Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167

Volume 6, Number 1, 2020, 73–90

GENERALIZED ACUTE POINT THEOREMS FOR GENERALIZED PSEUDOCONTRACTIONS IN A BANACH SPACE

TOSHIHARU KAWASAKI

Abstract. Acute point theorems require more assumptions on parameters than fixed point theorems. In this paper we generalize the concept of acute point and we introduce some acute point type theorems that holds under the same assumptions as fixed point theorems. Furthermore we show that fixed point theorems are derived from acute point type theorems.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is said to be generalized hybrid [20] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1-\beta)\|x - y\|^2$$

for any $x, y \in C$. Such a mapping is said to be (α, β)-generalized hybrid. The class of all generalized hybrid mappings is a new class of nonlinear mappings including nonexpansive mappings, nonspreading mappings [21] and hybrid mappings [23]. A mapping T from C into H is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for any $x, y \in C$; nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$; hybrid if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for any $x, y \in C$. Any nonexpansive mapping is $(1, 0)$-generalized hybrid; any nonspreading mapping is $(2, 1)$-generalized hybrid; any hybrid mapping is $\left(\frac{3}{2}, \frac{1}{2}\right)$-generalized hybrid.

Motivated these mappings, in [17] Kawasaki and Takahashi introduced a new very wider class of mappings, called widely more generalized hybrid mappings, than the class of all generalized hybrid mappings. A mapping T from C into H is widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned}&\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ &\quad + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0\end{aligned}$$

2010 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Fixed point theorem, Generalized acute point theorem, Generalized skew-acute point theorem, Generalized pseudocontraction, Banach space.

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$-widely more generalized hybrid. This class includes the class of all generalized hybrid mappings and also the class of all k-pseudocontractions [3] for $k \in [0, 1]$. A mapping T from C into H is called a k-pseudocontraction if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2$$

for any $x, y \in C$. Any (α, β)-generalized hybrid mapping is $(\alpha, 1 - \alpha, -\beta, \beta - 1, 0, 0, 0)$-widely more generalized hybrid; any k-pseudocontraction is $(1, 0, 0, -1, 0, 0, -k)$-widely more generalized hybrid. Furthermore they proved some fixed point theorems [6–11, 16–19] and some ergodic theorems [6, 7, 16–18].

There are some studies on Banach space related to these results. In [25] Takahashi, Wong and Yao introduced the generalized nonspreading mapping and the skew-generalized nonspreading mapping in a Banach space. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping T from C into E is said to be generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ &\quad \leq \varepsilon(\phi(Ty, Tx) - \phi(Ty, x)) + \zeta(\phi(y, Tx) - \phi(y, x)) \end{aligned}$$

for any $x, y \in C$, where J is the duality mapping on E and

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv\rangle + \|v\|^2.$$

Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-generalized nonspreading. A mapping T from C into E is said to be skew-generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ &\quad \leq \varepsilon(\phi(Ty, Tx) - \phi(y, Tx)) + \zeta(\phi(Ty, x) - \phi(y, x)) \end{aligned}$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$-skew-generalized nonspreading. These classes include the class of generalized hybrid mappings in a Hilbert space, however, it does not include the class of widely more generalized hybrid mappings.

Motivated these results, we introduced a new class of mappings [12–15] on Banach space corresponding to the class of all widely more generalized hybrid mappings on Hilbert space. Let E be a smooth Banach space and let C be a nonempty subset of E. A mapping T from C into E is called a generalized pseudocontraction if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1\phi(Tx, Ty) + \alpha_2\phi(Ty, Tx) + \beta_1\phi(x, Ty) + \beta_2\phi(Ty, x) \\ &\quad + \gamma_1\phi(Tx, y) + \gamma_2\phi(y, Tx) + \delta_1\phi(x, y) + \delta_2\phi(y, x) \\ &\quad + \varepsilon_1\phi(Tx, x) + \varepsilon_2\phi(x, Tx) + \zeta_1\phi(y, Ty) + \zeta_2\phi(Ty, y) \\ &\leq 0 \end{aligned}$$

for any $x, y \in C$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction. Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E and let C^* be a nonempty subset of

E^*. A mapping T^* from C^* into E^* is called a $*$-generalized pseudocontraction if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned}
&\alpha_1 \phi_*(T^*x^*, T^*y^*) + \alpha_2 \phi_*(T^*y^*, T^*x^*) + \beta_1 \phi_*(x^*, T^*y^*) + \beta_2 \phi_*(T^*y^*, x^*) \\
&\quad + \gamma_1 \phi_*(T^*x^*, y^*) + \gamma_2 \phi_*(y^*, T^*x^*) + \delta_1 \phi_*(x^*, y^*) + \delta_2 \phi_*(y^*, x^*) \\
&\quad + \varepsilon_1 \phi_*(T^*x^*, x^*) + \varepsilon_2 \phi_*(x^*, T^*x^*) + \zeta_1 \phi_*(y^*, T^*y^*) + \zeta_2 \phi_*(T^*y^*, y^*) \\
&\leq 0
\end{aligned}$$

for any $x^*, y^* \in C^*$, where

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^* \rangle + \|y^*\|^2$$

for any $x^*, y^* \in E^*$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-$*$-generalized pseudocontraction.

On the other hand, in [24] Takahashi and Takeuchi introduced a concept of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H. $x \in H$ is called an attractive point of T if

$$\|x - Ty\| \leq \|x - y\|$$

for any $y \in C$. Let

$$A(T) = \{x \in H \mid \|x - Ty\| \leq \|x - y\| \text{ for any } y \in C\}.$$

Furthermore they proved that the Baillon type ergodic theorem [2] for generalized hybrid mappings without convexity of C.

In [25] Takahashi, Wong and Yao introduced some extensions of attractive point and proved some attractive point theorems on Banach spaces. $x \in E$ is an attractive point of T if

$$\phi(x, Ty) \leq \phi(x, y)$$

for any $y \in C$; $x \in E$ is a skew-attractive point of T if

$$\phi(Ty, x) \leq \phi(y, x)$$

for any $y \in C$. Let

$$\begin{aligned}
A(T) &= \{x \in E \mid \phi(x, Ty) \leq \phi(x, y) \text{ for any } y \in C\}; \\
B(T) &= \{x \in E \mid \phi(Ty, x) \leq \phi(y, x) \text{ for any } y \in C\}.
\end{aligned}$$

In [1] Atsushiba, Iemoto, Kubota and Takeuchi introduced a concept of acute point as an extension of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H and $k \in [0, 1]$. $x \in H$ is called a k-acute point of T if

$$\|x - Ty\|^2 \leq \|x - y\|^2 + k\|y - Ty\|^2$$

for any $y \in C$. Let

$$\mathscr{A}_k(T) = \{x \in H \mid \|x - Ty\|^2 \leq \|x - y\|^2 + k\|y - Ty\|^2 \text{ for any } y \in C\}.$$

Furthermore, using a concept of acute point, they proved convergence theorems without convexity of C.

