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A WEAK CONVERGENCE THEOREM FOR TWO INFINITE FAMILIES OF EXTENDED GENERALIZED HYBRID MAPPINGS IN A BANACH SPACE AND APPLICATIONS

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Abstract. Let E be a real Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called extended generalized hybrid [6] if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

for all $x, y \in C$. In this paper, we prove a weak convergence theorem of Mann's type iteration for two infinite families of extended generalized hybrid mappings in a Banach space satisfying Opial's condition. This theorem generalizes a theorem by Hojo and Takahashi [8]. Using this result, we get well-known and new weak convergence theorems in a Banach space. In particular, we obtain a weak convergence theorem of Mann's type iteration for two finite families of extended generalized hybrid mappings in a Banach space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping $T : C \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. In 2010, Kocourek, Takahashi and Yao [11] defined a broad class of nonlinear mappings in a Hilbert space which covers nonexpansive mappings: Let C be a nonempty subset of H. A mapping $T : C \to H$ is called *generalized hybrid* [11] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{1.1}$$

for all $x, y \in C$; see also [1]. Such a mapping T is called (α, β)-*generalized hybrid.* Notice that the class of generalized hybrid mappings covers several well-known mappings in a Hilbert space. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is *nonspreading* [12, 13] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also *hybrid* [18] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [9]. Hojo and Takahashi [6] extended the concept of generalized hybrid mappings in a Hilbert space to that in a Banach space as follows: Let E be a Banach space and let C be a

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nonempty subset of E. A mapping $T : C \to E$ is called *extended generalized hybrid* [6] if there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0 \tag{1.2}$$

for all $x, y \in C$. We call such a mapping $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid. Hojo and Takahashi [7] proved the following weak convergence theorem for finding a common fixed point of two extended generalized hybrid mappings in a Banach space by using Mann's type iteration [14]; see also [19].

Theorem 1.1 ([7]). *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $\alpha', \beta', \gamma', \delta' \in \mathbb{R}$. Let S and T be $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma,' \delta')$-extended generalized hybrid mappings of C into itself such that $\beta \leq 0$ and $\gamma \leq 0$ and $\beta' \leq 0$ and $\gamma' \leq 0$, respectively. Suppose that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(\gamma_n S x_n + (1 - \gamma_n) T x_n\big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\gamma_n\}$ and $\{\alpha_n\}$ satisfy the following:

$$0 < a \leq \alpha_n \leq b < 1 \text{ and } 0 < c \leq \gamma_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in F(S) \cap F(T)$, where $F(S) \cap F(T)$ is the set of common fixed points of S and T.

In this paper, we prove a weak convergence theorem of Mann's type iteration for two infinite families of extended generalized hybrid mappings in a Banach space satisfying Opial's condition. This theorem generalizes a theorem by Hojo and Takahashi [8]. Using this result, we get well-known and new weak convergence theorems in a Banach space. In particular, we obtain a weak convergence theorem of Mann's type iteration for two finite families of extended generalized hybrid mappings in a Banach space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the topological dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \epsilon\right\}$$

for all ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for all $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. Let C be a nonempty subset of a Banach space E. A mapping $T : C \to E$ is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to E$ is quasi-nonexpansive if $F(T) \neq \emptyset$ and $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$, where $F(T)$ is the set of fixed points of T. If C is a nonempty, closed and convex subset of a strictly convex Banach space E and $T : C \to E$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [10]. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. The following result is in [17].

Lemma 2.1 ([17])**.** *Let E be a Banach space and let J be the duality mapping on E. Then, for any $x, y \in E$,*

$$\|x\|^2 - \|y\|^2 \geq 2\langle x - y, j \rangle,$$

where $j \in Jy$.

Let E be a Banach space and let $A \subset E \times E$. Then, A is accretive if for $(x_1, y_1), (x_2, y_2) \in A$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j \rangle \geq 0$, where J is the duality mapping of E. An accretive operator $A \subset E \times E$ is called m-accretive if $R(I + rA) = E$ for all $r > 0$, where I is the identity operator and $R(I + rA)$ is the range of $I + rA$. An accretive operator $A \subset E \times E$ is said to satisfy the range condition if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$, where $\overline{D(A)}$ is the closure of the domain $D(A)$ of A. An m-accretive operator satisfies the range condition. If C is a nonempty, closed and convex subset of a Banach space and T is a nonexpansive mapping of C into itself, then $A = I - T$ is an accretive operator and $C = D(A) \subset R(I + rA)$ for all $r > 0$; see [17, Theorem 4.6.4].

