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REFINED HERMITE-HADAMARD INEQUALITY AND WEIGHTED LOGARITHMIC MEAN

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Abstract. Inspired by the recent works by R.Pal et al., and Furuichi-Minculete, we give further refined inequalities for a convex Riemann integrable function, applying the refined Hermite Hadamard inequality. Our approach is different from their one in [10]. As corollaries, we give the refined two types of inequalities on the weighted logarithmic mean. At last we give corresponding operator inequalities.

1. Introduction

The inequalities on means attract many mathematicians for its developments. See [6] for example. Recently, in ([10], Theorem 2.2), the weighted logarithmic mean was introduced properly and the inequalities among weighted means were shown as

$$a\sharp_v b \le L_v(a,b) \le a \bigtriangledown_v b, \tag{1.1}$$

where the weighted geometric mean $a\sharp_v b = a^{1-v}b^v$, the weighted arithmetic mean $a \bigtriangledown_v b = (1-v)a + vb$ and the weighted logarithmic mean [10]:

$$L_v(a,b) = \frac{1}{\log a - \log b}\left(\frac{1-v}{v}(a - a^{1-v}b^v) + \frac{v}{1-v}(a^{1-v}b^v - b)\right) \tag{1.2}$$

for $a,b > 0$ and $v \in (0,1)$. We easily find that $L_{1/2}(a,b) = \dfrac{a-b}{\log a - \log b}$, $(a \ne b)$, with $L_{1/s}(a,a) = a$. This is the so-called logarithmic mean. We also find that $\lim_{v\to 0} L_v(a,b) = a$ and $\lim_{v \to 1} L_v(a,b) = b$. Thus the inequalities given in (1.1) recover the well-known relations:

$$\sqrt{ab} \le \frac{a-b}{\log a - \log b} \le \frac{a+b}{2},\ (a,b>0).$$

R.Pal et al. obtained the inequalities given in (1.1) by their general result given in ([10], Thorem 2.1) which can be regarded as the generalization of the famous Hermite-Hadamard inequality with weight $v \in [0,1]$:

$$f(a \bigtriangledown_v b) \le C_{f,v}(a,b) \le f(a) \bigtriangledown_v f(b), \tag{1.3}$$

where

$$C_{f,v}(a,b) \tag{1.4}$$

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$$= \left(\int_0^1 f(a \nabla_{vt} b)dt\right) \nabla_v \left(\int_0^1 f((1-v)(b-a)t + a \nabla_v b)dt\right)$$

for a convex Riemann integrable function, $a, b > 0$ and $v \in [0,1]$. By elementary calculations, we find that the inequalities given in (1.3) recover the standard Hermite-Hadamard inequalities:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}. \tag{1.5}$$

Recently Furuichi and Minculete [7] obtained refined Hermite-Hadamard inequality and gave the extended inequalities for weighted logarithmic mean. In this paper we extend the results of [7] by using the more refined Hermite-Hadamard inequality.

2. Refined Hermite Hadamard inequality

We give the refined Hermite Hadamard inequality.

Theorem 2.1. *Let $f(x)$ be a convex function on $[a,b]$. Then for any $n \in \mathbb{N} \cup \{0\}$*

$$\begin{aligned} \frac{1}{2^n}\sum_{k=1}^{2^n} f(a+(2k-1)\frac{h_n}{2}) &\le \frac{1}{b-a}\int_a^b f(x)dx \\ &\le \frac{1}{2^{n+1}}\{f(a)+f(b)+2\sum_{k=1}^{2^n-1} f(a+kh_n)\}, \end{aligned} \tag{2.1}$$

where $h_n = \frac{b-a}{2^n}$. By putting $n = 0$ in (2.1), (1.5) is obtained.

The proof is omitted.

Proposition 2.2. *The following properties hold.*

(1) $$\frac{1}{2^{n-1}}\sum_{k=1}^{2^{n-1}} f\left(a+(2k-1)\frac{h_{n-1}}{2}\right) \le \frac{1}{2^n}\sum_{k=1}^{2^n} f\left(a+(2k-1)\frac{h_n}{2}\right)$$

(2) $$\begin{aligned} &\frac{1}{2^{n+1}}\left\{f(a)+f(b)+2\sum_{k=1}^{2^n-1} f(a+kh_n)\right\} \\ &\le \frac{1}{2^n}\left\{f(a)+f(b)+2\sum_{k=1}^{2^{n-1}-1} f(a+kh_{n-1})\right\} \end{aligned}$$

