Linear and Nonlinear Analysis
Volume 6, Number 1, 2020, 91–101

Yokohama Publishers
ISSN 2188-8167

HALPERN ITERATION WITH TWO KINDS OF MAPPINGS IN A COMPLETE GEODESIC SPACE WITH CURVATURE BOUNDED ABOVE

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Abstract. In this paper, we show a strong convergence theorem for a Halpern iteration procedure with two kinds of mappings. The proposed method is different from known procedures with respect to the order of taking the convex combination.

1. Introduction

The fixed point approximation has been studied in a variety of ways and its results are useful for the other studies. In 1967, Halpern [4] introduced an iteration procedure for approximating fixed points of a nonexpansive mapping. In 1992, Wittmann [14] obtained that a Halpern iteration with nonexpansive mapping converges strongly to a fixed point in a Hilbert space. Later, this iteration method has been extended to Banach spaces and Hilbert spaces by many mathematicians; see [9, 12] for instance. In 2010, Saejung [10] introduced this iteration in CAT(0) spaces. In 2011, Piątek [8] proposed strong convergence theorems with single mapping in CAT(1) spaces. In 2013, Kimura and Satô [7] supplemented its proof and completed it. They also proved the convergence theorems in CAT(1) spaces with two mappings.

On the other hand, in 2016, Wada [13] proved the strong convergence theorem for two kinds of mappings in a complete CAT(0) space:

Theorem 1.1. *Let X be a complete* CAT(0) *space. Let R be a nonexpansive mapping from X into itself and S, T strongly quasinonexpansive and Δ-demiclosed mappings from X into itself with $F = F(R) \cap F(S) \cap F(T) \neq \emptyset$. For $r \in \,]0, 1/2[$, let $\{\alpha_n\} \subset [r, 1-r] \subset \,]0,1[$, $\{\beta_n\}, \{\gamma_n\} \subset \,]0,1[$ be real sequences satisfying $\beta_n \to \beta \in \,]0,1[$, $\gamma_n \to 0$, and $\sum_{n=1}^{\infty} \gamma_n = \infty$. Define $\{x_n\} \subset X$ by $x_1, u \in X$ and*

$$\begin{cases} s_n = \gamma_n u \oplus (1-\gamma_n) S x_n, \\ t_n = \gamma_n u \oplus (1-\gamma_n) T x_n, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) R(\beta_n s_n \oplus (1-\beta_n) t_n) \end{cases}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to a point in F.

In this paper, we show this result in the setting of complete CAT(1) spaces.

2010 *Mathematics Subject Classification.* 47H09.

Key words and phrases. Fixed point, strong convergence, Halpern type, geodesic space, quasinonexpansive mapping.

2. Preliminaries

Let X be a metric space. For $x, y \in X$, a mapping $c : [0, l] \to X$ is said to be a geodesic if c satisfies $c(0) = x, c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, l]$. The image $[x, y]$ of c is called a geodesic segment joining x and y. For $r > 0$, X is said to be an r-geodesic metric space if, for any $x, y \in X$ with $d(x, y) < r$, there exists a geodesic segment $[x, y]$. In particular, if a segment $[x, y]$ is unique for any $x, y \in X$ with $d(x, y) < r$, then X is said to be a uniquely r-geodesic metric space. In what follows, we always assume $d(x, y) < \pi/2$ for any $x, y \in X$. Thus, we say X is a geodesic metric space instead of a $\pi/2$-geodesic metric space. For more general cases, see [2].

Let X be a uniquely geodesic metric space. A geodesic triangle is defined by $\triangle(x, y, z) = [x, y] \cup [y, z] \cup [z, x]$. Let M be the two-dimensional unit sphere in $\mathbb{R}^3$. For $\bar{x}, \bar{y}, \bar{z} \in M$, a triangle $\triangle(\bar{x}, \bar{y}, \bar{z}) \subset M$ is called a comparison triangle of $\triangle(x, y, z)$ if $d(x, y) = d_M(\bar{x}, \bar{y})$, $d(y, z) = d_M(\bar{y}, \bar{z})$, and $d(z, x) = d_M(\bar{z}, \bar{x})$. Further, for any $x, y \in X$ and $t \in \,]0, 1[$, if $z \in [x, y]$ satisfies $d(x, z) = (1 - t)d(x, y)$ and $d(z, y) = td(x, y)$, then z is denoted by $z = tx \oplus (1 - t)y$. A point $\bar{z} \in [\bar{x}, \bar{y}]$ is called a comparison point of $z \in [x, y]$ if $d(x, z) = d_M(\bar{x}, \bar{z})$. X is said to be a CAT(1) space if, for any $x, y, z \in X$ and $p, q \in \triangle(x, y, z) \subset X$ and their comparison points $\bar{p}, \bar{q} \in \triangle(\bar{x}, \bar{y}, \bar{z}) \subset M$, the inequality $d(p, q) \leq d_M(\bar{p}, \bar{q})$ holds.

