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LOCAL BEHAVIOR OF NODE WITH EXTREME VALUE OF MODEL FUNCTION IN LEARNING PROCESSES OF BASIC SELF-ORGANIZING MAPS

MITSUHIRO HOSHINO

Abstract. In this paper, we deal with an essential mathematical model and its learning process for self-organizing maps referred to as Kohonen type algorithm. Self-organizing map algorithm is used to solve many problems in a wide range of fields. On self-organizing maps, by repeating learning, a model function defined as a mapping from a node set to a node-value set has some mathematically interesting properties such as regularity between the nodes and their values. In this paper, we mainly describe the local behavior on monotonization of model functions in the learning process of a basic self-organizing map with a one-dimensional indexed array. We give some sufficient and necessary conditions for the node with the extreme value of the renewed model function to shift by its learning.

1. A formulation of self-organizing map model

This paper deals with an essential mathematical model for self-organizing maps referred to as the Kohonen [6] type algorithm or the Kohonen Map, which has many applications, such as data analysis, pattern recognition, traveling-salesman problems [1], and so on. On self-organizing maps, by repeating learning, a model function defined as a mapping from a node set to a node-value set has several properties of mathematical interest such as regularity between the nodes and their values [2, 3, 4, 5].

One of the purposes of our research is to make a study of local behavior of model function in learning process. In this paper, we deal with a basic self-organizing map with a one-dimensionally indexed array and real valued nodes, and transition of model function which is a mapping from nodes to their values. We concentrate on a consideration of local behavior of monotonization of model function and give some sufficient and necessary conditions for the position of the node which has the extreme value of the renewed model function to shift by its learning. A numerical example and a numerical experiment are also presented.

We consider a model $(I, V, X, \{m_k(\cdot)\}_{k=0}^{\infty})$ with four elements which consist of the *nodes*, the *values of nodes*, *inputs* and *model functions* with some *learning processes*, in this paper. We suppose the following in this paper.

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(i) It is assumed that the model has nodes arranged according to a certain rule. Let I denote the set of all nodes, which is called the *node set*. We assume that I is a countable set metrized by a metric d.

(ii) Each node has a *value* and it is renewed by *learning*. V is the space of values of nodes. We assume that V is a real linear normed space with a norm $\|\cdot\|$. A mapping $m : I \to V$ transforming each node i to its value $m(i)$ is called a *model function*.

(iii) X is the *input set*. Let X be a subset of V. $x \in X$ is called an *input*.

(iv) The *learning process* is as follows. If an input is given, then the value of each node is renewed to a new value by the input. If an initial model function m_0 and a sequence $x_0, x_1, x_2, \ldots \in X$ of inputs are given, then the model functions $m_1, m_2, m_3, \ldots$ are generated sequentially according to

$$m_{k+1}(i) = (1 - \alpha_{m_k,x_k}(i))m_k(i) + \alpha_{m_k,x_k}(i)x_k, \quad k = 0, 1, 2, \ldots,$$

where α_{m_k,x_k} is the learning rate which satisfies $0 \le \alpha_{m_k,x_k} \le 1$.

2. An absorbing class

In this paper, we restrict our considerations to a fundamental self-organizing map with real-valued nodes and a one-dimensional array of nodes. We assume that $V = \mathbb{R}$, a set of values of nodes, where $\mathbb{R}$ is the set of all real numbers.

We consider a model

$$(I = \{1, 2, \ldots, N\}, V = \mathbb{R}, X \subset \mathbb{R}, \{m_k(\cdot)\}_{k=0}^{\infty}).$$

(i) Let $I = \{1, 2, \ldots, N\}$ be the node set with metric $d(i, j) = |i - j|$.

(ii) Assume $V = \mathbb{R}$, that is, each node is $\mathbb{R}$-valued.

(iii) $x_0, x_1, x_2, \ldots \in X \subset \mathbb{R}$ is an input sequence.

(iv) We assume a learning process defined by the following procedures. Learning process $\mathrm{L_A}$ with learning radius $r = 1$ is as follows.
(a) Areas of learning:

$$I(m_k, x_k) = \{i^* \in I \mid |m_k(i^*) - x_k| = \inf_{i \in I} |m_k(i) - x_k|\} \tag{2.1}$$

and $N_1(i) = \{j \in I \mid |j - i| \le 1\}$. (b) Learning-rate factor: $0 < \alpha < 1$. (c) Learning: let $N_1(I(m_k, x_k)) = \cup_{i^* \in I(m_k,x_k)} N_1(i^*)$ and $\{m_k\}$ is defined by the following. For a given initial model function m_0 and each $k = 0, 1, 2, \ldots$, if $i \in N_1(I(m_k, x_k))$ then

$$m_{k+1}(i) = (1 - \alpha)m_k(i) + \alpha x_k, \tag{2.2}$$

otherwise $m_{k+1}(i) = m_k(i)$.