Proof. (1)

$$\begin{aligned} RHS &= \frac{1}{2^n}\left\{f\left(a+\frac{h_n}{2}\right)+f\left(a+\frac{3}{2}h_n\right)+f\left(a+\frac{5}{2}h_n\right)\right. \\ &\quad \left. +\cdots+f\left(a+\frac{2^{n+1}-1}{2}h_n\right)\right\} \\ &\ge \frac{1}{2^{n-1}}\{f(a+h_n)+f(a+3h_n)+\cdots+f(a+(2^n-1)h_n)\} \\ &= \frac{1}{2^{n-1}}\left\{f\left(a+\frac{h_{n-1}}{2}\right)+f\left(a+\frac{3}{2}h_{n-1}\right)\right. \end{aligned}$$

$$+\cdots+f\left(a+\frac{2^n-1}{2}h_{n-1}\right)\Bigg\}=LHS.$$

(2) Since

$$f(a)+f(a+2h_n)\geq 2f(a+h_n),$$
$$f(a+2h_n)+f(a+4h_n)\geq 2f(a+3h_n),$$
$$f(a+(2^n-2)h_n)+f(b)\geq 2f(a+(2^n-1)h_n),$$

we obtain

$$\begin{aligned}
LHS &= \frac{1}{2^{n+1}}\{f(a)+2f(a+h_n)+2f(a+2h_n)+2f(a+3h_n)\\
&\quad+\cdots+2f(a+(2^n-1)h_n)+f(b)\}\\
&\leq \frac{1}{2^{n+1}}\{f(a)+f(a)+f(a+2h_n)+2f(a+2h_n)+f(a+2h_n)\\
&\quad+f(a+4h_n)+\cdots+f(a+(2^n-2)h_n)+f(b)+f(b)\}\\
&= \frac{1}{2^n}\{f(a)+f(b)+2f(a+2h_n)+2f(a+4h_n)\\
&\quad+\cdots+2f(a+(2^n-2)h_n)\}\\
&= \frac{1}{2^n}\{f(a)+f(b)+2f(a+h_{n-1})+2f(a+2h_{n-1})\\
&\quad+\cdots+2f(a+(2^{n-1}-1)h_{n-1})\}\\
&= RHS.
\end{aligned}$$

□

3. Main results 1

We give the refined inequalities for (1.3) by repeating use of the refined Hermite Hadamard inequalities given in (2.1).

Theorem 3.1. *For every convex Riemann integrable function $f:[a,b]\to\mathbb{R}$ and $v\in[0,1]$, we have*

$$\begin{aligned}
f(a\triangledown_v b) &\leq R^{(1)}_{f,v}(a,b)=R^{(1)}_{f,v,0}(a,b)\\
&\leq R^{(1)}_{f,v,1}(a,b)\leq R^{(1)}_{f,v,2}(a,b)\leq\cdots\leq R^{(1)}_{f,v,n}(a,b)\\
&\leq C_{f,v}(a,b)\\
&\leq R^{(2)}_{f,v,n}(a,b)\leq R^{(2)}_{f,v,n-1}(a,b)\leq\cdots\leq R^{(2)}_{f,v,1}(a,b)\\
&\leq R^{(2)}_{f,v,0}(a,b)=R^{(2)}_{f,v}(a,b)\leq f(a)\triangledown_v f(b),
\end{aligned}\tag{3.1}$$

where $h_n=\dfrac{v(b-a)}{2^n}$, $\ell_n=\dfrac{(1-v)(b-a)}{2^n}$,

$$\begin{aligned}
&R^{(1)}_{f,v,n}(a,b)\\
&= \frac{1}{2^n}\sum_{k=1}^{2^n}\left\{(1-v)f\left(a+(2k-1)\frac{h_n}{2}\right)+vf\left((1-v)a+vb+(2k-1)\frac{\ell_n}{2}\right)\right\}
\end{aligned}\tag{3.2}$$

$$= \frac{1}{2^n}\sum_{k=1}^{2^n} f(a \triangledown_{\frac{(2k-1)v}{2^{n+1}}} b) \triangledown_v f(a \triangledown_{v+\frac{(2k-1)(1-v)}{2^{n+1}}} b)$$