Let X be a geodesic metric space and $\{x_n\}$ a bounded sequence of X. For $x \in X$, we put $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius of $\{x_n\}$ is defined by $r(\{x_n\}) = \inf_{x\in X} r(x, \{x_n\})$. Further, the asymptotic center of $\{x_n\}$ is defined by $AC(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$. If, for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $AC(\{x_{n_k}\}) = \{x_0\}$, *i.e.*, their asymptotic center consists of the unique element x_0, then we say $\{x_n\}$ Δ-converges to x_0 and we denote it by $x_n \xrightarrow{\Delta} x_0$.

Let X be a metric space. A mapping $T : X \to X$ is said to be nonexpansive if T satisfies $d(Tx, Ty) \leq d(x, y)$ for any $x, y \in X$. The set of fixed points of T is denoted by $F(T) = \{z \in X : Tz = z\}$. Further, a mapping $T : X \to X$ with $F(T) \neq \emptyset$ is said to be strongly quasinonexpansive if, for any $x \in X$ and $z \in F(T)$, $d(Tx, z) \leq d(x, z)$ and if $d(x_n, Tx_n) \to 0$ for any bounded sequence $\{x_n\} \subset X$ and $z \in F(T)$ satisfying $\lim_{n\to\infty} \cos d(x_n, z)/ \cos d(Tx_n, z) = 1$. Moreover, T is said to be Δ-demiclosed if $x_0 \in F(T)$ for any bounded sequence $\{x_n\} \subset X$ and $x_0 \in X$ satisfying $d(x_n, Tx_n) \to 0$ and $x_n \xrightarrow{\Delta} x_0$. Let C be a nonempty closed convex subset of X. A mapping P_C is said to be metric projection from X onto C if P_C satisfies $d(u, P_C u) = \inf_{x\in C} d(u, x)$ for any $u \in X$.

In a Hilbert space, such P_C is also nonexpansive. On the other hand, in a complete CAT(1) space, P_C is not nonexpansive in general. It is strongly quasinonexpansive and Δ-demiclosed in a complete CAT(1) space. Thus, P_C is an example which our main result can be applied.

3. Tools for the main result

In this section, we introduce some tools for the main theorem.

Theorem 3.1 (Kimura and Satô [6]). *Let $\triangle(x,y,z)$ be a geodesic triangle in a* CAT(1) *space such that $d(x,y)+d(y,z)+d(z,x)<2\pi$. Let $u=tx\oplus(1-t)y$ for some $t\in[0,1]$. Then*

$$\cos d(u,z)\sin d(x,y)\geq\cos d(x,z)\sin td(x,y)+\cos d(y,z)\sin(1-t)d(x,y).$$

Corollary 3.2 (Kimura and Satô [7]). *Let $\triangle(x,y,z)$ be a geodesic triangle in a* CAT(1) *space such that $d(x,y)+d(y,z)+d(z,x)<2\pi$. Let $u=tx\oplus(1-t)y$ for some $t\in[0,1]$. Then*

$$\cos d(u,z)\geq t\cos d(x,z)+(1-t)\cos d(y,z).$$

Theorem 3.3 (Espínola and Fernández-León [3]). *Let X be a complete* CAT(1) *space and $\{x_n\}$ a sequence in X. If $r(\{x_n\})<\pi/2$, then the following hold.*

(*i*) *$AC(\{x_n\})$ consists of exactly one point;*
(*ii*) *$\{x_n\}$ has a Δ-convergent subsequence.*

Lemma 3.4 (Aoyama, Kimura, and Kohsaka [1], Saejung and Yotkaew [11]). *Let $\{a_n\}\subset[0,\infty[$, $\{d_n\}\subset\mathbb{R}$ and $\{\gamma_n\}\subset\,]0,1[$ such that $\sum_{n=1}^{\infty}\gamma_n=\infty$. Define a set $\Phi=\{\varphi:\mathbb{N}\to\mathbb{N},$ nondecreasing and $\lim_{i\to\infty}\varphi(i)=\infty\}$. Suppose that*

$$a_{n+1}\leq(1-\gamma_n)a_n+\gamma_n d_n$$

for any $n\in\mathbb{N}$. If $\overline{\lim}_{i\to\infty}d_{\varphi(i)}\leq 0$ for any $\varphi\in\Phi$ satisfying $\underline{\lim}_{i\to\infty}(a_{\varphi(i)+1}-a_{\varphi(i)})\geq 0$, then $\lim_{n\to\infty}a_n=0$.