Let E be a Banach space and let C be a nonempty subset of E. Then, a mapping $T : C \to E$ is said to be firmly nonexpansive [3] if

$$\|Tx - Ty\|^2 \leq \langle x - y, j \rangle,$$

for all $x, y \in C$, where $j \in J(Tx - Ty)$; see also [2, 5]. It is known that the resolvent of an accretive operator satisfying the range condition in a Banach space is a firmly nonexpansive mapping of the closure of the domain into itself. In fact, let $C = \overline{D(A)}$ and $r > 0$. Define the resolvent J_r of A as follows:

$$J_r x = \{z \in D(A) : x \in z + rAz\}$$

for all $x \in \overline{D(A)}$. It is known that such $J_r x$ is a singleton; see [17]. We have that for $x_1, x_2 \in \overline{D(A)}$, $x_1 = z_1 + ry_1,\ y_1 \in Az_1$ and $x_2 = z_2 + ry_2,\ y_2 \in Az_2$. Since A is accretive, we have that $\langle y_1 - y_2, j \rangle \geq 0$, where $j \in J(z_1 - z_2)$. So, we have

$$\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j \rangle \geq 0.$$

Furthermore, we have that

$$\begin{aligned} &\langle \frac{x_1 - z_1}{r} - \frac{x_2 - z_2}{r}, j \rangle \geq 0 \\ &\qquad \Longleftrightarrow \langle x_1 - z_1 - (x_2 - z_2), j \rangle \geq 0 \\ &\qquad \Longleftrightarrow \langle x_1 - x_2, j \rangle \geq \|z_1 - z_2\|^2. \end{aligned}$$

From $z_1 = J_r x_1$ and $z_2 = J_r x_2$, we have that J_r is a firmly nonexpansive mapping of C into itself; see also [3], [4] and [20]. Let E be a Banach space and let C be a nonempty subset of E. A mapping $T : C \to E$ is called extended generalized hybrid if it satisfies (1.2), that is, there are $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta > 0$ and

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \leq 0$$

for all $x, y \in C$. We call such a mapping $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid. We can also show that, in a Banach space, an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping is nonexpansive for $\alpha = 1$, $\beta = \gamma = 0$ and $\delta = -1$, nonspreading for $\alpha = 2$, $\beta = \gamma = -1$ and $\delta = 0$, and hybrid for $\alpha = 3$, $\beta = \gamma = -1$ and $\delta = -1$. Nonexpansive mappings, nonspreading mappings and hybrid mappings in a Banach space are deduced from firmly nonexpansive mappings as follows: Let T be a firmly nonexpansive mapping of C into E. Then we have that for $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\|Tx - Ty\|^2 \leq \langle x - y, j\rangle.$$

From Theorem 2.1 we have that

$$\begin{aligned} \|Tx-Ty\|^2 &\leq \langle x - y, j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x - Tx - (y - Ty), j\rangle \\ &\Longrightarrow 0 \leq \|x - y\|^2 - \|Tx - Ty\|^2 \\ &\Longleftrightarrow \|Tx - Ty\|^2 \leq \|x - y\|^2. \end{aligned} \tag{2.1}$$

Futhermore, we have that for $x, y \in C$ and $j \in J(Tx - Ty)$,

$$\begin{aligned} \|Tx-Ty\|^2 &\leq \langle x - y, j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x - Tx - (y - Ty), j\rangle \\ &\Longleftrightarrow 0 \leq 2\langle x - Tx, j\rangle + 2\langle Ty - y, j\rangle \\ &\Longrightarrow 0 \leq \|x - Ty\|^2 - \|Tx - Ty\|^2 + \|Tx - y\|^2 - \|Tx - Ty\|^2 \\ &\Longleftrightarrow 0 \leq \|x - Ty\|^2 + \|y - Tx\|^2 - 2\|Tx - Ty\|^2 \\ &\Longleftrightarrow 2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2. \end{aligned} \tag{2.2}$$

Therefore, using (2.1) and (2.2), we have that

$$\begin{aligned} \|Tx-Ty\|^2 &\leq \langle x - y, j\rangle \\ &\Longrightarrow 3\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2 + \|x - y\|^2. \end{aligned}$$

Hojo and Takahashi [6] proved the following result.

Lemma 2.2 ([6]). *Let E be a Banach space, let C be a nonempty, closed and convex subset of E. Then an extended generalized hybrid mapping which has a fixed point is quasi-nonexpansive.*

The following result was proved by Xu [21].

Lemma 2.3 ([21]). *Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\mu x + (1-\mu)y\|^2 \leq \mu\|x\|^2 + (1-\mu)\|y\|^2 - \mu(1-\mu)g(\|x-y\|)$$

for all $x, y \in B_r$ and μ with $0 \leq \mu \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a Banach space. Then, E satisfies Opial's condition [15] if for any $\{x_n\}$ of E such that $x_n \rightharpoonup x$ and $x \neq y$,

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

Let E be a Banach space. Let C be a nonempty, closed and convex subset of E. Let $T : C \to E$ be a mapping. Then, $p \in C$ is called an *asymptotic fixed point* of T [16] if there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T. A mapping $T : C \to E$ is said to be *demiclosed* if $\hat{F}(T) = F(T)$. We know the following result from Hojo and Takahashi [6].

Lemma 2.4 ([6]). *Let E be a Banach space satisfying Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and let T be an $(\alpha, \beta, \gamma, \delta)$-extended generalized hybrid mapping of C into E which satisfies $\beta \leq 0$ and $\gamma \leq 0$. Then $\hat{F}(T) = F(T)$, i.e., T is demiclosed.*

If E is a Banach space satisfying Opial's condition, then nonexpansive mappings, nonspreading mappngs and hybrid mappings are quasi-nonexpansive and demiclosed.