and

$$\begin{aligned}(3.3)\qquad & R^{(2)}_{f,v,n}(a,b) \\ = \;& \frac{1}{2^{n+1}}\{(1-v)f(a)+vf(b)+f((1-v)a+vb)\} \\ & +\frac{1}{2^n}\sum_{k=1}^{2^n-1}\{(1-v)f(a+kh_n)+vf((1-v)a+vb+k\ell_n)\} \\ = \;& \frac{1}{2^{n+1}}\{f(a)\triangledown_v f(b)+f(a\triangledown_v b)\} \\ & +\frac{1}{2^n}\sum_{k=1}^{2^n-1}\left\{f(a\triangledown_{\frac{kv}{2^n}} b)\triangledown_v f(a\triangledown_{v+\frac{k(1-v)}{2^n}} b)\right\} \\ = \;& \frac{1}{2^n}\Big\{(f(a)\triangledown_v f(b))\triangledown_{1/2}(f(a\triangledown_v b)) \\ & +\sum_{k=1}^{2^n-1} f(a\triangledown_{\frac{kv}{2^n}} b)\triangledown_v f(a\triangledown_{v+\frac{k(1-v)}{2^n}} b)\Big\}.\end{aligned}$$

In the case of $n = 0$ in (3.1), we have the results of ([7], Theorem 2.1).

Proof. Applying the refined Hermite Hadamard inequalities (2.1) on two intervals $[a,(1-v)a+vb]$ and $[(1-v)a+vb,b]$, we obtain respectively

$$\begin{aligned}(3.4)\qquad & \frac{1}{2^n}\sum_{k=1}^{2^n} f\left(a+(2k-1)\frac{h_n}{2}\right) \le \frac{1}{v(b-a)}\int_a^{(1-v)a+vb} f(x)dx \\ \le \;& \frac{1}{2^{n+1}}\left\{f(a)+f((1-v)a+b)+2\sum_{k=1}^{2^n-1} f(a+kh_n)\right\}\end{aligned}$$

and

$$\begin{aligned}(3.5)\qquad & \frac{1}{2^n}\sum_{k=1}^{2^n} f\left((1-v)a+vb+(2k-1)\frac{\ell_n}{2}\right) \\ \le \;& \frac{1}{(1-v)(b-a)}\int_{(1-v)a+vb}^{b} f(x)dx \\ \le \;& \frac{1}{2^{n+1}}\left\{f((1-v)a+vb)+f(b)+2\sum_{k=1}^{2^n-1} f((1-v)a+vb+k\ell_n)\right\}.\end{aligned}$$

Multiplying $(1-v)$ and v to the both sides in (3.4) and (3.5) respectively and summing each side, we obtain

$$(3.6)\qquad \frac{1}{2^n}\sum_{k=1}^{2^n}\left\{(1-v)f\left(a+(2k-1)\frac{h_n}{2}\right)+vf\left((1-v)a+vb+(2k-1)\frac{\ell_n}{2}\right)\right\}$$

$$\leq \frac{1-v}{v(b-a)}\int_a^{(1-v)a+vb} f(x)dx + \frac{v}{(1-v)(b-a)}\int_{(1-v)a+vb}^{b} f(x)dx$$
$$\leq \frac{1}{2^{n+1}}\{f(a)\nabla_v f(b) + f(a\nabla_v b)\nabla_v f(a\nabla_v b)\}$$
$$+\frac{1}{2^n}\sum_{k=1}^{2^n-1}\{(1-v)f(a+kh_n)+vf((1-v)a+vb+k\ell_n)\},$$

which is equivalent to

$$R^{(1)}_{f,v,n}(a,b) \leq C_{f,v}(a,b) \leq R^{(2)}_{f,v,n}(a,b), \tag{3.7}$$

by replacing the variables such as $x = v(b-a)s + a$ in the first term and $x = (1-v)(b-a)u + (1-v)a + vb$ in the second term of the integral parts in (3.6). Finally we estimate $R^{(1)}_{f,v,n}(a,b)$ and $R^{(2)}_{f,v,n}(a,b)$. By the same method in Proposition 2.2, it is easy to sow

$$R^{(1)}_{f,v,n-1}(a,b) \leq R^{(1)}_{f,v,n}(a,b),$$

and

$$R^{(2)}_{f,v,n}(a,b) \leq R^{(2)}_{f,v,n-1}(a,b).$$

□

Corollary 3.2. *For $a,b>0$ and $v\in(0,1)$, we have*

$$\frac{1}{2^n}\sum_{k=1}^{2^n}(a\sharp_{\frac{(2k-1)v}{2^{n+1}}}b)\nabla_v(a\sharp_{v+\frac{(2k-1)(1-v)}{2^{n+1}}}b) \tag{3.8}$$
$$\leq L_v(a,b)$$
$$\leq \frac{1}{2^n}\left[(a\nabla_v b)\nabla_{1/2}(a\sharp_v b) + \sum_{k=1}^{2^n-1}(a\sharp_{\frac{kv}{2^n}}b)\nabla_v(a\sharp_{v+\frac{k(1-v)}{2^n}}b)\right]$$