Lemma 3.5 (Kimura and Satô [7]). *Let X be a complete* CAT(1) *space such that $d(v,v')<\pi/2$ for every $v,v'\in X$. Let $\alpha\in[0,1]$ and $u,y,z\in X$. Then*

$$\begin{aligned}&1-\cos d(\alpha u\oplus(1-\alpha)y,z)\\&\quad\leq(1-\beta)(1-\cos d(y,z))+\beta\left(1-\frac{\cos d(u,z)}{\sin d(u,y)\tan\left(\frac{\alpha}{2}d(u,y)\right)+\cos d(u,y)}\right),\end{aligned}$$

where

$$\beta=\begin{cases}1-\dfrac{\sin((1-\alpha)d(u,y))}{\sin d(u,y)} & (u\neq y),\\ \alpha & (u=y).\end{cases}$$

Lemma 3.6 (He, Fang, Lopez, and Li [5]). *Let X be a complete* CAT(1) *space and $u\in X$. If a sequence $\{x_n\}$ in X satisfies that $\overline{\lim}_{n\to\infty}d(u,x_n)<\pi/2$ and $x_n\stackrel{\Delta}{\rightharpoonup}x\in X$, then*

$$\underline{\lim}_{n\to\infty}d(u,x_n)\geq d(u,x).$$

Lemma 3.7 (Kimura and Satô [7]). *Let X be a* CAT(1) *space with $d(u,v)<\pi/2$ for any $u,v\in X$. Let S and T be strongly quasinonexpansive and Δ-demiclosed mappings from X into itself with $F(S)\cap F(T)\neq\emptyset$. For $\beta\in\,]0,1[$, put $U=\beta S\oplus(1-\beta)T$. Then the following conditions hold;*

- *$F(U)=F(S)\cap F(T)$,*
- *U is strongly quasinonexpansive,*

- *U is Δ-demiclosed.*

Lemma 3.8 (Kimura and Satô [7])**.** *Let X be a* CAT(1) *space with $d(u,v) < \pi/2$ for any $u, v \in X$ and $\triangle(z,x,y) \subset X$. For $t \in \,]0,1[$, put $u = tz \oplus (1-t)x$ and $v = tz \oplus (1-t)y$. Then the following inequality holds:*

$$d(u,v) \leq \frac{\sin(1-t)M}{\sin M} d(x,y),$$

where $M = \sup_{p,q \in X} d(p,q) < \pi/2$.

Lemma 3.9. *Let $\{s_n\}$ and $\{t_n\} \subset \,]-\infty, 0]$. Suppose that $\lim_{n\to\infty}(s_n + t_n) = 0$. Then $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = 0$.*

Lemma 3.10. *Let $\{s_n\}$ and $\{t_n\} \subset [0, \infty[$. Then*

$$\overline{\lim_{n\to\infty}} (s_n \cdot t_n) \geq \overline{\lim_{n\to\infty}} s_n \cdot \underline{\lim_{n\to\infty}} t_n.$$

Lemma 3.11. *Let $\{D_n\} \subset \,]0, \pi/2[$ and $t \in \,]0,1[$. Suppose*

$$\lim_{n\to\infty} \frac{\sin(tD_n) + \sin((1-t)D_n)}{\sin D_n} = 1.$$

Then $\lim_{n\to\infty} D_n = 0$.

Proof. Assume $\{D_n\}$ does not converge to 0, *i.e.*, there exist a subsequence $\{D_{n_i}\} \subset \{D_n\}$ and $D_0 > 0$ such that $D_{n_i} \to D_0$. Then we get

$$\begin{aligned} 1 &= \lim_{n\to\infty} \frac{\sin(tD_n) + \sin((1-t)D_n)}{\sin D_n} \\ &= \lim_{i\to\infty} \frac{\sin(tD_{n_i}) + \sin((1-t)D_{n_i})}{\sin D_{n_i}} \\ &= \frac{\sin(tD_0) + \sin((1-t)D_0)}{\sin D_0}. \end{aligned}$$

Further, we have

$$\begin{aligned} \sin D_0 &= \sin(tD_0) + \sin((1-t)D_0) \\ &= 2\sin\frac{D_0}{2}\cos\frac{(2t-1)D_0}{2} \\ &= 2\sin\frac{D_0}{2}\cos\frac{|1-2t|D_0}{2}. \end{aligned}$$

Since $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$ and $D_0 > 0$, we obtain

$$\cos\frac{D_0}{2} = \cos\frac{|1-2t|D_0}{2}.$$

Then we get

$$\frac{D_0}{2} = \frac{|1-2t|D_0}{2},$$

and we have

$$(1 - |1-2t|)D_0 = 0.$$

Since $t \in \,]0,1[$, we have $1 - |1-2t| > 0$ and hence $D_0 = 0$. This is a contradiction. Thus, we obtain $\{D_n\}$ converges to 0. ∎

4. The main result

In this section, we show the main result.