3. Weak Convergence Theorems

In this section, we first prove a weak convergence theorem of Mann's type iteration [14] for an infinite family of extended generalized hybrid mappings in a Banach space satisfying Opial's condition.

Lemma 3.1. *Let E be a uniformly convex Banach and let C be a nonempty, closed and convex subset of E. Let $\{T_j\}$ be a family of quasi-nonexpansive and demiclosed mappings of C into E such that $\cap_{j=1}^{\infty} F(T_j) \neq \emptyset$ and let $\{\xi_j\} \subset (0, 1)$ be a family of real numbers such that $\sum_{j=1}^{\infty} \xi_j = 1$. Define*

$$Tx = \sum_{j=1}^{\infty} \xi_j T_j x, \quad \forall x \in C.$$

Then T is well-defined and quasi-nonexpansive. Furthermore, $F(T) = \cap_{j=1}^{\infty} F(T_j)$ and T is demiclosed.

Proof. Let $y_0 \in \cap_{j=1}^{\infty} F(T_j)$. Since T_j is quasi-nonexpansive, we have that

$$\|T_j x\| \leq \|T_j x - y_0\| + \|y_0\| \leq \|x - y_0\| + \|y_0\|$$

for all $x \in C$. For a family $\{\xi_j\} \subset (0,1)$ of real numbers such that $\sum_{j=1}^{\infty} \xi_j = 1$, define

$$Tx = \sum_{j=1}^{\infty} \xi_j T_j x$$

for all $x \in C$; see Bruck [3]. Then, $Tx = \sum_{j=1}^{\infty} \xi_j T_j x$ converges absolutely for all $x \in C$ and then T is well-defined. Since T_j is quasi-nonexpansive, it is obvious that T is quasi-nonexpansive. We show that T is demiclosed. Let $\{x_n\}$ be a sequence of C such that $x_n - Tx_n \to 0$ and $x_n \rightharpoonup v$. Let $w \in \cap_{j=1}^{\infty} F(T_j)$. For any $\varepsilon > 0$ and $i, k \in \mathbb{N}$ with $i \neq k$, we take $m \in \mathbb{N}$ such that $i, k \leq m$ and $\left\| \sum_{j=m+1}^{\infty} \xi_j (T_j x_n - w) \right\| \leq \varepsilon$. We have from Lemma 2.1 that, for $j(x_n - w) \in J(x_n - w)$,

$$\begin{aligned}
\|x_n - w\|^2 &= \|x_n - Tx_n + Tx_n - w\|^2 \\
&\leq \|Tx_n - w\|^2 + 2\langle x_n - Tx_n, j(x_n - w)\rangle \\
&= \Big\| \sum_{j=1}^{\infty} \xi_j T_j x_n - w \Big\|^2 + 2\langle x_n - Tx_n, j(x_n - w)\rangle \\
&= \Big\| \sum_{j=1}^{\infty} \xi_j (T_j x_n - w) \Big\|^2 + 2\langle x_n - Tx_n, j(x_n - w)\rangle \\
&\leq \Big(\Big\| \sum_{j=1}^{m} \xi_j (T_j x_n - w) \Big\| + \varepsilon \Big)^2 + 2\langle x_n - Tx_n, j(x_n - w)\rangle \\
&\leq \Big\| \sum_{j=1}^{m} \xi_j (T_j x_n - w) \Big\|^2 + 2r\varepsilon + \varepsilon^2 + 2\langle x_n - Tx_n, j(x_n - w)\rangle,
\end{aligned}$$

where $r = \sup\{\|T_j x_n - w\| : j \in \mathbb{N}\}$. From Theorem 2.3, there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\|\mu x + (1-\mu)y\|^2 \leq \mu\|x\|^2 + (1-\mu)\|y\|^2 - \mu(1-\mu)g(\|x - y\|)$$

for all $x, y \in B_r$ and μ with $0 \leq \mu \leq 1$, where $B_r = \{z \in E : \|z\| \leq r\}$. Set $\sigma = \frac{\xi_i}{\xi_i + \xi_k}$. Then we have that

$$\begin{aligned}
&\Big\| \sum_{j=1}^{m} \xi_j (T_j x_n - w) \Big\|^2 \\
&\quad = \Big\| (\xi_i + \xi_k)\big(\sigma(T_i x_n - w) + (1-\sigma)(T_k x_n - w)\big) + \sum_{j \neq i,k}^{m} \xi_j (T_j x_n - w) \Big\|^2 \\
&\quad \leq (\xi_i + \xi_k)\|\sigma(T_i x_n - w) + (1-\sigma)(T_k x_n - w)\|^2 + \sum_{j \neq i,k}^{m} \xi_j \|T_j x_n - w\|^2 \\
&\quad \leq (\xi_i + \xi_k)\Big(\sigma\|T_i x_n - w\|^2 + (1-\sigma)\|T_k x_n - w\|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&- (\xi_i + \xi_k)\sigma(1-\sigma)g(\|T_i x_n - T_k x_n\|) + \sum_{j\neq i,k}^{m} \xi_j \|T_j x_n - w\|^2 \\
&\leq (\xi_i + \xi_k)\big(\sigma\|x_n - w\|^2 + (1-\sigma)\|x_n - w\|^2\big) \\
&\quad - (\xi_i + \xi_k)\sigma(1-\sigma)g(\|T_i x_n - T_k x_n\|) + \sum_{j\neq i,k}^{m} \xi_j \|x_n - w\|^2 \\
&= \sum_{j=1}^{m} \xi_j \|x_n - w\|^2 - (\xi_i + \xi_k)\sigma(1-\sigma)g(\|T_i x_n - T_k x_n\|) \\
&\leq \|x_n - w\|^2 - (\xi_i + \xi_k)\sigma(1-\sigma)g(\|T_i x_n - T_k x_n\|).
\end{aligned}
$$