In the case of $n=0$ in (3.8), we have the results of ([7], Corollary 2.2).

Proof. Applying the convex function $f(t)=e^t$ in Theorem 3.1, we have for $a,b>0$

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\left\{(1-v)e^{(1-\frac{(2k-1)v}{2^{n+1}})a}e^{\frac{(2k-1)v}{2^{n+1}}b}\right\} \tag{3.9}$$
$$+\frac{1}{2^n}\sum_{k=1}^{2^n}\left\{ve^{(1-v-\frac{(2k-1)(1-v)}{2^{n+1}})a}e^{(v+\frac{(2k-1)(1-v)}{2^{n+1}})b}\right\}$$
$$\leq \frac{1-v}{v(b-a)}\{e^{(1-v)a+vb}-e^a\} + \frac{v}{(1-v)(b-a)}\{e^b - e^{(1-v)a+vb}\}$$
$$\leq \frac{1}{2^{n+1}}\left\{(1-v)e^a + ve^b + e^{(1-v)a}e^{vb}\right\}$$
$$+\frac{1}{2^n}\sum_{k=1}^{2^n-1}\{(1-v)e^{(1-\frac{kv}{2^n})a}e^{\frac{kv}{2^n}b} + ve^{(1-v-\frac{k(1-v)}{2^n})a}e^{(v+\frac{k(1-v)}{2^n})b}\}.$$

Replacing e^a and e^b with a and b respectively, we obtain the inequalities (3.8) for $b \geq a > 0$ and $v \in (0,1)$. $\square$

We give the inequalities on the weighted identric mean which was defined in [10] as

$$I_v(a,b) = \frac{1}{e}(a \nabla_v b)^{\frac{(1-2v)(a\nabla_v b)}{v(1-v)(b-a)}} \left(\frac{b^{\frac{vb}{1-v}}}{a^{\frac{(1-v)a}{v}}} \right)^{\frac{1}{b-a}}, \quad v \in (0,1). \tag{3.10}$$

It is easy to check that $I_{1/2}(a,b)$ recovers the usual identric mean $I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$, with $\lim_{v\to 0} I_v(a,b) = a$ and $\lim_{v\to 1} I_v(a,b) = b$.

Corollary 3.3. *For $a,b>0$ and $v \in (0,1)$, we have*

$$\begin{aligned} &\left\{ (a\sharp_v b)\sharp_{1/2}(a \nabla_v b) \prod_{k=1}^{2^n} (a \nabla_{\frac{kv}{2^n}} b)\sharp_v (a \nabla_{\frac{v+k(1-v)}{2^n}} b) \right\}^{\frac{1}{2^n}} \\ \leq\ & I_v(a,b) \\ \leq\ & \prod_{k=1}^{2^n} \left\{ (a \nabla_{\frac{(2k-1)v}{2^{n+1}}} b)\sharp_v (a \nabla_{v+\frac{(2k-1)(1-v)}{2^{n+1}}} b) \right\}^{\frac{1}{2^n}}. \end{aligned} \tag{3.11}$$

In the case of $n=0$ in (3.11), we have the results of ([7], Corollary 2.3).

Proof. We apply the convex function $f(t) = -\log t$ in Theorem 3.1. Since

$$\begin{aligned} & -(1-v)\log\left\{ \left(1 - \frac{(2k-1)v}{2^{n+1}}\right) a + \frac{(2k-1)v}{2^{n+1}} b \right\} \\ & -v \log\left\{ \left(1 - v - \frac{(2k-1)(1-v)}{2^{n+1}}\right) a + \left(v + \frac{(2k-1)(1-v)}{2^{n+1}}\right) b \right\} \\ =\ & -\log\left\{ \left(1 - \frac{(2k-1)v}{2^{n+1}}\right) a + \frac{(2k-1)v}{2^{n+1}} b \right\}^{1-v} \\ & \left\{ \left(1 - v - \frac{(2k-1)(1-v)}{2^{n+1}}\right) a + \left(v + \frac{(2k-1)(1-v)}{2^{n+1}}\right) b \right\}^{v}, \end{aligned}$$