Theorem 4.1. *Let X be a complete* CAT(1) *space with $M = \sup_{p,q\in X} d(p,q) < \pi/2$. Let R be a nonexpansive mapping from X into itself and S, T strongly quasinonexpansive and Δ-demiclosed mappings from X into itself with $F = F(R) \cap F(S) \cap F(T) \neq \emptyset$. For $r \in]0, 1/2[$, let $\{\alpha_n\} \subset [r, 1-r] \subset]0, 1[$, $\{\beta_n\}, \{\gamma_n\} \subset]0, 1[$ be real sequences satisfying $\beta_n \to \beta \in]0, 1[$, $\gamma_n \to 0$, and $\sum_{n=1}^{\infty} \gamma_n = \infty$. Define $\{x_n\} \subset X$ by $x_1, u \in X$ and*

$$\begin{cases} s_n = \gamma_n u \oplus (1-\gamma_n) S x_n, \\ t_n = \gamma_n u \oplus (1-\gamma_n) T x_n, \\ y_n = \beta_n s_n \oplus (1-\beta_n) t_n, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) R y_n \end{cases}$$

for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges to $P_F u \in F$.

Proof. Let $p = P_F u \in F$ and put

$$\begin{aligned}
a_n &= 1 - \cos d(x_n, p), \\
b_n &= 1 - \frac{\cos d(u,p)}{\sin d(u, Sx_n) \tan\left(\frac{\gamma_n}{2} d(u, Sx_n)\right) + \cos d(u, Sx_n)}, \\
c_n &= 1 - \frac{\cos d(u,p)}{\sin d(u, Tx_n) \tan\left(\frac{\gamma_n}{2} d(u, Tx_n)\right) + \cos d(u, Tx_n)}, \\
\sigma_n &= \begin{cases} 1 - \dfrac{\sin(1-\gamma_n) d(u, Sx_n)}{\sin d(u, Sx_n)} & (u \neq Sx_n), \\ \gamma_n & (u = Sx_n), \end{cases} \\
\tau_n &= \begin{cases} 1 - \dfrac{\sin(1-\gamma_n) d(u, Tx_n)}{\sin d(u, Tx_n)} & (u \neq Tx_n), \\ \gamma_n & (u = Tx_n) \end{cases}
\end{aligned}$$

for $n \in \mathbb{N}$. Then, by Lemma 3.5 and Corollary 3.2, we have

$$\begin{aligned}
a_{n+1} &= 1 - \cos d(x_{n+1}, p) \\
&\leq 1 - (\alpha_n \cos d(x_n, p) + (1-\alpha_n) \cos d(Ry_n, p)) \\
&\leq \alpha_n a_n + (1-\alpha_n)(1 - \cos d(y_n, p)) \\
&\leq \alpha_n a_n + (1-\alpha_n)(1 - (\beta_n \cos d(s_n, p) + (1-\beta_n) \cos d(t_n, p))) \\
&\leq \alpha_n a_n + (1-\alpha_n)(\beta_n((1-\sigma_n) a_n + \sigma_n b_n) \\
&\quad + (1-\beta_n)((1-\tau_n) a_n + \tau_n c_n)) \\
&= (\alpha_n + (1-\alpha_n)(\beta_n(1-\sigma_n) + (1-\beta_n)(1-\tau_n))) a_n \\
&\quad + (1-\alpha_n)(\beta_n \sigma_n b_n + (1-\beta_n) \tau_n c_n) \\
&= (1 - (1-\alpha_n)(\beta_n \sigma_n + (1-\beta_n)\tau_n)) a_n \\
&\quad + (1-\alpha_n)(\beta_n \sigma_n + (1-\beta_n)\tau_n) \frac{\beta_n \sigma_n b_n + (1-\beta_n)\tau_n c_n}{\beta_n \sigma_n + (1-\beta_n)\tau_n}.
\end{aligned}$$

To apply Lemma 3.4 to our sequence, we will show the following:

(i) $\sum_{n=1}^{\infty}(1-\alpha_n)(\beta_n\sigma_n+(1-\beta_n)\tau_n)=\infty$,

(ii) $\overline{\lim_{i\to\infty}} \dfrac{\beta_{\varphi(i)}\sigma_{\varphi(i)}b_{\varphi(i)}+(1-\beta_{\varphi(i)})\tau_{\varphi(i)}c_{\varphi(i)}}{\beta_{\varphi(i)}\sigma_{\varphi(i)}+(1-\beta_{\varphi(i)})\tau_{\varphi(i)}} \leq 0$
for any nondecreasing functions $\varphi : \mathbb{N} \to \mathbb{N}$ satisfying $\lim_{i\to\infty}\varphi(i) = \infty$ and $\underline{\lim}_{i\to\infty}(a_{\varphi(i)+1}-a_{\varphi(i)}) \geq 0$.