We introduced some extensions of acute point [12–15]. Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell \in \mathbb{R}$. $x \in E$ is called a (k, ℓ)-acute point of T if

$$\phi(x, Ty) \leq \phi(x, y) + k\phi(y, Ty) + \ell\phi(Ty, y)$$

for any $y \in C$. $x \in E$ is called a (k, ℓ)-skew-acute point of T if

$$\phi(Ty, x) \leq \phi(y, x) + k\phi(y, Ty) + \ell\phi(Ty, y)$$

for any $y \in C$. Let

$$\begin{aligned}
&\mathscr{A}_{k,\ell}(T) \\
&= \{x \in E \mid \phi(x, Ty) \leq \phi(x, y) + k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}; \\
&\mathscr{B}_{k,\ell}(T) \\
&= \{x \in E \mid \phi(Ty, x) \leq \phi(y, x) + k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}.
\end{aligned}$$

Furthermore we proved some fixed point and acute point theorems [12, 14], and some convergence theorems [13, 15]. However, acute point theorems require more assumptions on parameters than fixed point theorems.

In this paper we generalize the concept of acute point and we introduce some acute point type theorems that holds under the same assumptions as fixed point theorems. Furthermore we show that fixed point theorems are derived from acute point type theorems.

2. Preliminaries

We know that the following hold; for instance, see [4, 5, 22].

(T1) Let E be a Banach space, let E^* be the topological dual space of E and let J be the duality mapping on E defined by

$$J(x) = \{x^* \in E^* \mid \|x\|^2 = \langle x, x^* \rangle = \|x^*\|^2\}$$

for any $x \in E$. Then E is strictly convex if and only if J is injective, that is, $x \neq y$ implies $J(x) \cap J(y) = \emptyset$.

(T2) Let E be a Banach space, let E^* be the topological dual space of E and let J be the duality mapping on E. Then E is reflexive if and only if J is surjective, that is, $\bigcup_{x \in E} J(x) = E^*$.

(T3) Let E be a Banach space and let J be the duality mapping on E. Then E is smooth if and only if J is single-valued.

(T4) Let E be a Banach space and let J be the duality mapping on E. If J is single-valued, then J is norm-to-weak* continuous.

(T5) Let E be a Banach space and let J be the duality mapping on E. Then E is strictly convex if and only if

$$1 - \langle x, y^* \rangle > 0$$

for any $x, y \in E$ with $x \neq y$ and $\|x\| = \|y\| = 1$ and for any $y^* \in J(y)$.

(T6) Let E be a Banach space and let E^* be the topological dual space of E. Then E is reflexive if and only if E^* is reflexive.

(T7) Let E be a Banach space and let E^* be the topological dual space of E. If E^* is strictly convex, then E is smooth. Conversely, E is reflexive and smooth, then E^* is strictly convex.

(T8) Let E be a Banach space and let E^* be the topological dual space of E. If E^* is smooth, then E is strictly convex. Conversely, E is reflexive and strictly convex, then E^* is smooth.

Let E be a smooth Banach space, let J be the duality mapping on E and let ϕ be the mapping from $E \times E$ into $[0, \infty)$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for any $x, y \in E$. Since by (T3) J is single-valued, ϕ is well-defined. It is obvious that $x = y$ implies $\phi(x, y) = 0$. Conversely, by (T5)

(T9) If E is also strictly convex, then $\phi(x, y) = 0$ implies $x = y$.

Let E be a strictly convex and smooth Banach space. By (T1) an (T3) J is a bijective mapping from E onto $J(E)$. In particular, if E is also reflective, then by (T2) J is a bijective mapping from E onto E^*. Suppose that E is strictly convex, reflective and smooth. Let ϕ_* be the mapping from $E^* \times E^*$ into $[0, \infty)$ defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for any $x^*, y^* \in E^*$. Then

$$\phi_*(x^*, y^*) = \phi(J^{-1}y^*, J^{-1}x^*) \tag{2.1}$$

holds. Therefore

(T9)* $\phi_*(x^*, y^*) = 0$ if and only if $x^* = y^*$.

Let ℓ^∞ be the Banach space consists of all bounded sequences and $\mu \in (\ell^\infty)^*$. Sometimes we denote by $\mu_n x_n$ the value $\mu(\{x_n\}_{n=1}^\infty)$. If $\mu \in (\ell^\infty)^*$ satisfies $\mu(e) = \|\mu\| = 1$, where $e = \{1\}_{n=1}^\infty$, then μ is called a mean. If a mean μ satisfies $\mu_n x_{n+1} = \mu_n x_n$, then μ is called a Banach limit. We know that there exists some Banach limits. If $\{x_n\}_{n=1}^\infty \in \ell^\infty$ and μ is a mean, then the following holds:

$$\inf\{x_n \mid n \in \mathbb{N}\} \le \mu_n x_n \le \sup\{x_n \mid n \in \mathbb{N}\}.$$

The following lemma is introduced in [25]; see also [12–15].

Lemma 2.1. *Let E be a Banach space, let E^* be the topological dual space of E, let $\{x_n \mid n \in \mathbb{N}\}$ be a bounded sequence in E and let μ be a mean on ℓ^∞. Then there exists a unique $z_0 \in \overline{co}\{x_n \mid n \in \mathbb{N}\}$ such that $\mu_n\langle x_n, x^*\rangle = \langle z_0, x^*\rangle$ for any $x^* \in E^*$.*

The following lemmas are shown in [13–15].

Lemma 2.2. *Let E be a smooth Banach space, let C be a nonempty subset of E, let D be a nonempty convex subset of E, let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into D and let $\lambda \in [0, 1]$. Then T is a $((1-\lambda)\alpha_1 + \lambda\alpha_2), \lambda\alpha_1 + (1-\lambda)\alpha_2, (1-\lambda)\beta_1 + \lambda\gamma_2, \lambda\gamma_1 + (1-\lambda)\beta_2, (1-\lambda)\gamma_1 + \lambda\beta_2,$*

$\lambda\beta_1 + (1-\lambda)\gamma_2, (1-\lambda)\delta_1 + \lambda\delta_2, \lambda\delta_1 + (1-\lambda)\delta_2, (1-\lambda)\varepsilon_1 + \lambda\zeta_2, \lambda\zeta_1 + (1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1 + \lambda\varepsilon_2, \lambda\varepsilon_1 + (1-\lambda)\zeta_2)$*-generalized pseudocontraction from C into D.*

Lemma 2.3. *Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^*, let D^* be a nonempty convex subset of E^*, let T^* be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-*-generalized pseudocontraction from C^* into D^* and let $\lambda \in [0,1]$. Then T^* is a $((1-\lambda)\alpha_1 + \lambda\alpha_2), \lambda\alpha_1 + (1-\lambda)\alpha_2, (1-\lambda)\beta_1 + \lambda\gamma_2, \lambda\gamma_1 + (1-\lambda)\beta_2, (1-\lambda)\gamma_1 + \lambda\beta_2, \lambda\beta_1 + (1-\lambda)\gamma_2, (1-\lambda)\delta_1 + \lambda\delta_2, \lambda\delta_1 + (1-\lambda)\delta_2, (1-\lambda)\varepsilon_1 + \lambda\zeta_2, \lambda\zeta_1 + (1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1 + \lambda\varepsilon_2, \lambda\varepsilon_1 + (1-\lambda)\zeta_2)$-*-generalized pseudocontraction from C^* into D^*.*

Lemma 2.4. *Let E be a strictly convex, reflexive and smooth Banach space, let E^* be the topological dual space of E, let C and D be nonempty subsets of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into D. Put $T^* = JTJ^{-1}$, where J is the duality mapping on E. Then T^* is an $(\alpha_2, \alpha_1, \beta_2, \beta_1, \gamma_2, \gamma_1, \delta_2, \delta_1, \varepsilon_2, \varepsilon_1, \zeta_2, \zeta_1)$-*-generalized pseudocontraction from $J(C)$ into $J(D)$.*