By repeating learning, the values of all the nodes gradually have certain regularity such as monotonicity. The following is a well-known property [6].

Theorem 2.1. *We consider a self-organizing map model*

$$(\{1, 2, \ldots, N\}, \mathbb{R}, X \subset \mathbb{R}, \{m_k(\cdot)\}_{k=0}^{\infty})$$

with Learning process $L_A(r = 1)$. For model functions m_1, m_2, ..., the following statements hold.

(i) *if m_k is increasing on $I = \{1, 2, \ldots, N\}$, that is $m_k(i) \leq m_k(i+1)$ for all i, then m_{k+1} is increasing on I;*
(ii) *if m_k is decreasing on I, that is $m_k(i) \geq m_k(i+1)$ for all i, then m_{k+1} is decreasing on I;*
(iii) *if m_k is strictly increasing on I, that is $m_k(i) < m_k(i+1)$ for all i, then m_{k+1} is strictly increasing on I;*
(iv) *if m_k is strictly decreasing on I, that is $m_k(i) > m_k(i+1)$ for all i, then m_{k+1} is strictly decreasing on I.*

A property like monotonicity is called an *absorbing state* or a *closed class* of states in a self-organizing map model in the sense that once a model function is in this state, it does not become any other state for any input.

3. A numerical example for a transition of model function

By Theorem 2.1, model function m_k turn to a monotone state from a non-monotone state after a sufficient number of renewals.

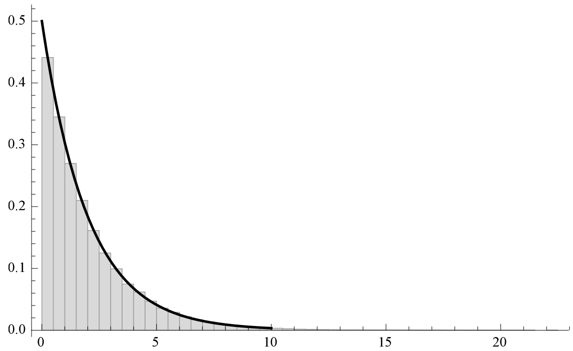

FIGURE 1. The histogram of input values and an exponential distribution for generating them.

Example 3.1. We give a numerical example of a learning process in a one-dimensional arrayed self-organizing map model with 100 real-valued nodes. Figure 2 shows the transition of the values of nodes in the model with a learning process from random inputs which are generated by an exponential distribution with its mean 2 shown in Figure 1. The initial values of nodes shown in the left of Figure 2 (step 0), are generated by the discrete uniform distribution on $\{0, 1, \ldots, 12\}$. We can observe that model function turns to be monotone gradually in Figure 2.

4. Local behavior of model function

In this section, we discuss local behavior when a model function gradually monotonizes in the learning process and describe some conditions for a model function to monotonize. As observed in Example 3.1, usually, the position of the