First, we show (i). If $u = Sx_n$, then we have $\sigma_n = \gamma_n \geq \gamma_n \cos M$. For any $n \in \mathbb{N}$ satisfying $d(u, Sx_n) > 0$, we get

$$
\begin{aligned}
\sigma_n &= \frac{\sin d(u,Sx_n)-\sin(1-\gamma_n)d(u,Sx_n)}{\sin d(u,Sx_n)} \\
&= \frac{2}{\sin d(u,Sx_n)}\sin\left(\frac{\gamma_n}{2}d(u,Sx_n)\right)\cos\left(\left(1-\frac{\gamma_n}{2}\right)d(u,Sx_n)\right) \\
&\geq \gamma_n\cos\left(\left(1-\frac{\gamma_n}{2}\right)d(u,Sx_n)\right) \\
&\geq \gamma_n\cos M.
\end{aligned}
$$

Thus, we have $\sigma_n \geq \gamma_n \cos M$ for any $n \in \mathbb{N}$. We also get $\tau_n \geq \gamma_n \cos M$. Therefore, we obtain

$$
\begin{aligned}
(1-\alpha_n)(\beta_n\sigma_n+(1-\beta_n)\tau_n) &\geq r(\beta_n\gamma_n\cos M+(1-\beta_n)\gamma_n\cos M) \\
&\geq \gamma_n\cdot r\cos M
\end{aligned}
$$

for any $n \in \mathbb{N}$. Then (i) holds.

Next, we consider (ii). Let $n_i = \varphi(i)$ for any $i \in \mathbb{N}$. For any $i \in \mathbb{N}$ satisfying $\underline{\lim}_{i\to\infty}(a_{n_i+1}-a_{n_i}) \geq 0$, from Corollary 3.2 and the definition of strong quasinonexpansiveness, we have

$$
\begin{aligned}
0 &\leq \underline{\lim}_{i\to\infty}(a_{n_i+1}-a_{n_i}) \\
&= \underline{\lim}_{i\to\infty}(\cos d(x_{n_i},p)-\cos d(x_{n_i+1},p)) \\
&\leq \underline{\lim}_{i\to\infty}(\cos d(x_{n_i},p)-(\alpha_{n_i}\cos d(x_{n_i},p) \\
&\quad +(1-\alpha_{n_i})\cos d(Ry_{n_i},p))) \\
&= \underline{\lim}_{i\to\infty}((1-\alpha_{n_i})(\cos d(x_{n_i},p)-\cos d(Ry_{n_i},p)) \\
&\leq \underline{\lim}_{i\to\infty}(1-\alpha_{n_i})(\cos d(x_{n_i},p)-\cos d(y_{n_i},p)) \\
&\leq \underline{\lim}_{i\to\infty}(1-\alpha_{n_i})(\cos d(x_{n_i},p)-(\beta_{n_i}\cos d(s_{n_i},p) \\
&\quad +(1-\beta_{n_i})\cos d(t_{n_i},p))) \\
&= \underline{\lim}_{i\to\infty}(1-\alpha_{n_i})(\beta_{n_i}(\cos d(x_{n_i},p)-\cos d(s_{n_i},p)) \\
&\quad +(1-\beta_{n_i})(\cos d(x_{n_i},p)-\cos d(t_{n_i},p)))
\end{aligned}
$$

$$
\begin{aligned}
&\le \varliminf_{i\to\infty}(1-\alpha_{n_i})(\beta_{n_i}(\cos d(x_{n_i},p)-(\gamma_{n_i}\cos d(u,p)\\
&\quad+(1-\gamma_{n_i})\cos d(Sx_{n_i},p)))\\
&\quad+(1-\beta_{n_i})(\cos d(x_{n_i},p)-(\gamma_{n_i}\cos d(u,p)\\
&\quad+(1-\gamma_{n_i})\cos d(Tx_{n_i},p))))\\
&\le \varliminf_{i\to\infty}(1-\alpha_{n_i})(\beta(\cos d(x_{n_i},p)-\cos d(Sx_{n_i},p))\\
&\quad+(1-\beta)(\cos d(x_{n_i},p)-\cos d(Tx_{n_i},p)))\\
&\le \varlimsup_{i\to\infty}(1-\alpha_{n_i})(\beta(\cos d(x_{n_i},p)-\cos d(Sx_{n_i},p))\\
&\quad+(1-\beta)(\cos d(x_{n_i},p)-\cos d(Tx_{n_i},p)))\\
&\le 0.
\end{aligned}
$$