3. Generalized acute and skew-acute point

Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell, s \in \mathbb{R}$. $x \in E$ is called a (k, ℓ, s)-generalized acute point of T if

$$s(\phi(x, Ty) - \phi(x, y)) \leq k\phi(y, Ty) + \ell\phi(Ty, y) \tag{3.1}$$

for any $y \in C$. $x \in E$ is called a (k, ℓ, s)-generalized skew-acute point of T if

$$s(\phi(Ty, x) - \phi(y, x)) \leq k\phi(y, Ty) + \ell\phi(Ty, y) \tag{3.2}$$

for any $y \in C$. Let

$$\begin{aligned} &\mathscr{A}_{k,\ell,s}(T) \\ &= \{x \in E \mid s(\phi(x, Ty) - \phi(x, y)) \leq k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}; \\ &\mathscr{B}_{k,\ell,s}(T) \\ &= \{x \in E \mid s(\phi(Ty, x) - \phi(y, x)) \leq k\phi(y, Ty) + \ell\phi(Ty, y) \text{ for any } y \in C\}. \end{aligned}$$

It is obvious that

$$\mathscr{A}_{k_1,\ell_1,s_1}(T) \subset \mathscr{A}_{k_2,\ell_2,s_2}(T),\ \mathscr{B}_{k_1,\ell_1,s_2}(T) \subset \mathscr{B}_{k_2,\ell_2,s_2}(T)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{R}$ and for any $s_1, s_2 \in (0, \infty)$ with $\frac{k_1}{s_1} \leq \frac{k_2}{s_2}$ and $\frac{\ell_1}{s_1} \leq \frac{\ell_2}{s_2}$;

$$\mathscr{A}_{k_1,\ell_1,s_1}(T) \supset \mathscr{A}_{k_2,\ell_2,s_2}(T),\ \mathscr{B}_{k_1,\ell_1,s_2}(T) \supset \mathscr{B}_{k_2,\ell_2,s_2}(T)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{R}$ and for any $s_1, s_2 \in (-\infty, 0)$ with $\frac{k_1}{s_1} \leq \frac{k_2}{s_2}$ and $\frac{\ell_1}{s_1} \leq \frac{\ell_2}{s_2}$. Furthermore

$$\mathscr{A}_{k,\ell,0}(T) = \mathscr{B}_{k,\ell,0}(T) = E$$

for any $(k, \ell) \in [0, \infty) \times [0, \infty)$;

$$\mathscr{A}_{k,\ell,0}(T) = \mathscr{B}_{k,\ell,0}(T) = \emptyset$$

for any $(k,\ell) \in (-\infty,0] \times (-\infty,0] \setminus \{(0,0)\}$; otherwise,

$$\mathscr{A}_{k,\ell,0}(T) = E \text{ or } \emptyset, \ \mathscr{B}_{k,\ell,0}(T) = E \text{ or } \emptyset;$$

however, it is generally unknown which case holds. In this way, $\mathscr{A}_{k,\ell,0}(T)$ and $\mathscr{B}_{k,\ell,0}(T)$ may be empty. However, in later discussions, under some assumptions, such cases will be properly ruled out.

The following lemmas are important property characterizing them.

Lemma 3.1. *Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k,\ell,s \in \mathbb{R}$. Then $\mathscr{A}_{k,\ell,s}(T)$ is closed and convex.*

Proof. Since

$$\phi(u,v) = \phi(u,w) + \phi(w,v) + 2\langle u-w, Jw - Jv\rangle \tag{3.3}$$

for any $u,v,w \in E$, (3.1) is equivalent to

$$2s\langle x, Jy - JTy\rangle \le (k-s)\phi(y,Ty) + \ell\phi(Ty,y) + 2s\langle y, Jy - JTy\rangle.$$

Therefore $\mathscr{A}_{k,\ell,s}(T)$ is closed and convex. □

Lemma 3.2. *Let E be a smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k,\ell,s \in \mathbb{R}$. Then $\mathscr{B}_{k,\ell,s}(T)$ is closed.*

Proof. (3.2) is equivalent to

$$2s\langle y - Ty, Jx\rangle \le k\phi(y,Ty) + (\ell - s)\phi(Ty,y) + 2s\langle y - Ty, Jy\rangle$$

from (3.3). Furthermore by (T4) J is norm-to-weak* continuous. Therefore $\mathscr{B}_{k,\ell,s}(T)$ is closed. □

Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k,\ell,s \in \mathbb{R}$. $x^* \in E^*$ is called a (k,ℓ,s)-generalized-*-acute point of T^* if

$$s(\phi_*(x^*,T^*y^*) - \phi_*(x^*,y^*)) \le k\phi_*(y^*,T^*y^*) + \ell\phi_*(T^*y^*,y^*) \tag{3.4}$$

for any $y^* \in C^*$. $x^* \in E^*$ is called a (k,ℓ,s)-generalized-*-skew-acute point of T^* if

$$s(\phi_*(T^*y^*,x^*) - \phi_*(y^*,x^*)) \le k\phi_*(y^*,T^*y^*) + \ell\phi_*(T^*y^*,y^*) \tag{3.5}$$

for any $y^* \in C^*$. Let

$$\begin{aligned}&\mathscr{A}^*_{k,\ell,s}(T^*)\\ &= \left\{ x^* \in E^* \;\middle|\; \begin{array}{l} s(\phi_*(x^*,T^*y^*) - \phi_*(x^*,y^*)) \le k\phi_*(y^*,T^*y^*) + \ell\phi_*(T^*y^*,y^*) \\ \text{for any } y^* \in C^* \end{array} \right\};\\ &\mathscr{B}^*_{k,\ell,s}(T^*)\\ &= \left\{ x^* \in E^* \;\middle|\; \begin{array}{l} s(\phi_*(T^*y^*,x^*) - \phi_*(y^*,x^*)) \le k\phi_*(y^*,T^*y^*) + \ell\phi_*(T^*y^*,y^*) \\ \text{for any } y^* \in C^* \end{array} \right\}.\end{aligned}$$

Lemma 3.3. *Let E^* be the topological dual space of a strictly convex, reflective and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell, s \in \mathbb{R}$. Then $\mathscr{A}^*_{k,\ell,s}(T^*)$ is closed and convex.*

Proof. (3.4) is equivalent to

$$\begin{aligned}&2s\langle J^{-1}y^* - J^{-1}T^*y^*, x^*\rangle \\ &\quad \le (k-s)\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*) + 2s\langle J^{-1}y^* - J^{-1}T^*y^*, y^*\rangle\end{aligned}$$

from (3.3) and (2.1), $\mathscr{A}^*_{k,\ell,s}(T^*)$ is closed and convex. □

Lemma 3.4. *Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell, s \in \mathbb{R}$. Then $\mathscr{B}^*_{k,\ell,s}(T^*)$ is closed.*

Proof. (3.5) is equivalent to

$$\begin{aligned}&2s\langle J^{-1}x^*, y^* - T^*y^*\rangle \\ &\quad \le k\phi_*(y^*, T^*y^*) + (\ell-s)\phi_*(T^*y^*, y^*) + 2s\langle J^{-1}y^*, y^* - T^*y^*\rangle\end{aligned}$$

from (3.3) and (2.1). Furthermore by (T4) J^{-1} is norm-to-weak* continuous. Therefore $\mathscr{B}^*_{k,\ell,s}(T^*)$ is closed. □

Lemma 3.5. *Let E be a strictly convex, reflective and smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E, let $T^* = JTJ^{-1}$ and let $k, \ell, s \in \mathbb{R}$. Then*

$$\mathscr{A}^*_{k,\ell,s}(T^*) = J(\mathscr{B}_{\ell,k,s}(T)),\ \mathscr{B}^*_{k,\ell,s}(T^*) = J(\mathscr{A}_{\ell,k,s}(T)).$$

In particular, $J(\mathscr{B}_{k,\ell,s}(T))$ is closed and convex and $J(\mathscr{A}_{k,\ell,s}(T))$ is closed.