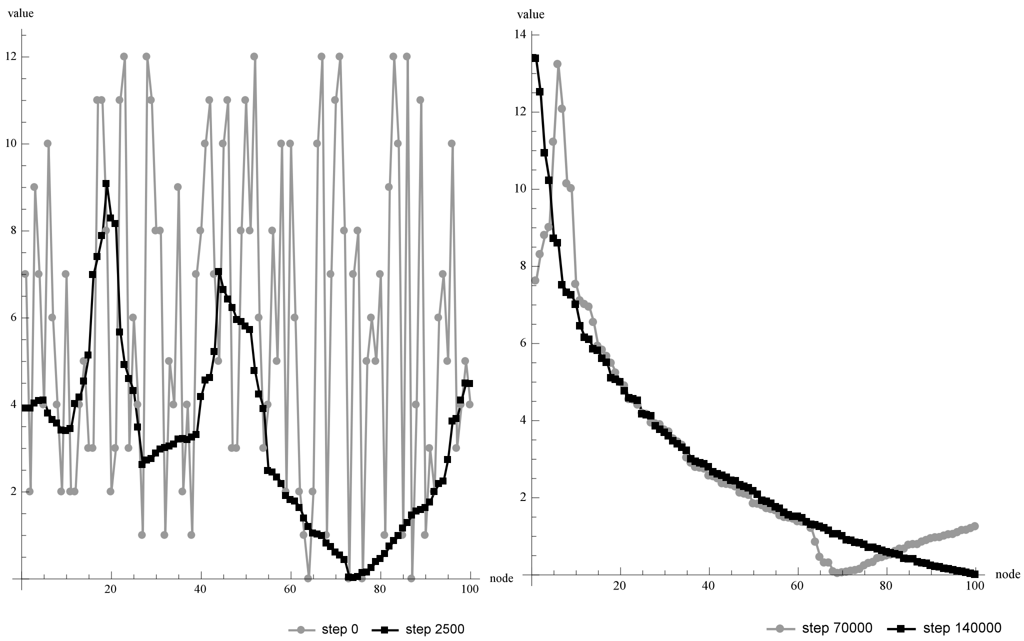

FIGURE 2. The transition of the values of nodes (iteration steps: step 0, 2500, 70000, 140000). The horizontal axis and the vertical axis represent the node index and the value of each node, respectively.

node which has the minimum value or the maximum value transitions gradually from the inside to the one end. The following result give the behavior just before a model function becomes monotone, and give a condition on state of model function and input for monotonization.

Theorem 4.1. *We consider a self-organizing map model*

$$(\{1, 2, \ldots, N\}, V \subset \mathbb{R}, X \subset \mathbb{R}, \{m_k(\cdot)\}_{k=0}^{\infty})$$

with Learning process $L_A(r = 1, 0 < \alpha < 1)$*. It is assumed that*

$$m(1) > m(2),\ m(2) < m(3) < \cdots < m(N) \tag{4.1}$$

and

$$m(i) \neq m(j) \text{ for } i \neq j$$

hold for the model function m *after several updates. Let* m' *be the updated model function of* m *when learning from input* x*. Then, the following holds.*

(i) *If* $m(1) \geq m(3)$*, then* m' *does not increasing on* $\{1, 2, \ldots, N\}$ *for any* x*.*
(ii) *If* $m(1) < m(3)$*, then* m' *is strictly increasing on* $\{1, 2, \ldots, N\}$ *if and only if input* x *satisfies inequality*

$$\max\left\{\frac{m(1)+m(3)}{2}, \left(1 - \frac{1}{\alpha}\right)m(2) + \frac{1}{\alpha}m(1)\right\} < x \leq \frac{m(3)+m(4)}{2}. \tag{4.2}$$

A proof of Theorem 4.1 is in [5].

We give a condition on state of model function and input for the node which takes the extreme value of the model function to shift by one node to the left or the right.

Theorem 4.2. *We consider a self-organizing map model*

$$(\{1,2,\ldots,N\}, V \subset \mathbb{R}, X \subset \mathbb{R}, \{m_k(\cdot)\}_{k=0}^{\infty})$$

with Learning process $L_A(r=1, 0<\alpha<1)$. *It is assumed that*

$$m(1) > m(2) > \cdots > m(q),\ m(q) < m(q+1) < \cdots < m(N) \tag{4.3}$$

and

$$m(i) \neq m(j) \text{ for } i \neq j$$

hold for the model function m *after several updates and some node* q, *where* $3 \leq q \leq N-2$. *Let* m' *be the updated model function of* m *using input* x.

Then, the extreme point of the model function shifts to the left by one node, that is,

$$m'(1) > m'(2) > \cdots > m'(q-1) \text{ and } m'(q-1) < m'(q) < \cdots < m'(N) \tag{4.4}$$

hold if and only if m *and* x *satisfy at least one of the following conditions (i)-(iv).*

(i) *For* $s_- = \max\{s \geq 0 \mid m(q-s) < m(q+1)\}$, $s_- = 1$ *and*

$$\max\left\{\frac{m(q-1)+m(q+1)}{2},\ \left(1-\frac{1}{\alpha}\right)m(q)+\frac{1}{\alpha}m(q-1)\right\} < x < \frac{m(q+1)+\min\{m(q-2),\, m(q+2)\}}{2}; \tag{4.5}$$