Hence we get

$$
\begin{aligned}
\lim_{i\to\infty}(1-\alpha_{n_i})(\beta(\cos d(x_{n_i},p)-\cos d(Sx_{n_i},p))&\\
+(1-\beta)(\cos d(x_{n_i},p)-\cos d(Tx_{n_i},p)))&=0.
\end{aligned}
$$

From Lemma 3.9, we have

$$\lim_{i\to\infty}(\cos d(x_{n_i},p)-\cos d(Sx_{n_i},p))=0 \text{ and } \lim_{i\to\infty}(\cos d(x_{n_i},p)-\cos d(Tx_{n_i},p))=0.$$

It follows that

$$\lim_{i\to\infty}\frac{\cos d(x_{n_i},p)}{\cos d(Sx_{n_i},p)}=\lim_{i\to\infty}\frac{\cos d(x_{n_i},p)}{\cos d(Tx_{n_i},p)}=1.$$

Since S and T are strongly quasinonexpansive, we get

$$\lim_{i\to\infty}d(x_{n_i},Sx_{n_i})=\lim_{i\to\infty}d(x_{n_i},Tx_{n_i})=0. \tag{4.1}$$

On the other hand, by Corollary 3.2, we have

$$
\begin{aligned}
\cos d(y_{n_i},p)&\ge \beta_{n_i}\cos d(s_{n_i},p)+(1-\beta_{n_i})\cos d(t_{n_i},p)\\
&\ge \beta_{n_i}(\gamma_{n_i}\cos d(u,p)+(1-\gamma_{n_i})\cos d(Sx_{n_i},p))\\
&\quad+(1-\beta_{n_i})(\gamma_{n_i}\cos d(u,p)+(1-\gamma_{n_i})\cos d(Tx_{n_i},p))\\
&\ge \gamma_{n_i}\cos d(u,p)+(1-\gamma_{n_i})\cos d(x_{n_i},p).
\end{aligned}
$$

To show $\lim_{i\to\infty}d(x_{n_i},Ry_{n_i})=0$, we may assume $d(x_{n_i},Ry_{n_i})>0$. Then, by Theorem 3.1 and Corollary 3.2 again, we get

$$
\begin{aligned}
&\cos d(x_{n_i+1},p)\sin d(x_{n_i},Ry_{n_i})\\
&\ge \cos d(x_{n_i},p)\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})+\cos d(Ry_{n_i},p)\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})\\
&\ge \cos d(x_{n_i},p)\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})+\cos d(y_{n_i},p)\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})\\
&\ge \cos d(x_{n_i},p)\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})\\
&\quad+(\gamma_{n_i}\cos d(u,p)+(1-\gamma_{n_i})\cos d(x_{n_i},p))\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})\\
&=\cos d(x_{n_i},p)(\sin\alpha_{n_i}d(x_{n_i},Ry_{n_i})+\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i}))\\
&\quad+\gamma_{n_i}\sin(1-\alpha_{n_i})d(x_{n_i},Ry_{n_i})(\cos d(u,p)-\cos d(x_{n_i},p)).
\end{aligned}
$$

Thus, we obtain

$$\begin{aligned}
1 &\geq \frac{\cos d(x_{n_i},p)}{\cos d(x_{n_i+1},p)} \cdot \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\quad + \gamma_{n_i} \cdot \frac{\sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \cdot \frac{1}{\cos d(x_{n_i+1},p)} \cdot (\cos d(u,p) - \cos d(x_{n_i},p)) \\
&\geq \frac{\cos d(x_{n_i},p)}{\cos d(x_{n_i+1},p)} \cdot \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\quad - \gamma_{n_i} \cdot \frac{\sin(1-r) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \cdot \frac{1}{\cos M} \cdot |\cos d(u,p) - \cos d(x_{n_i},p)| \\
&\geq \frac{\cos d(x_{n_i},p)}{\cos d(x_{n_i+1},p)} \cdot \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\quad - \gamma_{n_i} \cdot (1-r) \cdot \frac{d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \cdot \frac{\sin(1-r) d(x_{n_i}, Ry_{n_i})}{(1-r) d(x_{n_i}, Ry_{n_i})} \cdot \frac{1}{\cos M} \cdot 2
\end{aligned}$$