Proof. Let $x^* \in \mathscr{A}^*_{k,\ell,s}(T^*)$. Then

$$s(\phi_*(x^*, T^*y^*) - \phi_*(x^*, y^*)) \le k\phi_*(y^*, T^*y^*) + \ell\phi_*(T^*y^*, y^*)$$

for any $y^* \in J(C)$. From (2.1)

$$\begin{aligned}&s(\phi(J^{-1}T^*y^*, J^{-1}x^*) - \phi(J^{-1}y^*, J^{-1}x^*)) \\ &\quad \le k\phi(J^{-1}T^*y^*, J^{-1}y^*) + \ell\phi(J^{-1}y^*, J^{-1}T^*y^*)\end{aligned}$$

for any $y^* \in J(C)$. Since $J^{-1}T^* = TJ^{-1}$, putting $y = J^{-1}y^*$, we obtain

$$s(\phi(Ty, J^{-1}x^*) - \phi(y, J^{-1}x^*)) \le \ell\phi(y, Ty) + k\phi(Ty, y).$$

Therefore $J^{-1}x^* \in \mathscr{B}_{\ell,k,s}(T)$ and hence $\mathscr{A}^*_{k,\ell,s}(T^*) = J(\mathscr{B}_{\ell,k,s}(T))$.

$\mathscr{B}^*_{k,\ell,s}(T^*) = J(\mathscr{A}_{\ell,k,s}(T))$ can be shown similarly.

Furthermore, by Lemma 3.3 $J(\mathscr{B}_{k,\ell,s}(T))$ is closed and convex and by Lemma 3.4 $J(\mathscr{A}_{k,\ell,s}(T))$ is closed. □

Lemma 3.6. *Let E be a strictly convex and smooth Banach space, let C be a nonempty subset of E, let T be a mapping from C into E and let $k, \ell, s \in \mathbb{R}$. Then the following hold.*

(1) *If $(k, \ell) \in (-\infty, s] \times (-\infty, 0] \setminus \{(s, 0)\}$, then $C \cap \mathscr{A}_{k,\ell,s}(T)$ is a subset of the set of all fixed points of T;*

(2) *If $(k,\ell) \in (-\infty,0] \times (-\infty,s] \setminus \{(0,s)\}$, then $C \cap \mathscr{B}_{k,\ell,s}(T)$ is a subset of the set of all fixed points of T.*

Proof. Let $x \in C \cap \mathscr{A}_{k,\ell,s}(T)$. Then (3.1) holds for any $y \in C$. Putting $y = x$, we obtain $(s-k)\phi(x,Tx) - \ell\phi(Tx,x) \le 0$. If $(k,\ell) \in (-\infty,s] \times (-\infty,0] \setminus \{(s,0)\}$, then by (T9) we obtain $x = Tx$.

Let $x \in C \cap \mathscr{B}_{k,\ell,s}(T)$. Then (3.2) holds for any $y \in C$. Putting $y = x$, we obtain $-k\phi(x,Tx) + (s-\ell)\phi(Tx,x) \le 0$. If $(k,\ell) \in (-\infty,0] \times (-\infty,s] \setminus \{(0,s)\}$, then by (T9) we obtain $x = Tx$. □

Lemma 3.7. *Let E^* be a strictly convex and smooth topological dual space of a Banach space, let C^* be a nonempty subset of E^*, let T^* be a mapping from C^* into E^* and let $k, \ell \in \mathbb{R}$. Then the following hold.*

(1) *If $(k,\ell) \in (-\infty,s] \times (-\infty,0] \setminus \{(s,0)\}$, then $C \cap \mathscr{A}^*_{k,\ell,s}(T^*)$ is a subset of the set of all fixed points of T^*;*

(2) *If $(k,\ell) \in (-\infty,0] \times (-\infty,s] \setminus \{(0,1)\}$, then $C \cap \mathscr{B}^*_{k,\ell,s}(T^*)$ is a subset of the set of all fixed points of T^*.*

Proof. Let $x^* \in C^* \cap \mathscr{A}^*_{k,\ell,s}(T^*)$. Then (3.4) holds for any $y^* \in C^*$. Putting $y^* = x^*$, by we obtain $(s-k)\phi_*(x^*,T^*x^*) - \ell\phi_*(T^*x^*,x^*) \le 0$. If $(k,\ell) \in (-\infty,s] \times (-\infty,0] \setminus \{(s,0)\}$, then by (T9)* we obtain $x^* = T^*x^*$.

Let $x^* \in C^* \cap \mathscr{B}^*_{k,\ell,s}(T^*)$. Then (3.5) holds for any $y^* \in C^*$. Putting $y^* = x^*$, by we obtain $-k\phi_*(x^*,T^*x^*) + (s-\ell)\phi_*(T^*x^*,x^*) \le 0$. If $(k,\ell) \in (-\infty,0] \times (-\infty,s] \setminus \{(0,s)\}$, then by (T9)* we obtain $x^* = T^*x^*$. □

4. Generalized acute and skew-acute point theorems

Theorem 4.1. *Let E be a reflexive and smooth Banach space, let C be a nonempty subset of E and let T be an $(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma_1,\gamma_2,\delta_1,\delta_2,\varepsilon_1,\varepsilon_2,\zeta_1,\zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and suppose that there exists $\lambda \in [0,1]$ such that*

$$
\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1) + \lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \ge 0;\\
&\lambda(\alpha_1+\gamma_1) + (1-\lambda)(\alpha_2+\beta_2) \ge 0;\\
&\lambda(\beta_1+\delta_1) + (1-\lambda)(\gamma_2+\delta_2) \ge 0;\\
&(1-\lambda)\varepsilon_1 + \lambda\zeta_2 \ge 0;\\
&\lambda\zeta_1 + (1-\lambda)\varepsilon_2 \ge 0.
\end{aligned}
$$

Then there exists a $(-((1-\lambda)\zeta_1+\lambda\varepsilon_2), -(\lambda\varepsilon_1+(1-\lambda)\zeta_2), (1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))$-generalized acute point.