Then, we have that

$$
\begin{aligned}
\|x_n - w\|^2 &\leq \Big\|\sum_{j=1}^{m} \xi_j (T_j x_n - w)\Big\|^2 + 2r\varepsilon + \varepsilon^2 + 2\langle x_n - Tx_n, j(x_n - w)\rangle \\
&\leq \|x_n - w\|^2 - (\xi_i + \xi_k)\sigma(1-\sigma)g(\|T_i x_n - T_k x_n\|) \\
&\qquad + 2\langle x_n - Tx_n, j(x_n - w)\rangle + 2r\varepsilon + \varepsilon^2
\end{aligned}
$$

and hence

$$
(\xi_i + \xi_k)\sigma(1-\sigma)g(\|T_i x_n - T_k x_n\|) \leq 2\langle x_n - Tx_n, j(x_n - w)\rangle + 2r\varepsilon + \varepsilon^2.
$$

Since $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, we have that

$$
(\xi_i + \xi_k)\sigma(1-\sigma)\limsup_{n\to\infty} g(\|T_i x_n - T_k x_n\|) \leq 2r\varepsilon + \varepsilon^2.
$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$
(\xi_i + \xi_k)\sigma(1-\sigma)\limsup_{n\to\infty} g(\|T_i x_n - T_k x_n\|) \leq 0.
$$

Since $(\xi_i + \xi_k)\sigma(1-\sigma) > 0$, we have that $\limsup_{n\to\infty} g(\|T_i x_n - T_k x_n\|) \leq 0$ and then

$$
\lim_{n\to\infty} g(\|T_i x_n - T_k x_n\|) = 0.
$$

From the properties of g, we have that

$$
\lim_{n\to\infty} \|T_i x_n - T_k x_n\| = 0.
$$

For any $\varepsilon > 0$ and $i \in \mathbb{N}$, we take $m \in \mathbb{N}$ such that

$$
\Big\|\sum_{j=m+1}^{\infty} \xi_j (T_j x_n - T_i x_n)\Big\| \leq \varepsilon.
$$

Then we have that

$$
\begin{aligned}
\|x_n - T_i x_n\| &= \|x_n - Tx_n + Tx_n - T_i x_n\| \\
&\leq \|x_n - Tx_n\| + \|Tx_n - T_i x_n\| \\
&= \|x_n - Tx_n\| + \Big\|\sum_{j=1}^{\infty} \xi_j (T_j x_n - T_i x_n)\Big\|
\end{aligned}
$$

$$\leq \|x_n - Tx_n\| + \Big\|\sum_{j=1}^{m} \xi_j(T_jx_n - T_ix_n)\Big\| + \varepsilon.$$

From $\|x_n - Tx_n\| \to 0$ and $T_jx_n - T_ix_n \to 0$ for all $j \in \mathbb{N}$, we have that

$$\limsup_{n\to\infty} \|x_n - T_ix_n\| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have that

$$\lim_{n\to\infty} \|x_n - T_ix_n\| = 0, \quad \forall i \in \mathbb{N}. \tag{3.1}$$

Since $x_n \rightharpoonup v$ and T_i is demiclosed for all $i \in \mathbb{N}$, we have that $v \in \cap_{j=1}^{\infty} F(T_j)$. Finally, to prove that T is demiclosed, we show that $F(T) = \cap_{j=1}^{\infty} F(T_j)$. In fact, it is obvious that $\cap_{j=1}^{\infty} F(T_j) \subset F(T)$. We show that $F(T) \subset \cap_{j=1}^{\infty} F(T_j)$. For $y_0 \in \cap_{j=1}^{\infty} F(T_j)$ and $w \in F(T)$, we have that

$$w - y_0 = Tw - y_0 = \sum_{j=1}^{\infty} \xi_j T_jw - y_0 = \sum_{j=1}^{\infty} \xi_j(T_jw - y_0)$$

and hence

$$\|w - y_0\| \leq \sum_{j=1}^{\infty} \xi_j \|T_jw - y_0\| \leq \sum_{j=1}^{\infty} \xi_j \|w - y_0\| = \|w - y_0\|.$$

