aligned}
&d\left(\frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big), x_3\right) - d\left(\frac{1-t}{2}x_2 \oplus \frac{1+t}{2}x_3, x_3\right)\\
&= \frac{1}{2}d\big((1-t)x_1 \oplus tx_2, x_3\big) - \frac{1-t}{2}d(x_2, x_3)\\
&= \frac{1}{2}\left\{d\big((1-t)x_1 \oplus tx_2, m\big) + d(m, x_3)\right\} - \frac{1-t}{2}d(x_2, x_3)\\
&= \frac{1}{2}\left\{d\big((1-t)x_1 \oplus tx_2, m\big) + d(m, x_2)\right\} - \frac{1-t}{2}d(x_2, x_3)\\
&= \frac{1}{2}d\big((1-t)x_1 \oplus tx_2, x_2\big) - \frac{1-t}{2}d(x_2, x_3)\\
&= \frac{1-t}{2}d(x_1, x_2) - \frac{1-t}{2}d(x_1, x_2) = 0.
\end{aligned}$$

By a uniqueness of a point p belonging to a unique geodesic segment $[m, x_3]$ such that $d(p, x_3) = (1-t)d(m, x_3) = \big((1-t)/2\big)d(x_2, x_3)$, therefore for $0 \leq t \leq 1/2$,

$$\frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big) = \frac{1-t}{2}x_2 \oplus \frac{1+t}{2}x_3.$$

We next prove the theorem for $1/2 \leq t \leq 1$. Obviously,

$$\frac{t}{2}x_2 \oplus \frac{2-t}{2}x_3 \in [m, z] \subset [m, x_3].$$

By (4.1), we get

$$\begin{aligned}
&d\left(\frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big), x_3\right) - d\left(\frac{t}{2}x_2 \oplus \frac{2-t}{2}x_3, x_3\right)\\
&= \frac{1}{2}d\big((1-t)x_1 \oplus tx_2, x_3\big) - \frac{t}{2}d(x_2, x_3)\\
&= \frac{1}{2}\left\{d\big((1-t)x_1 \oplus tx_2, m\big) + d(m, x_3)\right\} - \frac{t}{2}d(x_2, x_3)\\
&= \frac{1}{2}\left\{d\big((1-t)x_1 \oplus tx_2, m\big) + d(m, x_1)\right\} - \frac{t}{2}d(x_2, x_3)\\
&= \frac{1}{2}d\big((1-t)x_1 \oplus tx_2, x_1\big) - \frac{t}{2}d(x_2, x_3)\\
&= \frac{t}{2}d(x_1, x_2) - \frac{t}{2}d(x_1, x_2) = 0.
\end{aligned}$$

By a uniqueness of a point p belonging to a unique geodesic segment $[m, x_3]$ such that $d(p, x_3) = td(m, x_3) = (t/2)d(x_2, x_3)$, therefore for $1/2 \leq t \leq 1$,

$$\frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big) = \frac{t}{2}x_2 \oplus \frac{2-t}{2}x_3.$$

$\square$

Theorem 4.5. Let (X,d) be an $\mathbb{R}$-tree. Let $\triangle(x_1,x_2,x_3)\subset X$ be an equilateral triangle such that $d(x_i,x_j)=r$ for $i,j\in\{1,2,3\}$ with $i\neq j$. Then

$$d\left(x_1,\frac{1}{2}x_3\oplus\frac{1}{2}\big((1-t)x_1\oplus tx_2\big)\right)=\begin{cases}\dfrac{1+t}{2}r, & 0\le t\le \frac{1}{2},\\[2ex] \dfrac{2-t}{2}r, & \frac{1}{2}\le t\le 1.\end{cases}\tag{4.3}$$

Proof. We have

$$\begin{aligned}&d\left(x_1,\frac{1}{2}x_3\oplus\frac{1}{2}\big((1-t)x_1\oplus tx_2\big)\right)\\&=d\left(x_1,\frac{1}{2}x_1\oplus\frac{1}{2}x_3\right)+d\left(\frac{1}{2}x_1\oplus\frac{1}{2}x_3,\frac{1}{2}x_3\oplus\frac{1}{2}\big((1-t)x_1\oplus tx_2\big)\right)\\&=\frac{1}{2}r+d\left(\frac{1}{2}x_2\oplus\frac{1}{2}x_3,\frac{1}{2}x_3\oplus\frac{1}{2}\big((1-t)x_1\oplus tx_2\big)\right).\end{aligned}\tag{4.4}$$