Proof. Suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. By Lemma 2.2 T is a $((1-\lambda)\alpha_1+\lambda\alpha_2, \lambda\alpha_1+(1-\lambda)\alpha_2, (1-\lambda)\beta_1+\lambda\gamma_2, \lambda\gamma_1+(1-\lambda)\beta_2,$ $(1-\lambda)\gamma_1+\lambda\beta_2, \lambda\beta_1+(1-\lambda)\gamma_2, (1-\lambda)\delta_1+\lambda\delta_2, \lambda\delta_1+(1-\lambda)\delta_2, (1-\lambda)\varepsilon_1+\lambda\zeta_2,$ $\lambda\zeta_1+(1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1+\lambda\varepsilon_2, \lambda\varepsilon_1+(1-\lambda)\zeta_2)$-generalized pseudocontraction. From (3.3) we obtain

$$((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty) + (\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)$$

$$
\begin{aligned}
&+((1-\lambda)\beta_1+\lambda\gamma_2)\phi(x,Ty)+(\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&+((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y)+(\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&+((1-\lambda)\delta_1+\lambda\delta_2)\phi(x,y)+(\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&+((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x)+(\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&+((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi(y,Ty)+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
=\;&((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty)+(\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&-((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(x,Ty)\\
&+((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))(\phi(x,y)+\phi(y,Ty)+2\langle x-y,Jy-JTy\rangle)\\
&+(\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&+((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y)+(\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&+((1-\lambda)\delta_1+\lambda\delta_2)\phi(x,y)+(\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&+((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x)+(\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&+((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi(y,Ty)+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
=\;&((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty)+(\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&-((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(x,Ty)+(\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&+((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y)+(\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&+((1-\lambda)(\alpha_1+\beta_1+\delta_1)+\lambda(\alpha_2+\gamma_2+\delta_2))\phi(x,y)\\
&+(\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&+((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x)+(\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)\\
&+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y,Jy-JTy\rangle.
\end{aligned}
$$

Since

$$
\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\delta_1)+\lambda(\alpha_2+\beta_2+\delta_2)\geq-((1-\lambda)\gamma_1+\lambda\gamma_2);\\
&\lambda\gamma_1+(1-\lambda)\beta_2\geq-(\lambda\alpha_1+(1-\lambda)\alpha_2);\\
&\lambda\delta_1+(1-\lambda)\delta_2\geq-(\lambda\beta_1+(1-\lambda)\gamma_2);\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2\geq 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2\geq 0,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(Tx,Ty)+(\lambda\alpha_1+(1-\lambda)\alpha_2)\phi(Ty,Tx)\\
&\qquad-((1-\lambda)\alpha_1+\lambda\alpha_2)\phi(x,Ty)+(\lambda\gamma_1+(1-\lambda)\beta_2)\phi(Ty,x)\\
&\qquad+((1-\lambda)\gamma_1+\lambda\beta_2)\phi(Tx,y)+(\lambda\beta_1+(1-\lambda)\gamma_2)\phi(y,Tx)\\
&\qquad+((1-\lambda)(\alpha_1+\beta_1+\delta_1)+\lambda(\alpha_2+\gamma_2+\delta_2))\phi(x,y)\\
&\qquad+(\lambda\delta_1+(1-\lambda)\delta_2)\phi(y,x)\\
&\qquad+((1-\lambda)\varepsilon_1+\lambda\zeta_2)\phi(Tx,x)+(\lambda\zeta_1+(1-\lambda)\varepsilon_2)\phi(x,Tx)
\end{aligned}
$$

$$
\begin{aligned}
&+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y, Jy-JTy\rangle\\
\geq\ & ((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi(Tx,Ty)-\phi(x,Ty))\\
&+(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi(Ty,Tx)-\phi(Ty,x))\\
&+((1-\lambda)\gamma_1+\lambda\beta_2)(\phi(Tx,y)-\phi(x,y))\\
&+(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi(y,Tx)-\phi(y,x))\\
&+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y, Jy-JTy\rangle.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi(Tx,Ty)-\phi(x,Ty))\\
&\quad+(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi(Ty,Tx)-\phi(Ty,x))\\
&\quad+((1-\lambda)\gamma_1+\lambda\beta_2)(\phi(Tx,y)-\phi(x,y))\\
&\quad+(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi(y,Tx)-\phi(y,x))\\
&\quad+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle x-y, Jy-JTy\rangle\\
&\leq 0.
\end{aligned}
$$

Replacing x by T^nz, we obtain

$$
\begin{aligned}
&((1-\lambda)\alpha_1+\lambda\alpha_2)(\phi(T^{n+1}z,Ty)-\phi(T^nz,Ty))\\
&\quad+(\lambda\alpha_1+(1-\lambda)\alpha_2)(\phi(Ty,T^{n+1}z)-\phi(Ty,T^nz))\\
&\quad+((1-\lambda)\gamma_1+\lambda\beta_2)(\phi(T^{n+1}z,y)-\phi(T^nz,y))\\
&\quad+(\lambda\beta_1+(1-\lambda)\gamma_2)(\phi(y,T^{n+1}z)-\phi(y,T^nz))\\
&\quad+((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle T^nz-y, Jy-JTy\rangle\\
&\leq 0.
\end{aligned}
$$

Applying a Banach limit μ to both sides of this inequality, we obtain

$$
\begin{aligned}
&((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\mu_n\langle T^nz-y, Jy-JTy\rangle\\
&\leq 0.
\end{aligned}
$$

By Lemma 2.1 there exists a unique $z_0 \in \overline{co}\{T^n z \mid n \in \mathbb{N}\}$ such that

$$\mu_n \langle T^n z - y, Jy - JTy\rangle = \langle z_0 - y, Jy - JTy\rangle.$$

Therefore we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle z_0-y, Jy-JTy\rangle\\
&\le 0.
\end{aligned}$$

for any $y \in C$. From (3.3) we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi(y,Ty)\\
&\quad+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\quad+2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle z_0-y, Jy-JTy\rangle\\
&=((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\phi(z_0,Ty)\\
&\quad-((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\phi(z_0,y)\\
&\quad+((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi(y,Ty)+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)\\
&\le 0.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\phi(z_0,Ty)-\phi(z_0,y))\\
&\quad\le -((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi(y,Ty)-(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi(Ty,y)
\end{aligned}$$

and hence z_0 is a $(-((1-\lambda)\zeta_1+\lambda\varepsilon_2), -(\lambda\varepsilon_1+(1-\lambda)\zeta_2), (1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))$-generalized acute point. □

Theorem 4.2. *Let E^* be the topological dual space of a strictly convex, reflexive and smooth Banach space E, let C^* be a nonempty subset of E^* and let T^* be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-*-generalized pseudocontraction from C^* into itself. Suppose that there exists $z^* \in C^*$ such that $\{(T^*)^n z^* \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and suppose that there exists $\lambda \in [0,1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1)+\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2)\ge 0;\\
&\lambda(\alpha_1+\gamma_1)+(1-\lambda)(\alpha_2+\beta_2)\ge 0;\\
&\lambda(\beta_1+\delta_1)+(1-\lambda)(\gamma_2+\delta_2)\ge 0;\\
&(1-\lambda)\varepsilon_1+\lambda\zeta_2\ge 0;\\
&\lambda\zeta_1+(1-\lambda)\varepsilon_2\ge 0.
\end{aligned}$$

*Then there exists a $(-((1-\lambda)\zeta_1+\lambda\varepsilon_2), -(\lambda\varepsilon_1+(1-\lambda)\zeta_2), (1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))$-generalized *-acute point.*

Proof. By Lemma 2.3 T^* is a $((1-\lambda)\alpha_1+\lambda\alpha_2), \lambda\alpha_1+(1-\lambda)\alpha_2, (1-\lambda)\beta_1+\lambda\gamma_2, \lambda\gamma_1+(1-\lambda)\beta_2, (1-\lambda)\gamma_1+\lambda\beta_2, \lambda\beta_1+(1-\lambda)\gamma_2, (1-\lambda)\delta_1+\lambda\delta_2, \lambda\delta_1+(1-\lambda)\delta_2, (1-\lambda)\varepsilon_1+\lambda\zeta_2,$