Since $\gamma_n \to 0$ and $\underline{\lim}_{i\to\infty}(\cos d(x_{n_i},p) - \cos d(x_{n_i+1},p)) \geq 0$, or equivalently,

$$\varliminf_{i\to\infty} \frac{\cos d(x_{n_i},p)}{\cos d(x_{n_i+1},p)} \geq 1,$$

by Lemma 3.10, we get

$$\begin{aligned}
1 &\geq \varlimsup_{i\to\infty} \left(\frac{\cos d(x_{n_i},p)}{\cos d(x_{n_i+1},p)} \cdot \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \right) \\
&\geq \varliminf_{i\to\infty} \frac{\cos d(x_{n_i},p)}{\cos d(x_{n_i+1},p)} \cdot \varlimsup_{i\to\infty} \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\geq \varlimsup_{i\to\infty} \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\geq \varliminf_{i\to\infty} \frac{\sin \alpha_{n_i} d(x_{n_i}, Ry_{n_i}) + \sin(1-\alpha_{n_i}) d(x_{n_i}, Ry_{n_i})}{\sin d(x_{n_i}, Ry_{n_i})} \\
&\geq 1.
\end{aligned}$$

Since $\{\alpha_{n_i}\} \subset [r, 1-r]$ and by Lemma 3.11, we have

$$\lim_{i\to\infty} d(x_{n_i}, Ry_{n_i}) = 0.$$

Let $U = \beta S \oplus (1-\beta)T$. Then U satisfies three conditions in Lemma 3.7 and from Lemma 3.8, we get

$$\begin{aligned}
&d(y_{n_i}, Ux_{n_i}) \\
&= d(\beta_{n_i} s_{n_i} \oplus (1-\beta_{n_i}) t_{n_i}, \beta S x_{n_i} \oplus (1-\beta) T x_{n_i}) \\
&\leq d(\beta_{n_i} s_{n_i} \oplus (1-\beta_{n_i}) t_{n_i}, \beta_{n_i} S x_{n_i} \oplus (1-\beta_{n_i}) t_{n_i}) \\
&\quad + d(\beta_{n_i} S x_{n_i} \oplus (1-\beta_{n_i}) t_{n_i}, \beta_{n_i} S x_{n_i} \oplus (1-\beta_{n_i}) T x_{n_i}) \\
&\qquad + d(\beta_{n_i} S x_{n_i} \oplus (1-\beta_{n_i}) T x_{n_i}, \beta S x_{n_i} \oplus (1-\beta) T x_{n_i})
\end{aligned}$$

$$\leq \frac{\sin \beta_{n_i} M}{\sin M} d(s_{n_i}, Sx_{n_i}) + \frac{\sin(1-\beta_{n_i})M}{\sin M} d(t_{n_i}, Tx_{n_i}) + |\beta_{n_i} - \beta| d(Sx_{n_i}, Tx_{n_i})$$
$$\leq \gamma_{n_i} \left(\frac{\sin \beta_{n_i} M}{\sin M} d(u, Sx_{n_i}) + \frac{\sin(1-\beta_{n_i})M}{\sin M} d(u, Tx_{n_i}) \right) + |\beta_{n_i} - \beta| d(Sx_{n_i}, Tx_{n_i}).$$

Hence we have $\lim_{i\to\infty} d(y_{n_i}, Uy_{n_i}) = 0$. Further, since

$$d(x_{n_i}, RUx_{n_i}) \leq d(x_{n_i}, Ry_{n_i}) + d(Ry_{n_i}, RUx_{n_i}) \leq d(x_{n_i}, Ry_{n_i}) + d(y_{n_i}, Ux_{n_i}),$$

we also get $\lim_{i\to\infty} d(x_{n_i}, RUx_{n_i}) = 0$. Moreover, since

$$d(x_{n_i}, p) \leq d(x_{n_i}, RUx_{n_i}) + d(RUx_{n_i}, p) \leq d(x_{n_i}, RUx_{n_i}) + d(Ux_{n_i}, p)$$

for $p \in F$, we obtain

$$\begin{aligned} 0 &\leq |\cos d(x_{n_i}, p) - \cos d(Ux_{n_i}, p)| \\ &= 2 \sin \frac{d(x_{n_i}, p) + d(Ux_{n_i}, p)}{2} \left| \sin \frac{d(x_{n_i}, p) - d(Ux_{n_i}, p)}{2} \right| \\ &\leq 2 \sin M \sin \frac{d(x_{n_i}, RUx_{n_i})}{2} \\ &\to 0 \end{aligned}$$

as $i \to \infty$. It follows $\lim_{i\to\infty} \cos d(x_{n_i}, p)/\cos d(Ux_{n_i}, p) = 1$. Since U is strongly quasinonexpansive, we get $\lim_{i\to\infty} d(x_{n_i}, Ux_{n_i}) = 0$. Thus, we have

$$d(x_{n_i}, Rx_{n_i}) \leq d(x_{n_i}, RUx_{n_i}) + d(Ux_{n_i}, x_{n_i}) \to 0 \tag{4.2}$$

as $i \to \infty$.