Then, $\|T_jw - y_0\| = \|w - y_0\|$ for all $j \in \mathbb{N}$. Assume that $T_iw - y_0 \neq T_kw - y_0$ for some $i, k \in \mathbb{N}$. Since E is a strictly convex Banach space, there exists $\delta > 0$ such that

$$\Big\|\tau(T_iw - y_0) + (1-\tau)(T_kw - y_0)\Big\| = \|w - y_0\| - \delta,$$

where $\tau = \frac{\xi_i}{\xi_i + \xi_k}$. Then, we have that, for any $m \in \mathbb{N}$ with $i, k \leq m$,

$$\begin{aligned}
&\Big\|\sum_{j=1}^{m} \xi_j(T_jw - y_0)\Big\| \\
&\quad = \Big\|(\xi_i + \xi_k)(\tau(T_iw - y_0) + (1-\tau)(T_kw - y_0)) + \sum_{j\neq i,k}^{m} \xi_j(T_jw - y_0)\Big\| \\
&\quad \leq (\xi_i + \xi_k)\|\tau(T_iw - y_0) + (1-\tau)(T_kw - y_0)\| + \sum_{j\neq i,k}^{m} \xi_j \|T_jw - y_0\| \\
&\quad \leq (\xi_i + \xi_k)(\|w - y_0\| - \delta) + \sum_{j\neq i,k}^{m} \xi_j \|w - y_0\| \\
&\quad = \sum_{j=1}^{m} \xi_j \|w - y_0\| - (\xi_i + \xi_k)\delta \\
&\quad \leq \|w - y_0\| - (\xi_i + \xi_k)\delta.
\end{aligned}$$

Since $w - y_0 = \sum_{j=1}^{\infty} \xi_j(T_jw - y_0)$, we have that

$$\|w - y_0\| \leq \|w - y_0\| - (\xi_1 + \xi_i)\delta.$$

This is a contradiction. Therefore, we have that $T_i w - y_0 = T_k w - y_0$ for all $i, k \in \mathbb{N}$. From $w - y_0 = \sum_{j=1}^{\infty} \xi_j (T_j w - y_0)$, we have that $w - y_0 = T_i w - y_0$ for all $i \in \mathbb{N}$ and hence $w = T_i w$ for all $i \in \mathbb{N}$. This implies that $F(T) \subset \cap_{j=1}^{\infty} F(T_j)$. Therefore, $F(T) = \cap_{j=1}^{\infty} F(T_j)$ and then T is demiclosed. □

Lemma 3.2. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let S and T be quasi-nonexpansive and demiclosed mappings of C into itself such that*

$$F(S) \cap F(T) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n)\big(\mu_n S x_n + (1 - \mu_n) T x_n\big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the following:

$$0 < a \le \lambda_n \le b < 1 \quad \text{and} \quad 0 < c \le \mu_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in F(S) \cap F(T)$.

Proof. Since S and T are quasi-nonexpansive, we have that $F(S) \cap F(T)$ is closed and convex. Put

$$T_n = \mu_n S + (1 - \mu_n) T$$

for all $n \in \mathbb{N}$ and let w be a point of $F(S) \cap F(T)$. We have that

$$\begin{aligned} \|T_n x_n - w\| &= \|(\mu_n S + (1 - \mu_n) T) x_n - w\| \\ &\le \mu_n \|S x_n - w\| + (1 - \mu_n) \|T x_n - w\| \\ &\le \mu_n \|x_n - w\| + (1 - \mu_n) \|x_n - w\| \\ &= \|x_n - w\|. \end{aligned} \tag{3.2}$$

Using (3.2), we have that

$$\begin{aligned} \|x_{n+1} - w\| &= \|\lambda_n x_n + (1 - \lambda_n) T_n x_n - w\| \\ &\le \lambda_n \|x_n - w\| + (1 - \lambda_n) \|T_n x_n - w\| \\ &\le \lambda_n \|x_n - w\| + (1 - \lambda_n) \|x_n - w\| \\ &= \|x_n - w\|. \end{aligned} \tag{3.3}$$

Then $\lim_{n\to\infty} \|x_n - w\|$ exists. Thus we have that the sequence $\{x_n\}$ is bounded. This implies that $\{T_n x_n\}$ is bounded. Let

$$r = \max\{\sup_{n\in\mathbb{N}} \|x_n - w\|, \sup_{n\in\mathbb{N}} \|T_n x_n - w\|\}.$$

Then, from Theorem 2.3, there exists a strictly increasing, continuous and convex function $g : [0, \infty) \to [0, \infty)$ such that $g(0) = 0$ and

$$\|\mu x + (1 - \mu) y\|^2 \le \mu \|x\|^2 + (1 - \mu) \|y\|^2 - \mu(1 - \mu) g(\|x - y\|)$$

for all $x, y \in B_r$ and μ with $0 \le \mu \le 1$, where $B_r = \{z \in E : \|z\| \le r\}$. Then we have that for $w \in F(S) \cap F(T)$ and $n \in \mathbb{N}$,

$$\|x_{n+1} - w\|^2 = \|\lambda_n x_n + (1 - \lambda_n) T_n x_n - w\|^2$$

$$\begin{aligned}
&= \|\lambda_n(x_n - w) + (1-\lambda_n)(T_n x_n - w)\|^2 \\
&\le \lambda_n \|x_n - w\|^2 + (1-\lambda_n)\|T_n x_n - w\|^2 - \lambda_n(1-\lambda_n)g(\|x_n - T_n x_n\|) \\
&\le \lambda_n \|x_n - w\|^2 + (1-\lambda_n)\|x_n - w\|^2 - \lambda_n(1-\lambda_n)g(\|x_n - T_n x_n\|) \\
&= \|x_n - w\|^2 - \lambda_n(1-\lambda_n)g(\|x_n - T_n x_n\|)
\end{aligned}$$

and hence

$$\lambda_n(1-\lambda_n)g(\|x_n - T_n x_n\|) \le \|x_n - w\|^2 - \|x_{n+1} - w\|^2.$$