$\lambda\zeta_1 + (1-\lambda)\varepsilon_2, (1-\lambda)\zeta_1 + \lambda\varepsilon_2, \lambda\varepsilon_1 + (1-\lambda)\zeta_2)$-*-generalized pseudocontraction. From (2.1) and (3.3) we obtain

$$\phi_*(u^*, v^*) = \phi_*(u^*, w^*) + \phi_*(w^*, v^*) + 2\langle J^{-1}w^* - J^{-1}v^*, u^* - w^*\rangle \tag{4.1}$$

for any $u^*, v^*, w^* \in E^*$. Similarly to the proof of Theorem 4.1 we obtain

$$\begin{aligned}
&((1-\lambda)\alpha_1 + \lambda\alpha_2)(\phi_*(T^*x^*, T^*y^*) - \phi_*(x^*, T^*y^*)) \\
&\quad +(\lambda\alpha_1 + (1-\lambda)\alpha_2)(\phi_*(T^*y^*, T^*x^*) - \phi_*(T^*y^*, x^*)) \\
&\quad +((1-\lambda)\gamma_1 + \lambda\beta_2)(\phi_*(T^*x^*, y^*) - \phi_*(x^*, y^*)) \\
&\quad +(\lambda\beta_1 + (1-\lambda)\gamma_2)(\phi_*(y^*, T^*x^*) - \phi_*(y^*, x^*)) \\
&\quad +((1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2))\phi_*(y^*, T^*y^*) \\
&\quad +(\lambda\varepsilon_1 + (1-\lambda)\zeta_2)\phi_*(T^*y^*, y^*) \\
&\quad +2((1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2))\langle J^{-1}y^* - J^{-1}T^*y^*, x^* - y^*\rangle \\
&\leq 0.
\end{aligned}$$

Replacing x^* by $(T^*)^n z^*$, we obtain

$$\begin{aligned}
&((1-\lambda)\alpha_1 + \lambda\alpha_2)(\phi_*((T^*)^{n+1}z^*, T^*y^*) - \phi_*((T^*)^n z^*, T^*y^*)) \\
&\quad +(\lambda\alpha_1 + (1-\lambda)\alpha_2)(\phi_*(T^*y^*, (T^*)^{n+1}z^*) - \phi_*(T^*y^*, (T^*)^n z^*)) \\
&\quad +((1-\lambda)\gamma_1 + \lambda\beta_2)(\phi_*((T^*)^{n+1}z^*, y^*) - \phi_*((T^*)^n z^*, y^*)) \\
&\quad +(\lambda\beta_1 + (1-\lambda)\gamma_2)(\phi_*(y^*, (T^*)^{n+1}z^*) - \phi_*(y^*, (T^*)^n z^*)) \\
&\quad +((1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2))\phi_*(y^*, T^*y^*) \\
&\quad +(\lambda\varepsilon_1 + (1-\lambda)\zeta_2)\phi_*(T^*y^*, y^*) \\
&\quad +2((1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2))\langle J^{-1}y^* - J^{-1}T^*y^*, (T^*)^n z^* - y^*\rangle \\
&\leq 0.
\end{aligned}$$

Applying a Banach limit μ to both sides of this inequality, we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2))\phi_*(y^*, T^*y^*) \\
&\quad +(\lambda\varepsilon_1 + (1-\lambda)\zeta_2)\phi_*(T^*y^*, y^*) \\
&\quad +2((1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2))\mu_n\langle J^{-1}y^* - J^{-1}T^*y^*, (T^*)^n z^* - y^*\rangle \\
&\leq 0.
\end{aligned}$$

By Lemma 2.1 there exists a unique $z_0^* \in \overline{co}\{(T^*)^n z^* \mid n \in \mathbb{N}\}$ such that

$$\mu_n\langle J^{-1}y^* - J^{-1}T^*y^*, (T^*)^n z^* - y^*\rangle = \langle J^{-1}y^* - J^{-1}T^*y^*, z_0^* - y^*\rangle.$$

Therefore we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2))\phi_*(y^*, T^*y^*) \\
&\quad +(\lambda\varepsilon_1 + (1-\lambda)\zeta_2)\phi_*(T^*y^*, y^*) \\
&\quad +2((1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2))\langle J^{-1}y^* - J^{-1}T^*y^*, z_0^* - y^*\rangle \\
&\leq 0.
\end{aligned}$$

for any $y^* \in C^*$. By (4.1) we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1+\beta_1+\zeta_1)+\lambda(\alpha_2+\gamma_2+\varepsilon_2))\phi_*(y^*,T^*y^*) \\
&\quad +(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi_*(T^*y^*,y^*) \\
&\quad +2((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\langle J^{-1}y^*-J^{-1}T^*y^*, z_0^*-y^*\rangle \\
&= ((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\phi_*(z_0^*,T^*y^*) \\
&\quad -((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))\phi_*(z_0^*,y^*) \\
&\quad +((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi_*(y^*,T^*y^*)+(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi_*(T^*y^*,y^*) \\
&\le 0.
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&((1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))(\phi_*(z_0^*,T^*y^*)-\phi_*(z_0^*,y^*)) \\
&\quad \le -((1-\lambda)\zeta_1+\lambda\varepsilon_2)\phi_*(y^*,T^*y^*)-(\lambda\varepsilon_1+(1-\lambda)\zeta_2)\phi_*(T^*y^*,y^*)
\end{aligned}$$

and hence z_0^* is a $(-((1-\lambda)\zeta_1+\lambda\varepsilon_2), -(\lambda\varepsilon_1+(1-\lambda)\zeta_2), (1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2))$-generalized *-acute point. □

By Theorem 4.2 we obtain the following.

Theorem 4.3. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded and suppose that there exists $\lambda \in [0,1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2)+\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) \ge 0; \\
&\lambda(\alpha_2+\gamma_2)+(1-\lambda)(\alpha_1+\beta_1) \ge 0; \\
&\lambda(\beta_2+\delta_2)+(1-\lambda)(\gamma_1+\delta_1) \ge 0; \\
&(1-\lambda)\varepsilon_2+\lambda\zeta_1 \ge 0; \\
&\lambda\zeta_2+(1-\lambda)\varepsilon_1 \ge 0.
\end{aligned}$$

Then there exists a $(-(\lambda\varepsilon_2+(1-\lambda)\zeta_1), -((1-\lambda)\zeta_2+\lambda\varepsilon_1), (1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1))$-generalized skew-acute point.