we have

$$\begin{aligned} & -\frac{1}{2^n}\sum_{k=1}^{2^n} \log\left\{ \left(1 - \frac{(2k-1)v}{2^{n+1}}\right) a + \frac{(2k-1)v}{2^{n+1}} b \right\}^{1-v} \\ & \left\{ \left(1 - v - \frac{(2k-1)(1-v)}{2^{n+1}}\right) a + \left(v + \frac{(2k-1)(1-v)}{2^{n+1}}\right) b \right\}^{v} \\ =\ & -\frac{1}{2^n} \log \prod_{k=1}^{2^n} \left\{ \left(1 - \frac{(2k-1)v}{2^{n+1}}\right) a + \frac{(2k-1)v}{2^{n+1}} b \right\}^{1-v} \\ & \left\{ \left(1 - v - \frac{(2k-1)(1-v)}{2^{n+1}}\right) a + \left(v + \frac{(2k-1)(1-v)}{2^{n+1}}\right) b \right\}^{v} \end{aligned} \tag{3.12}$$

$$= \quad -\log \prod_{k=1}^{2^n} \left\{ (a \bigtriangledown_{\frac{(2k-1)v}{2^{n+1}}} b) \sharp_v (a \bigtriangledown_{v+\frac{(2k-1)(1-v)}{2^{n+1}}} b) \right\}^{\frac{1}{2^n}}.$$

Since

$$\begin{aligned} & -(1-v)\log a - v\log b - \log((1-v)a+vb) \\ & -2\sum_{k=1}^{2^n-1} \left\{ (1-v)\log\left(\left(1-\frac{kv}{2^n}\right) a + \frac{kv}{2^n} b \right) \right. \\ & \left. +v\log\left(\left(1-v-\frac{k(1-v)}{2^n}\right) a + \left(v+\frac{k(1-v)}{2^n}\right) b \right) \right\} \\ = \; & -\log a^{1-v} b^v ((1-v)a+vb) \\ & -2\sum_{k=1}^{2^n-1} \log \left\{ \left(1-\frac{kv}{2^n}\right) a + \frac{kv}{2^n} b \right\}^{1-v} \\ & \left\{ \left(1-v-\frac{k(1-v)}{2^n}\right) a + \left(v+\frac{k(1-v)}{2^n}\right) b \right\}^{v}, \end{aligned}$$

we have

$$\begin{aligned} \text{(3.13)} \qquad & -\frac{1}{2^{n+1}}\log a^{1-v} b^v ((1-v)a+vb) \\ & -\frac{1}{2^n}\sum_{k=1}^{2^n-1} \log \left\{ \left(1-\frac{kv}{2^n}\right) a + \frac{kv}{2^n} b \right\}^{1-v} \\ & \left\{ \left(1-v-\frac{k(1-v)}{2^n}\right) a + \left(v+\frac{k(1-v)}{2^n}\right) b \right\}^{v}, \\ = \; & -\frac{1}{2^n}\log a^{\frac{1-v}{2}} b^{\frac{v}{2}} ((1-v)a+vb)^{1/2} \\ & -\frac{1}{2^n}\log \prod_{k=1}^{2^n-1} \left\{ \left(1-\frac{kv}{2^n}\right) a + \frac{kv}{2^n} b \right\}^{1-v} \\ & \left\{ \left(1-v-\frac{k(1-v)}{2^n}\right) a + \left(v+\frac{k(1-v)}{2^n}\right) b \right\}^{v} \\ = \; & -\log \left\{ (a\sharp_v b)\sharp_{1/2}(a \bigtriangledown_v b) \prod_{k=1}^{2^n} (a \bigtriangledown_{\frac{kv}{2^n}} b)\sharp_v (a \bigtriangledown_{v+\frac{k(1-v)}{2^n}} b) \right\}^{\frac{1}{2^n}}. \end{aligned}$$