Since $\lim_{n\to\infty} \|x_n - w\|^2$ exists, we have from $0 < a \le \lambda_n \le b < 1$ that

$$\lim_{n\to\infty} g(\|x_n - T_n x_n\|) = 0.$$

From the properties of g, we have

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{3.4}$$

We have from Lemma 2.1 that, for $w \in F(S) \cap F(T)$,

$$\begin{aligned}
\|x_n - w\|^2 &= \|x_n - T_n x_n + T_n x_n - w\|^2 \\
&\le \|T_n x_n - w\|^2 + 2\langle x_n - T_n x_n, j(x_n - w)\rangle \\
&= \|\mu_n S x_n + (1-\mu_n) T x_n - w\|^2 + 2\langle x_n - T_n x_n, J(x_n - w)\rangle \\
&\le \mu_n \|S x_n - w\|^2 + (1-\mu_n)\|T x_n - w\|^2 \\
&\quad - \mu_n(1-\mu_n)g(\|S x_n - T x_n\|) + 2\langle x_n - T_n x_n, J(x_n - w)\rangle \\
&\le \mu_n \|x_n - w\|^2 + (1-\mu_n)\|x_n - w\|^2 \\
&\quad - \mu_n(1-\mu_n)g(\|S x_n - T x_n\|) + 2\langle x_n - T_n x_n, j(x_n - w)\rangle \\
&= \|x_n - w\|^2 - \mu_n(1-\mu_n)g(\|S x_n - T x_n\|) + 2\langle x_n - T_n x_n, j(x_n - w)\rangle
\end{aligned}$$

and hence

$$\mu_n(1-\mu_n)g(\|S x_n - T x_n\|) \le 2\langle x_n - T_n x_n, j(x_n - w)\rangle.$$

Since $x_n - T_n x_n \to 0$ and $\{x_n\}$ is bounded, we have from $0 < c \le \mu_n \le d < 1$ that $S x_n - T x_n \to 0$. Then we have that

$$\begin{aligned}
\|x_n - S x_n\| &= \|x_n - T_n x_n + T_n x_n - S x_n\| \\
&\le \|x_n - T_n x_n\| + \|T_n x_n - S x_n\| \\
&= \|x_n - T_n x_n\| + (1-\mu_n)\|T x_n - S x_n\| \\
&\to 0.
\end{aligned}$$

Similarly, we have that $\|x_n - T x_n\| \to 0$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in C$. Since S and T are demiclosed, we have that v is a point of $F(S) \cap F(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$. We have that $u, v \in F(S) \cap F(T)$. Suppose $u \neq v$. From

$u, v \in F(S) \cap F(T)$, we know that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. Since E satisfies Opial's condition, we have that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{i\to\infty} \|x_{n_i} - u\| \\ &< \lim_{i\to\infty} \|x_{n_i} - v\| = \lim_{n\to\infty} \|x_n - v\| \\ &= \lim_{j\to\infty} \|x_{n_j} - v\| < \lim_{j\to\infty} \|x_{n_j} - u\| \\ &= \lim_{n\to\infty} \|x_n - u\|.\end{aligned}$$

This is a contradiction. Thus we must have $u = v$. This implies that $\{x_n\}$ converges weakly to a point of $F(S) \cap F(T)$. This completes the proof. □

Using Lemmas 3.1 an 3.2, we can prove the following weak convergence theorem for two infinite families of extended generalized hybrid mappings in Banach spaces.

Theorem 3.3. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $a_i, b_i, c_i, d_i \in \mathbb{R}$ for all $i \in \mathbb{N}$ and let $\{S_i\}$ be a sequence of (a_i, b_i, c_i, d_i)-extended generalized hybrid mappings of C into itself such that $b_i \le 0$ and $c_i \le 0$ for all $i \in \mathbb{N}$. Let $\{\xi_i\}$ be a family of real numbers in $(0,1)$ such that $\sum_{i=1}^{\infty} \xi_i = 1$. Let $\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{R}$ for all $j \in \mathbb{N}$ and let $\{T_j\}$ be a sequence of $(\alpha_j, \beta_j, \gamma_j, \delta_j)$-extended generalized hybrid mappings of C into itself such that $\beta_j \le 0$ and $\gamma_j \le 0$ for all $j \in \mathbb{N}$. Let $\{\sigma_j\}$ be a family of real numbers in $(0,1)$ such that $\sum_{j=1}^{\infty} \sigma_j = 1$. Suppose that*

$$\Omega := \cap_{i=1}^{\infty} F(S_i) \cap (\cap_{j=1}^{\infty} F(T_j)) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)\Big(\mu_n \sum_{i=1}^{\infty} \xi_i S_i + (1-\mu_n) \sum_{j=1}^{\infty} \sigma_j T_j\Big) x_n, \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$ and $\{\lambda_n\}, \{\mu_n\} \subset (0,1)$ satisfy the following:

$$0 < a \le \lambda_n \le b < 1 \quad \text{and} \quad 0 < c \le \mu_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in \Omega$.