(ii) $s_- = 1$, $m(q-2) > m(q+2)$ *and*

$$\left(1-\frac{1}{\alpha}\right)m(q)+\frac{1}{\alpha}m(q-1) < x = \frac{m(q+1)+m(q+2)}{2}; \tag{4.6}$$

(iii) $s_- \geq 2$ *and*

$$\max\left\{\frac{m(q-s_-)+m(q+1)}{2},\ \left(1-\frac{1}{\alpha}\right)m(q)+\frac{1}{\alpha}m(q-1)\right\} < x \leq \frac{m(q+1)+\min\{m(q-s_--1),\, m(q+2)\}}{2}. \tag{4.7}$$

If $s_- = q-1$, *replace the right side of (4.7) with* $\frac{m(q+1)+m(q+2)}{2}$;

(iv) $s_- \geq 3$ *and*

$$\left(1-\frac{1}{\alpha}\right)m(q)+\frac{1}{\alpha}m(q-1) < x = \frac{m(q-s_-)+m(q+1)}{2}. \tag{4.8}$$

Moreover, the extreme point of the model function shifts to the right by one node, that is,

$$m'(1) > m'(2) > \cdots > m'(q+1) \text{ and } m'(q+1) < m'(q+2) < \cdots < m'(N) \tag{4.9}$$

hold if and only if m *and* x *satisfy at least one of the following conditions (v)-(viii).*

(v) *For* $s_+ = \max\{s \geq 0 \mid m(q+s) < m(q-1)\}$, $s_+ = 1$ *and*

$$\max\left\{\frac{m(q+1)+m(q-1)}{2},\ \Big(1-\frac{1}{\alpha}\Big)m(q)+\frac{1}{\alpha}m(q+1)\right\}$$

$$< x < \frac{m(q-1)+\min\{m(q+2),\, m(q-2)\}}{2}; \tag{4.10}$$

(vi) $s_+ = 1$, $m(q-2) < m(q+2)$ *and*

$$\Big(1-\frac{1}{\alpha}\Big)m(q)+\frac{1}{\alpha}m(q+1) < x = \frac{m(q-1)+m(q-2)}{2}; \tag{4.11}$$

(vii) $s_+ \geq 2$ *and*

$$\max\left\{\frac{m(q+s_+)+m(q-1)}{2},\ \Big(1-\frac{1}{\alpha}\Big)m(q)+\frac{1}{\alpha}m(q+1)\right\}$$

$$< x \leq \frac{m(q-1)+\min\{m(q+s_++1),\, m(q-2)\}}{2}. \tag{4.12}$$

If $s_+ = N-q$, *replace the right side of (4.12) with* $\frac{m(q-1)+m(q-2)}{2}$;

(viii) $s_+ \geq 3$ *and*

$$\Big(1-\frac{1}{\alpha}\Big)m(q)+\frac{1}{\alpha}m(q+1) < x = \frac{m(q+s_+)+m(q-1)}{2}. \tag{4.13}$$

Proof. Let $s_- = 0$. If $q \in I(m,x)$ or $q-1 \in I(m,x)$, then $m'(q-1) > m'(q)$. If $q+1 \in I(m,x)$ and $q, q-1 \notin I(m,x)$, then it follows from $x < m(q-1)$ that

$$\begin{aligned} m'(q-1)-m'(q) &= m(q-1)-(1-\alpha)m(q)-\alpha x \\ &> m(q-1)-(1-\alpha)m(q)-\alpha m(q-1) \\ &= (1-\alpha)(m(q-1)-m(q)) > 0. \end{aligned}$$

Therefore $m'(q-1) > m'(q)$. If $i \in I(m,x)$ and $q-2, q-1, q+1 \notin I(m,x)$, where $i \geq q+2$, then $m'(q-1) > m'(q)$. If $q-2 \in I(m,x)$ and $q-1 \notin I(m,x)$, then it follows from $x > m(q-1)$ that

$$\begin{aligned} m'(q-1)-m'(q) &= (1-\alpha)m(q-1)+\alpha x - m(q) \\ &> m(q-1)-m(q) > 0. \end{aligned}$$

Therefore $m'(q-1) > m'(q)$. If $i \in I(m,x)$ and $q-2 \notin I(m,x)$, where $i \geq q-3$, then $m'(q-1) > m'(q)$. Thus, if $s_- = 0$, then (4.4) does not hold for any input x.