We calculate the following

$$\begin{aligned} \text{(3.14)} \qquad & -\frac{1-v}{v(b-a)}\{(a \bigtriangledown_v b)\log(a \bigtriangledown_v b) - a \bigtriangledown_v b - a\log a + a\} \\ & -\frac{v}{(1-v)(b-a)}\{b\log b - b - (a \bigtriangledown_v b)\log(a \bigtriangledown_v b) + a \bigtriangledown_v b\} \\ = \; & -\log\{(1-v)a+vb\}^{\frac{(1-2v)((1-v)a+vb)}{v(1-v)(b-a)}} b^{\frac{vb}{(1-v)(b-a)}} a^{-\frac{(1-v)a}{v(b-a)}} - 1 \end{aligned}$$

$$= \quad -\log \frac{1}{e}\{(1-v)a+vb)\}^{\frac{(1-2v)((1-v)a+vb)}{v(1-v)(b-a)}} \left(\frac{b^{\frac{vb}{1-v}}}{a^{\frac{(1-v)a}{v}}} \right)^{\frac{1}{b-a}}.$$

Thus we complete the proof for any $a, b > 0$ by the similar way to the proof of Corollary 3.2. □

4. Main results 2

We give the refined inequalities for (1.3) by repeating use of the refined Hermite Hadamard inequalities given in (2.1). The obtained inequalities are different from the inequalities in Theorem 3.1.

Theorem 4.1. *For every convex Riemann integralble function $f : [a, b] \to \mathbb{R}$ and $v \in [0, 1]$, we have*

$$\begin{aligned} f(\frac{a+b}{2}) &\leq r^{(1)}_{f,v}(a,b) = r^{(1)}_{f,v,0}(a,b) \\ &\leq r^{(1)}_{f,v,1}(a,b) \leq r^{(1)}_{f,v,2}(a,b) \leq \cdots \leq r^{(1)}_{f,v,n}(a,b) \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq r^{(2)}_{f,v,n}(a,b) \leq r^{(2)}_{f,v,n-1}(a,b) \leq \cdots \leq r^{(2)}_{f,v,1}(a,b) \\ &\leq r^{(2)}_{f,v,0}(a,b) = r^{(2)}_{f,v}(a,b) \leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{4.1}$$

where $h_n = \dfrac{v(b-a)}{2^n}$, $\ell_n = \dfrac{(1-v)(b-a)}{2^n}$,

$$\begin{aligned} & r^{(1)}_{f,v,n}(a,b) \\ &= \frac{1}{2^n}\sum_{k=1}^{2^n}\left\{ vf\left(a+(2k-1)\frac{h_n}{2}\right) + (1-v)f\left((1-v)a+vb+(2k-1)\frac{\ell_n}{2}\right)\right\} \\ &= \frac{1}{2^n}\sum_{k=1}^{2^n} f(a \bigtriangledown_{v+\frac{(2k-1)(1-v)}{2^{n+1}}} b) \bigtriangledown_v f(a \bigtriangledown_{\frac{(2k-1)v}{2^{n+1}}} b) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} & r^{(2)}_{f,v,n}(a,b) \\ &= \frac{1}{2^{n+1}}\{vf(a)+(1-v)f(b)+f((1-v)a+vb)\} \\ &\quad + \frac{1}{2^n}\sum_{k=1}^{2^n-1}\{vf((a+kh_n)+(1-v)f((1-v)a+vb+k\ell_n)\} \\ &= \frac{1}{2^{n+1}}\{f(b) \bigtriangledown_v f(a) + f(a \bigtriangledown_v b)\} \\ &\quad + \frac{1}{2^n}\sum_{k=1}^{2^n-1} f(a \bigtriangledown_{v+\frac{k(1-v)}{2^n}} b) \bigtriangledown_v f(a \bigtriangledown_{\frac{kv}{2^n}} b) \end{aligned} \tag{4.3}$$

$$= \frac{1}{2^n}\{(f(b)\bigtriangledown_v f(a))\bigtriangledown_{1/2}(f(a\bigtriangledown_v b)) + \sum_{k=1}^{2^n-1} f(a\bigtriangledown_{v+\frac{k(1-v)}{2^n}} b)\bigtriangledown_v f(a\bigtriangledown_{\frac{kv}{2^n}} b)\}.$$

Proof. Multiplying v and $(1-v)$ to the both sides in (3.4) and (3.5) respectively and summing each side, we obtain