Proof. Let $T^* = JTJ^{-1}$. By Lemma 2.4 T^* is an $(\alpha_2, \alpha_1, \beta_2, \beta_1, \gamma_2, \gamma_1, \delta_2, \delta_1, \varepsilon_2, \varepsilon_1, \zeta_2, \zeta_1)$-*-generalized pseudocontraction from $J(C)$ into itself. By Theorem 4.2, T^* has a $(-((1-\lambda)\zeta_2+\lambda\varepsilon_1), -(\lambda\varepsilon_2+(1-\lambda)\zeta_1), (1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1))$-generalized *-acute point $z_0^* \in \overline{co}\{(T^*)^n Jz \mid n \in \mathbb{N}\}$. By Lemma 3.5 $J^{-1}z_0^*$ is a $(-(\lambda\varepsilon_2+(1-\lambda)\zeta_1), -((1-\lambda)\zeta_2+\lambda\varepsilon_1), (1-\lambda)(\alpha_2+\beta_2)+\lambda(\alpha_1+\gamma_1))$-generalized skew-acute point of T. □

Remark 4.1. In the proof by using the concept of acute or skew-acute point we needed the assumption $(1-\lambda)(\alpha_1+\beta_1)+\lambda(\alpha_2+\gamma_2) > 0$ or $(1-\lambda)(\alpha_2+\beta_2)+$

$\lambda(\alpha_1 + \gamma_1) > 0$ in addition to

$$\begin{aligned}
&(1-\lambda)(\alpha_1 + \beta_1 + \gamma_1 + \delta_1) + \lambda(\alpha_2 + \beta_2 + \gamma_2 + \delta_2) \geq 0;\\
&\lambda(\alpha_1 + \gamma_1) + (1-\lambda)(\alpha_2 + \beta_2) \geq 0;\\
&\lambda(\beta_1 + \delta_1) + (1-\lambda)(\gamma_2 + \delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1 + \lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1 + (1-\lambda)\varepsilon_2 \geq 0,
\end{aligned}$$

or

$$\begin{aligned}
&(1-\lambda)(\alpha_2 + \beta_2 + \gamma_2 + \delta_2) + \lambda(\alpha_1 + \beta_1 + \gamma_1 + \delta_1) \geq 0;\\
&\lambda(\alpha_2 + \gamma_2) + (1-\lambda)(\alpha_1 + \beta_1) \geq 0;\\
&\lambda(\beta_2 + \delta_2) + (1-\lambda)(\gamma_1 + \delta_1) \geq 0;\\
&(1-\lambda)\varepsilon_2 + \lambda\zeta_1 \geq 0;\\
&\lambda\zeta_2 + (1-\lambda)\varepsilon_1 \geq 0;
\end{aligned}$$

see [14]. However, by using the concept of generalized acute and skew-acute point we do not need the assumptions $(1-\lambda)(\alpha_1 + \beta_1) + \lambda(\alpha_2 + \gamma_2) > 0$ and $(1-\lambda)(\alpha_2 + \beta_2) + \lambda(\alpha_1 + \gamma_1) > 0$.

5. Fixed point theorems

In this section we show that fixed point theorems, introduced in [14], are derived from generalized acute and skew-acute point theorems.

Theorem 5.1. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty, closed and convex subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_1 + \beta_1 + \gamma_1 + \delta_1) + \lambda(\alpha_2 + \beta_2 + \gamma_2 + \delta_2) \geq 0;\\
&\lambda(\alpha_1 + \gamma_1) + (1-\lambda)(\alpha_2 + \beta_2) \geq 0;\\
&\lambda(\beta_1 + \delta_1) + (1-\lambda)(\gamma_2 + \delta_2) \geq 0;\\
&(1-\lambda)\varepsilon_1 + \lambda\zeta_2 \geq 0;\\
&\lambda\zeta_1 + (1-\lambda)\varepsilon_2 \geq 0
\end{aligned}$$

and suppose that one of the following holds:

(1) $(1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2) > 0$ *and* $\lambda\varepsilon_1 + (1-\lambda)\zeta_2 \geq 0$*;*
(2) $(1-\lambda)(\alpha_1 + \beta_1 + \zeta_1) + \lambda(\alpha_2 + \gamma_2 + \varepsilon_2) \geq 0$ *and* $\lambda\varepsilon_1 + (1-\lambda)\zeta_2 > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Furthermore, if $(1-\lambda)(\alpha_1 + \beta_1 + \gamma_1 + \delta_1) + \lambda(\alpha_2 + \beta_2 + \gamma_2 + \delta_2) > 0$ or $\lambda(\alpha_1 + \beta_1 + \gamma_1 + \delta_1) + (1-\lambda)(\alpha_2 + \beta_2 + \gamma_2 + \delta_2) > 0$, then the fixed point of T is unique.

Proof. If T has a fixed point z, then $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. By Theorem 4.1 there exits $z_0 \in \overline{co}\{T^n z \mid n \in \mathbb{N}\} \subset C$ such that it is a $(-((1-\lambda)\zeta_1 + \lambda\varepsilon_2), -(\lambda\varepsilon_1 + (1-\lambda)\zeta_2), (1-\lambda)(\alpha_1+\beta_1) + \lambda(\alpha_2+\gamma_2))$-generalized acute point. Furthermore, if (1) holds, then $-((1-\lambda)\zeta_1 + \lambda\varepsilon_2) < (1-\lambda)(\alpha_1+\beta_1) + \lambda(\alpha_2+\gamma_2)$ and $-(\lambda\varepsilon_1 + (1-\lambda)\zeta_2) \leq 0$; if (2) holds, then $-((1-\lambda)\zeta_1 + \lambda\varepsilon_2) \leq (1-\lambda)(\alpha_1+\beta_1) + \lambda(\alpha_2+\gamma_2)$ and $-(\lambda\varepsilon_1 + (1-\lambda)\zeta_2) < 0$. By Lemma 3.6 (1) z_0 is a fixed point of T.

Suppose that z_1 and z_2 are fixed points of T. Then we obtain

$$\begin{aligned}
&((1-\lambda)\alpha_1 + \lambda\alpha_2)\phi(Tz_1, Tz_2) + (\lambda\alpha_1 + (1-\lambda)\alpha_2)\phi(Tz_2, Tz_1) \\
&\quad +((1-\lambda)\beta_1 + \lambda\gamma_2)\phi(z_1, Tz_2) + (\lambda\gamma_1 + (1-\lambda)\beta_2)\phi(Tz_2, z_1) \\
&\quad +((1-\lambda)\gamma_1 + \lambda\beta_2)\phi(Tz_1, z_2) + (\lambda\beta_1 + (1-\lambda)\gamma_2)\phi(z_2, Tz_1) \\
&\quad +((1-\lambda)\delta_1 + \lambda\delta_2)\phi(z_1, z_2) + (\lambda\delta_1 + (1-\lambda)\delta_2)\phi(z_2, z_1) \\
&\quad +((1-\lambda)\varepsilon_1 + \lambda\zeta_2)\phi(Tz_1, z_1) + (\lambda\zeta_1 + (1-\lambda)\varepsilon_2)\phi(z_1, Tz_1) \\
&\quad +((1-\lambda)\zeta_1 + \lambda\varepsilon_2)\phi(z_2, Tz_2) + (\lambda\varepsilon_1 + (1-\lambda)\zeta_2)\phi(Tz_2, z_2) \\
&= ((1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1) + \lambda(\alpha_2+\beta_2+\gamma_2+\delta_2))\phi(z_1, z_2) \\
&\quad +(\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) + (1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2))\phi(z_2, z_1) \\
&\leq 0.
\end{aligned}$$

By assumption $(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1) + \lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) \geq 0$ and $\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) + (1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2) \geq 0$ hold. Furthermore, if $(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1) + \lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) > 0$ or $\lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) + (1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2) > 0$, then by (T9) we obtain $z_1 = z_2$ and hence the fixed point of T is unique. □