Proof. Since S_i is quasi-nonexpansive from Lemma 2.2, we have that $F(S_i)$ is closed and convex. Therefore, $\cap_{i=1}^{\infty} F(S_i)$ is closed and convex. For a family $\{\xi_i\} \subset (0,1)$ of real numbers such that $\sum_{i=1}^{\infty} \xi_i = 1$, define

$$Sx = \sum_{i=1}^{\infty} \xi_i S_i x$$

for all $x \in C$. Then we havs from Lemma 3.1 that S is well defined and quasi-nonexpansive. Furthermore, we have from Lemma 3.1 that $F(S) = \cap_{i=1}^{\infty} F(S_i)$ and S is demiclosed. Similarly, for a family $\{\sigma_j\} \subset (0,1)$ of real numbers such that $\sum_{j=1}^{\infty} \sigma_j = 1$, define

$$Tx = \sum_{j=1}^{\infty} \sigma_j T_j x$$

for all $x \in C$. We havs from Lemma 3.1 that T is well defined and quasi-nonexpansive. Furthermore, we have that $F(T) = \cap_{j=1}^{\infty} F(T_j)$ and T is demiclosed. Thus, for $x_1 = x \in C$, the sequence $\{x_n\}$ in Theorem 3.3 is as follows:

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)\Big(\mu_n S + (1-\mu_n)T\Big)x_n, \quad \forall n \in \mathbb{N}.$$

Using Lemmas 3.1 an 3.2, we have the desired result. □

Using Theorem 3.3, we obtain the following weak convergence theorem for two finite families of extended generalized hybrid mappings in a Banach space satisfying Opial's condition.

Theorem 3.4. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $a_i, b_i, c_i, d_i \in \mathbb{R}$ for all $i \in \{1, 2, \dots, M\}$ and let $\{S_i\}_{i=1}^M$ be a sequence of (a_i, b_i, c_i, d_i)-extended generalized hybrid mappings of C into itself such that $b_i \le 0$ and $c_i \le 0$ for all $i \in \{1, 2, \dots, M\}$. Let $\{\xi_i\}$ be a family of real numbers in $(0,1)$ such that $\sum_{i=1}^M \xi_i = 1$. Let $\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{R}$ for all $j \in \{1, 2, \dots, N\}$ and let $\{T_j\}_{j=1}^N$ be a sequence of $(\alpha_j, \beta_j, \gamma_j, \delta_j)$-extended generalized hybrid mappings of C into itself such that $\beta_j \le 0$ and $\gamma_j \le 0$ for all $j \in \mathbb{N}$. Let $\{\sigma_j\}$ be a family of real numbers in $(0,1)$ such that $\sum_{j=1}^N \sigma_j = 1$. Suppose that*

$$\Omega := \cap_{i=1}^M F(S_i) \cap (\cap_{j=1}^N F(T_j)) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)\Big(\mu_n \sum_{i=1}^M \xi_i S_i + (1-\mu_n)\sum_{j=1}^N \sigma_j T_j\Big)x_n, \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$ and $\{\lambda_n\}, \{\mu_n\} \subset (0,1)$ satisfy the following:

$$0 < a \le \lambda_n \le b < 1 \quad \textit{and} \quad 0 < c \le \mu_n \le d < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in \Omega$.

Proof. From $\xi_M S_M = \frac{\xi_M}{2} S_M + \frac{\xi_M}{2^2} S_M + \cdots$, we have that

$$\sum_{j=1}^M \xi_j T_j = \sum_{j=1}^{M-1} \xi_j S_j + \frac{\xi_M}{2} S_M + \frac{\xi_M}{2^2} S_M + \cdots.$$

From $\sigma_N T_N = \frac{\sigma_N}{2} T_N + \frac{\sigma_N}{2^2} T_N + \cdots$, we have that

$$\sum_{j=1}^N \sigma_j T_j = \sum_{j=1}^{N-1} \sigma_j T_j + \frac{\sigma_N}{2} T_N + \frac{\sigma_N}{2^2} T_N + \cdots.$$

Thus, we have the desired result from Theorem 3.3. □

Using Theorem 3.3, we also obtain the following weak convergence theorem by Hojo and Takahashi [8].

Theorem 3.5 ([8]). *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\alpha_j, \beta_j, \gamma_j, \delta_j \in \mathbb{R}$ for all $j \in \mathbb{N}\}$ and let $\{T_j\}$ be an infinite family of $(\alpha_j, \beta_j, \gamma_j, \delta_j)$-extended generalized hybrid mappings of C into itself such that $\beta_j \leq 0$ and $\gamma_j \leq 0$ for all $j \in \mathbb{N}\}$. Let $\{\sigma_j\}$ be a family of real numbers in $(0,1)$ such that $\sum_{j=1}^{\infty} \sigma_j = 1$. Suppose that $\cap_{j=1}^{\infty} F(T_j) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and*

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n) \sum_{j=1}^{\infty} \sigma_j T_j x_n, \quad \forall n \in \mathbb{N},$$

where $a, b \in \mathbb{R}$ and $\{\lambda_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n \leq b < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in \cap_{j=1}^{\infty} F(T_j)$.