Let $s_- = 1$. If $q \in I(m,x)$ or $q-1 \in I(m,x)$, then $m'(q-1) > m'(q)$. If $q-2 \in I(m,x)$ and $q+1 \notin I(m,x)$, then it follows from $x > m(q+1) > m(q-1)$ that

$$\begin{aligned} m'(q-1)-m'(q) &= (1-\alpha)m(q-1)+\alpha x - m(q) \\ &> (1-\alpha)m(q-1)+\alpha m(q-1)-m(q) \\ &= m(q-1)-m(q) > 0. \end{aligned}$$

Therefore $m'(q-1) > m'(q)$. If $i \in I(m,x)$ and $q-2 \notin I(m,x)$, where $i \leq q-3$, then $m'(q-1) > m'(q)$. If $i \in I(m,x)$ and $q+1, q-2 \notin I(m,x)$, where $i \geq q+2$, then $m'(q-1) > m'(q)$. Thus, if $s_- = 1$ and $I(m,x) \neq \{q+1\}, \{q+1, q+2\}$, then (4.4) does not hold for any input x.

If $s_- = 1$ and $I(m,x) = \{q+1\}$, then $m'(q) > m'(q-1)$ is equivalent to $x > (1-\frac{1}{\alpha})m(q)+\frac{1}{\alpha}m(q-1)$. Therefore, $s_- = 1$ and (4.5) imply $m'(q) > m'(q-1)$. Since $x < m(q+2)$, we obtain $m'(q+3) - m'(q+2) > m(q+3) - m(q+2) > 0$. Thus, $s_- = 1$ and (4.5) imply (4.4).

If $s_- = 1$ and $I(m,x) = \{q+1, q+2\}$, then $m(q+2) < m(q-2)$. Moreover, (4.6) implies $m'(q) > m'(q-1)$. Since $x < m(q+3)$, we obtain $m'(q+4) - m'(q+3) > m(q+4) - m(q+3) > 0$. Thus $s_- = 1$, $m(q+2) < m(q-2)$ and (4.6) imply (4.4).

Let $s_- \geq 2$. If $q \in I(m,x)$ or $q-1 \in I(m,x)$, then $m'(q-1) > m'(q)$. If $q-2 \in I(m,x)$ and $q-1, q+1 \notin I(m,x)$, since $x > m(q-1)$,

$$\begin{aligned} m'(q-1) - m'(q) &= (1-\alpha)m(q-1) + \alpha x - m(q) \\ &> m(q-1) - m(q) > 0. \end{aligned}$$

Therefore $m'(q-1) > m'(q)$. If $i \in I(m,x)$ and $q-2, q+1 \notin I(m,x)$, where $i \leq q-3$, then $m'(q-1) > m'(q)$. If $i \in I(m,x)$, where $i \geq q+2$, then $m'(q-1) > m'(q)$. If $I(m,x) = \{q-2, q+1\}$, then $m'(q-1) > m'(q)$.

If $s_- \geq 2$ and $I(m,x) = \{q+1\}$, by the same argument of the case of $s_- = 1$,

$$\begin{aligned} \max\left\{\frac{m(q-s_-) + m(q+1)}{2}, \left(1-\frac{1}{\alpha}\right)m(q) + \frac{1}{\alpha}m(q-1)\right\} \\ < x < \frac{m(q+1) + \min\{m(q-s_- -1),\, m(q+2)\}}{2} \end{aligned}$$

implies (4.4). If $s_- \geq 2$ and $I(m,x) = \{q+1, i^*\}$, where

$$m(i^*) = \min\{m(q-s_- -1), m(q+2)\},$$

then

$$\begin{aligned} &\left(1-\frac{1}{\alpha}\right)m(q) + \frac{1}{\alpha}m(q-1) \\ &\qquad < x = \frac{m(q+1) + \min\{m(q-s_- -1),\, m(q+2)\}}{2} \end{aligned}$$

implies (4.4). Therefore, if $s_- \geq 2$ and (4.7) hold, then (4.4) holds.

Moreover, if $s_- \geq 3$ and $I(m,x) = \{q-s_-, q+1\}$, then (4.8) implies (4.4).

Thus, (4.4) holds if and only if m and x satisfy at least one of (i)-(iv).

By the same argument, we obtain the latter statement. □

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Mitsuhiro Hoshino
Department of Management Science and Engineering, Faculty of Systems Science and Technology, Akita Prefectural University, 84-4 Ebinokuchi Tsuchiya Yurihonjo, Akita 015-0055, Japan
E-mail address: `hoshino@akita-pu.ac.jp`