Theorem 5.2. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty subset of E satisfying $J(C)$ is closed and convex and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$-generalized pseudocontraction from C into itself. Suppose that there exists $\lambda \in [0, 1]$ such that*

$$\begin{aligned}
&(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2) + \lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) \geq 0; \\
&\lambda(\alpha_2+\gamma_2) + (1-\lambda)(\alpha_1+\beta_1) \geq 0; \\
&\lambda(\beta_2+\delta_2) + (1-\lambda)(\gamma_1+\delta_1) \geq 0; \\
&(1-\lambda)\varepsilon_2 + \lambda\zeta_1 \geq 0; \\
&\lambda\zeta_2 + (1-\lambda)\varepsilon_1 \geq 0
\end{aligned}$$

and suppose that one of the following holds:

(1) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2) + \lambda(\alpha_1+\gamma_1+\varepsilon_1) > 0$ *and* $\lambda\varepsilon_2 + (1-\lambda)\zeta_1 \geq 0$*;*

(2) $(1-\lambda)(\alpha_2+\beta_2+\zeta_2) + \lambda(\alpha_1+\gamma_1+\varepsilon_1) \geq 0$ *and* $\lambda\varepsilon_2 + (1-\lambda)\zeta_1 > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Furthermore, if $(1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2) + \lambda(\alpha_1+\beta_1+\gamma_1+\delta_1) > 0$ or $\lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) + (1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1) > 0$, then the fixed point of T is unique.

Proof. If T has a fixed point z, then $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. By Theorem 4.3 there exits $z_0 \in J^{-1}(\overline{co}\{(T^*)^n Jz \mid n \in \mathbb{N}\}) \subset C$ such that it is a $(-(\lambda\varepsilon_2 + (1-\lambda)\zeta_1), -((1-\lambda)\zeta_2 + \lambda\varepsilon_1), (1-\lambda)(\alpha_2+\beta_2) + \lambda(\alpha_1+\gamma_1))$-generalized skew-acute point. Furthermore, if (1) holds, then $-(\lambda\varepsilon_2 + (1-\lambda)\zeta_1) \leq 0$ and $-((1-\lambda)\zeta_2 + \lambda\varepsilon_1) < (1-\lambda)(\alpha_2+\beta_2) + \lambda(\alpha_1+\gamma_1)$; if (2) holds, then $-(\lambda\varepsilon_2 + (1-\lambda)\zeta_1) < 0$ and $-((1-\lambda)\zeta_2 + \lambda\varepsilon_1) \leq (1-\lambda)(\alpha_2+\beta_2) + \lambda(\alpha_1+\gamma_1)$. By Lemma 3.6 (2) z_0 is a fixed point of T.

Furthermore, if $(1-\lambda)(\alpha_1+\beta_1+\gamma_1+\delta_1) + \lambda(\alpha_2+\beta_2+\gamma_2+\delta_2) > 0$ or $\lambda(\alpha_1 + \beta_1+\gamma_1+\delta_1) + (1-\lambda)(\alpha_2+\beta_2+\gamma_2+\delta_2) > 0$, then we can show similarly to the proof of Theorem 5.1 that the fixed point of T is unique. □

References

[1] S. Atsushiba, S. Iemoto, R. Kubota, and Y. Takeuchi, *Convergence theorems for some classes of nonlinear mappings in Hilbert spaces*, Linear and Nonlinear Analysis **2** (2016), 125–154.

[2] J.-B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. Séries A et B **280** (1975), 1511–1514.

[3] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, Journal of Mathematical Analysis and Applications **20** (1967), 197–226.

[4] D. Butnariu, A. N. Iusem, and E. Resmerita, *Total convexity for powers of the norm in uniformly convex Banach spaces*, Journal of Convex Analysis **7** (2000), 319–334.

[5] I. Ciorănescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Mathematics and its applications, vol. 62, Kluwer Academic Publishers, Dordrecht, 1990.

[6] T. Kawasaki, *An extension of existence and mean approximation of fixed points of generalized hybrid non-self mappings in Hilbert spaces*, preprint.

[7] T. Kawasaki, *Fixed points theorems and mean convergence theorems for generalized hybrid self mappings and non-self mappings in Hilbert spaces*, Pacific Journal of Optimization **12** (2016), 133–150.

[8] T. Kawasaki, *Fixed point theorem for widely more generalized hybrid demicontinuous mappings in Hilbert spaces*, Proceedings of Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, to appear.

[9] T. Kawasaki, *Fixed point theorems for widely more generalized hybrid mappings in metric spaces, Banach spaces and Hilbert spaces*, Journal of Nonlinear and Convex Analysis **19** (2018), 1675–1683.

[10] T. Kawasaki, *On convergence of orbits to a fixed point for widely more generalized hybrid mappings*, Nihonkai Mathematical Journal **27** (2016), 89–97.

[11] T. Kawasaki, *Fixed point theorems for widely more generalized hybrid mappings in a metric space, a Banach space and a Hilbert space*, Journal of Nonlinear and Convex Analysis **19** (2018), 1675–1683.

[12] T. Kawasaki, *Fixed point and acute point theorems for new mappings in a Banach space*, Journal of Mathematics **2019** (2019), 12 pages.

[13] T. Kawasaki, *Mean convergence theorems for new mappings in a Banach space*, Journal of Nonlinear and Variational Analysis **3** (2019), 61–78.

[14] T. Kawasaki, *Fixed point and acute point theorems for generalized pseudocontractions in a Banach space*, Proceedings of Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, submitted.

[15] T. Kawasaki, *Weak convergence theorems for new mappings in a Banach space*, Linear and Nonlinear Analysis **5** (2019), 147–171.

[16] T. Kawasaki and T. Kobayashi, *Existence and mean approximation of fixed points of generalized hybrid non-self mappings in Hilbert spaces*, Scientiae Mathematicae Japonicae **77 (Online Version: e-2014)** (2014), 13–26 (Online Version: 29–42).

[17] T. Kawasaki and W. Takahashi, *Existence and mean approximation of fixed points of generalized hybrid mappings in Hilbert spaces*, Journal of Nonlinear and Convex Analysis **14** (2013), 71–87.

[18] T. Kawasaki and W. Takahashi, *Fixed point and nonlinear ergodic theorems for widely more generalized hybrid mappings in Hilbert spaces and applications*, Proceedings of Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, to appear.

[19] T. Kawasaki and W. Takahashi, *Fixed point theorems for generalized hybrid demicontinuous mappings in Hilbert spaces*, Linear and Nonlinear Analysis **1** (2015), 125–138.

[20] P. Kocourek, W. Takahashi, and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese Journal of Mathematics **14** (2010), 2497–2511.

[21] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Archiv der Mathematik **91** (2008), 166–177.

[22] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[23] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, Journal of Nonlinear and Convex Analysis **11** (2010), 79–88.

[24] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, Journal of Nonlinear and Convex Analysis **12** (2011), 399–406.

[25] W. Takahashi, N.-C. Wong, and J.-C. Yao, *Attractive point and mean convergence theorems for new generalized nonspreading mappings in Banach spaces*, Infinite Products of Operators and Their Applications (S. Reich and A. J. Zaslavski, eds.), Contemporary Mathematics, vol. 636, American Mathematical Society, Providence, 2015, pp. 225–248.

Manuscript received 15 January 2020
revised 22 March 2020

Toshiharu Kawasaki
College of Engineering, Nihon University, Fukushima 963–8642, Japan; Faculty of Engineering, Tamagawa University, Tokyo 194–8610, Japan
E-mail address: `toshiharu.kawasaki@nifty.ne.jp`