Furthermore, there exists a subsequence $\{x_{n_{i_j}}\} \subset \{x_{n_i}\}$ such that

$$\overline{\lim_{i\to\infty}} \frac{\beta_{n_i}\sigma_{n_i}b_{n_i} + (1-\beta_{n_i})\tau_{n_i}c_{n_i}}{\beta_{n_i}\sigma_{n_i} + (1-\beta_{n_i})\tau_{n_i}} = \lim_{j\to\infty} \frac{\beta_{n_{i_j}}\sigma_{n_{i_j}}b_{n_{i_j}} + (1-\beta_{n_{i_j}})\tau_{n_{i_j}}c_{n_{i_j}}}{\beta_{n_{i_j}}\sigma_{n_{i_j}} + (1-\beta_{n_{i_j}})\tau_{n_{i_j}}}.$$

From Theorem 3.3(ii), there exists a subsequence $\{z_k\} \subset \{x_{n_{i_j}}\}$ such that it Δ-converges to $x_0 \in X$. Then from the definition of Δ-demiclosedness with (4.1) and (4.2), we have $x_0 \in F$. Further, we can take a subsequence $\{z_{k_l}\} \subset \{z_k\}$ satisfying

$$\delta = \lim_{l\to\infty} d(u, z_{k_l}) = \underline{\lim}_{k\to\infty} d(u, z_k) \geq d(u, x_0) \geq d(u, p) \tag{4.3}$$

by Lemma 3.6. Moreover, we get

$$\begin{aligned} \frac{\sigma_{k_l}}{\gamma_{k_l}} &= \frac{\sin d(u, Sz_{k_l}) - \sin(1-\gamma_{k_l}) d(u, Sz_{k_l})}{\gamma_{k_l} \sin d(u, Sz_{k_l})} \\ &= \frac{1 - \cos \gamma_{k_l} d(u, Sz_{k_l})}{\gamma_{k_l}} + \frac{d(u, Sz_{k_l})}{\tan d(u, Sz_{k_l})} \cdot \frac{\sin \gamma_{k_l} d(u, Sz_{k_l})}{\gamma_{k_l} d(u, Sz_{k_l})} \\ &\to 0 + \frac{\delta}{\tan \delta} \cdot 1 = \frac{\delta}{\tan \delta} \end{aligned}$$

as $l \to \infty$. Similarly, we have $\tau_{k_l}/\gamma_{k_l} \to \delta/\tan\delta$. Then we obtain $\sigma_{k_l}/\tau_{k_l} \to 1$. Thus, using (4.3), we get

$$\begin{aligned}
&\overline{\lim_{i\to\infty}} \frac{\beta_{\varphi(i)}\sigma_{\varphi(i)}b_{\varphi(i)} + (1-\beta_{\varphi(i)})\tau_{\varphi(i)}c_{\varphi(i)}}{\beta_{\varphi(i)}\sigma_{\varphi(i)} + (1-\beta_{\varphi(i)})\tau_{\varphi(i)}} \\
&= \overline{\lim_{i\to\infty}} \frac{\beta_{n_i}\sigma_{n_i}b_{n_i} + (1-\beta_{n_i})\tau_{n_i}c_{n_i}}{\beta_{n_i}\sigma_{n_i} + (1-\beta_{n_i})\tau_{n_i}} \\
&= \lim_{l\to\infty} \frac{\beta_{k_l}\sigma_{k_l}b_{k_l} + (1-\beta_{k_l})\tau_{k_l}c_{k_l}}{\beta_{k_l}\sigma_{k_l} + (1-\beta_{k_l})\tau_{k_l}} \\
&= \lim_{l\to\infty} \frac{\beta_{k_l}\frac{\sigma_{k_l}}{\tau_{k_l}}b_{k_l} + (1-\beta_{k_l})c_{k_l}}{\beta_{k_l}\frac{\sigma_{k_l}}{\tau_{k_l}} + (1-\beta_{k_l})} \\
&= \beta\left(1 - \frac{\cos d(u,p)}{\cos\delta}\right) + (1-\beta)\left(1 - \frac{\cos d(u,p)}{\cos\delta}\right) \\
&\le 1 - \frac{\cos d(u,p)}{\cos d(u,x_0)} \\
&\le 0
\end{aligned}$$

for any φ satisfying $\underline{\lim}_{i\to\infty}(a_{\varphi(i)+1} - a_{\varphi(i)}) \ge 0$. Therefore, from Lemma 3.4, we obtain that $a_n \to 0$ *i.e.*, $\{x_n\}$ converges to $p = P_F u \in F$. □

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Manuscript received 15 January 2020
revised 22 March 2020

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