Proof. Putting $S_j = T_j$, $\xi_j = \sigma_j$ and $\mu_n = \frac{1}{2}$ for all $n \in \mathbb{N}$ in Theorem 3.3, we have that for any $x_1 = x \in C$,

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n) \sum_{j=1}^{\infty} \sigma_j T_j x_n, \quad \forall n \in \mathbb{N}.$$

Thus, we have the desired result from Theorem 3.3. □

Using Theorems 3.3 and 3.4, we can also prove the following weak convergence theorems in a Banach space.

Theorem 3.6. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\{S_i\}$ be a sequence of nonexpansive mappings of C into itself. Let $\{\xi_i\}$ be a family of real numbers in $(0,1)$ such that $\sum_{i=1}^{\infty} \xi_i = 1$. Let $\{T_j\}$ be a sequence of nonspreading mappings of C into itself. Let $\{\sigma_j\}$ be a family of real numbers in $(0,1)$ such that $\sum_{j=1}^{\infty} \sigma_j = 1$. Suppose that*

$$\Omega := \cap_{i=1}^{\infty} F(S_i) \cap (\cap_{j=1}^{\infty} F(T_j)) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence in C generated by $x_1 = x \in C$ and

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)\Big(\mu_n \sum_{i=1}^{\infty} \xi_i S_i + (1-\mu_n) \sum_{j=1}^{\infty} \sigma_j T_j\Big) x_n, \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$ and $\{\lambda_n\}, \{\mu_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \mu_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in \Omega$.

Proof. Nonexpansive mappings and nonspreading mappings are contained in the class of extended generalized hybrid mappings satisfying the conditions of Theorem 3.3. In particular, nonexpansive mappings and nonspreading mappings in a Banach space satisfying Opial's condition are quasi-nonexpansive and demiclosed. Then, we obtain the desired result from Theorem 3.3. □

Theorem 3.7. *Let E be a uniformly convex Banach space which satisfies Opial's condition and let C be a nonempty, closed and convex subset of E. Let $\{S_i\}_{i=1}^M$ be a sequence of nonexpansive mappings of C into itself. Let $\{\xi_i\}$ be a family of real numbers in $(0,1)$ such that $\sum_{i=1}^M \xi_i = 1$. Let $\{T_j\}_{j=1}^N$ be a sequence of hybrid mappings of C into itself. Let $\{\sigma_j\}$ be a family of real numbers in $(0,1)$ such that $\sum_{j=1}^N \sigma_j = 1$. Suppose that*

$$\Omega := \cap_{i=1}^M F(S_i) \cap (\cap_{j=1}^N F(T_j)) \neq \emptyset.$$

Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \lambda_n x_n + (1-\lambda_n)\Big(\mu_n \sum_{i=1}^M \xi_i S_i + (1-\mu_n)\sum_{j=1}^N \sigma_j T_j\Big)x_n, \quad \forall n \in \mathbb{N},$$

where $a,b,c,d \in \mathbb{R}$ and $\{\lambda_n\},\{\mu_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \lambda_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \mu_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then, the sequence $\{x_n\}$ converges weakly to an element $z \in \Omega$.

Proof. Nonexpansive mappings and hybrid mappings are contained in the class of extended generalized hybrid mappings satisfying the conditions of Theorem 3.4. In particular, nonexpansive mappings and hybrid mappings in a Banach space satisfying Opial's condition are quasi-nonexpansive and demiclosed. Then, we obtain the desired result from Theorem 3.4. □

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References

[1] K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, *Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 335–343.

[2] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201–225.

[3] R. E. Bruck, *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math. **47** (1973), 341–355.

[4] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), 459–470.

[5] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.

[6] M. Hojo and W. Takahashi, *Fixed point and weak convergence theorems for nonlinear hybrid mappings in Banach spaces*, Linear Nonlinear Anal. **3** (2017), 61–72.

[7] M. Hojo and W. Takahashi, *Fixed point and weak convergence theorems for noncommutative two extended generalized hybrid mappings in Banach spaces*, J. Nonlinear Convex Anal. **21** (2020), 184–194.

[8] M. Hojo and W. Takahashi, *Weak convergence theorem for infinite families of extended generalized hybrid mappings in Banach spaces*, to appear.

[9] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2008, pp. 75–85.

[10] S. Itoh and W. Takahashi, *The common fixed point theory of single-valued mappings and multi-valued mappings*, Pacific J. Math. **79** (1978), 493–508.

[11] P. Kocourek, W. Takahashi and J. -C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[12] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.

[13] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166–177.

[14] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[15] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591–597.

[16] S. Reich, *A weak convergence theorem for the alternating method with Bregman distances*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 313–318.

[17] W. Takahashi, *Nonlinear Functional Analysis,* Yokohoma Publishers, Yokohoma, 2000.

[18] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.

[19] W. Takahashi, *Weak and strong convergence theorems for noncommutative two generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **19** (2018), 867–880.

[20] W. Takahashi and J. -C. Yao, *Nonlinear operators of monotone type and convergence theorems with equilibrium problems in Banach spaces*, Taiwanese J. Math. **15** (2011), 787–818.

[21] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1981), 1127–1138.

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