Combining (2.2) and (4.4) with Proposition 4.4, we obtain for $0\le t\le 1/2$,

$$\begin{aligned}d\left(x_1,\frac{1}{2}x_3\oplus\frac{1}{2}\big((1-t)x_1\oplus tx_2\big)\right)&=\frac{1}{2}r+d\left(\frac{1}{2}x_2\oplus\frac{1}{2}x_3,\frac{1-t}{2}x_2\oplus\frac{1+t}{2}x_3\right)\\&=\frac{1}{2}r+\left|\frac{1}{2}-\frac{1-t}{2}\right|d(x_2,x_3)=\frac{1+t}{2}r.\end{aligned}$$

Similarly, we have for $1/2\le t\le 1$,

$$\begin{aligned}d\left(x_1,\frac{1}{2}x_3\oplus\frac{1}{2}\big((1-t)x_1\oplus tx_2\big)\right)&=\frac{1}{2}r+d\left(\frac{1}{2}x_2\oplus\frac{1}{2}x_3,\frac{t}{2}x_2\oplus\frac{2-t}{2}x_3\right)\\&=\frac{1}{2}r+\left|\frac{1}{2}-\frac{t}{2}\right|d(x_2,x_3)=\frac{2-t}{2}r.\end{aligned}$$

$\square$

Remark 3. In an $\mathbb{R}$-tree, let $\triangle(x_1,x_2,x_3)$ be an equilateral triangle such that the length r of its one side. We easily have $d(x_3,(1-t)x_1\oplus tx_2)=(1-t)r$ for $0\le t\le 1/2$ and $d(x_3,(1-t)x_1\oplus tx_2)=tr$ for $1/2\le t\le 1$. Using Theorem 3.5 and Lemma 4.2, notice that $\epsilon=1$, we obtain

$$\begin{aligned}&d\left(x_1,\frac{1}{2}x_3\oplus\frac{1}{2}\big((1-t)x_1\oplus tx_2\big)\right)\\&\le r\left(\frac{1+t}{2}-t\cdot\frac{\epsilon}{2}\right)+\frac{1}{2}\sqrt{(1-t)^2r^2+t\epsilon^2r^2-d(x_3,(1-t)x_1\oplus tx_2)^2}\\&\le\begin{cases}\dfrac{r}{2}+\dfrac{1}{2}\sqrt{(1-t)^2r^2+tr^2-\big((1-t)r\big)^2}=\dfrac{r}{2}\left(1+\sqrt{t}\right), & 0\le t\le\frac{1}{2},\\[2ex]\dfrac{r}{2}+\dfrac{1}{2}\sqrt{(1-t)^2r^2+tr^2-\big(tr\big)^2}=\dfrac{r}{2}\left(1+\sqrt{1-t}\right), & \frac{1}{2}\le t\le 1.\end{cases}\end{aligned}$$

If $0 \leq t \leq 1/2$, then $1+t \leq 1+\sqrt{t}$. Moreover if $1/2 \leq t \leq 1$, then $2-t \leq 1+\sqrt{1-t}$. Combining these results and Theorem 4.5, we get

$$d\left(x_1, \frac{1}{2}x_3 \oplus \frac{1}{2}\big((1-t)x_1 \oplus tx_2\big)\right) = \begin{cases} \dfrac{1+t}{2}r \leq \dfrac{r}{2}\Big(1+\sqrt{t}\Big), & 0 \leq t \leq \frac{1}{2}, \\ \dfrac{2-t}{2}r \leq \dfrac{r}{2}\big(1+\sqrt{1-t}\big), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The above are results of equilateral triangles in $\mathbb{R}$-trees. Considering theorems in Section 4, a separate question arises:

Question 1. Let $\triangle\,(x_1, x_2, x_3)$ be a geodesic triangle in an $\mathbb{R}$-tree such that $d(x_1, x_2) = d(x_1, x_3) = r > 0$ and $d(x_2, x_3) = \epsilon r$ for $\epsilon \in (0, 2]$. How long are the distances between convex combinations inside this geodesic triangle?

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