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\left\{vf\left(a+(2k-1)\frac{h_n}{2}\right)+(1-v)f\left((1-v)a+vb+(2k-1)\frac{\ell_n}{2}\right)\right\} \tag{4.4}$$
$$\leq \frac{1}{b-a}\int_a^{(1-v)a+vb} f(x)dx + \frac{1}{b-a}\int_{(1-v)a+vb}^{b} f(x)dx$$
$$\leq \frac{1}{2^{n+1}}\{f(b)\bigtriangledown_v f(a) + vf(a\bigtriangledown_v b) + (1-v)f(a\bigtriangledown_v b)\} + \frac{1}{2^n}\sum_{k=1}^{2^n-1}\{vf(a+kh_n)+(1-v)f((1-v)a+vb+k\ell_n)\},$$

which is equivalent to

$$r^{(1)}_{f,v,n}(a,b) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq r^{(2)}_{f,v,n}(a,b). \tag{4.5}$$

Finally we estimate $r^{(1)}_{f,v,n}(a,b)$ and $r^{(2)}_{f,v,n}(a,b)$. By the same method in Proposition 2.2, it is easy to sow

$$r^{(1)}_{f,v,n-1}(a,b) \leq r^{(1)}_{f,v,n}(a,b),$$

and

$$r^{(2)}_{f,v,n}(a,b) \leq r^{(2)}_{f,v,n-1}(a,b).$$

□

Corollary 4.2. *For $a,b>0$ and $v\in(0,1)$, we have*

$$\frac{1}{2^n}\sum_{k=1}^{2^n}(a\sharp_{v+\frac{(2k-1)(1-v)}{2^{n+1}}} b)\bigtriangledown_v (a\sharp_{\frac{(2k-1)v}{2^{n+1}}} b) \tag{4.6}$$
$$\leq L_{1/2}(a,b) = \frac{b-a}{\log b - \log a}$$
$$\leq \frac{1}{2^n}\left[(b\bigtriangledown_v a)\bigtriangledown_{1/2}(a\sharp_v b) + \sum_{k=1}^{2^n-1}(a\sharp_{v+\frac{k(1-v)}{2^n}} b)\bigtriangledown_v (a\sharp_{\frac{kv}{2^n}} b)\right]$$

Proof. After we apply the convex function $f(t)=e^t$ in Theorem 4.1, we replace e^a and e^b with a and b respectively. Then we obtain the inequalities (4.6) for $b>a>0$ and $v\in(0,1)$. □

Corollary 4.3. *For $a, b > 0$ and $v \in (0,1)$, we have*

$$\left\{(b\sharp_v a)\sharp_{1/2}(a \bigtriangledown_v b)\prod_{k=1}^{2^n}(a \bigtriangledown_{v+\frac{k(1-v)}{2^n}} b)\sharp_v(a \bigtriangledown_{\frac{kv}{2^n}} b)\right\}^{\frac{1}{2^n}} \tag{4.7}$$
$$\leq I_{1/2}(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$
$$\leq \prod_{k=1}^{2^n}\{(a \bigtriangledown_{v+\frac{(2k-1)(1-v)}{2^{n+1}}} b)\sharp_v(a \bigtriangledown_{\frac{(2k-1)v}{2^{n+1}}} b)\}^{\frac{1}{2^n}}.$$

Proof. Applying the convex function $f(t) = -\log t$ in Theorem 4.1, we obtain inequalities (4.7) for $b > a > 0$ and $v \in (0,1)$. □

5. Related results

Our obtained results in this paper can be extended to the operator inequalities. We give operator inequalities corresponding to Corollary 3.2 and 4.2. For strictly positive operators A and B, the weighted geometric operator mean and arithmetic operator mean are defined as

$$A\sharp_v B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^v A^{1/2},\ A \bigtriangledown_v B = (1-v)A + vB.$$

It is known that an operator mean $M(A,B)$ is associated with the representing function $f(t) = m(1,t)$ with a mean $m(a,b)$ for positive numbers a, b, in the following

$$M(A,B) = A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2}$$

in the general operator mean theory by Kubo-Ando [8]. Thus it is understood that the weighted logarithmic operator mean $A\ell_v B$ is defined by through the representing function $L_v(1,t)$ for $v \in (0,1)$.

Theorem 5.1. *For any $v \in (0,1)$ and strictly positive operators A and B, we have*

$$\frac{1}{2^n}\sum_{k=1}^{2^n}\left[(A\sharp_{\frac{(2k-1)v}{2^{n+1}}} B) \bigtriangledown_v (A\sharp_{v+\frac{(2k-1)(1-v)}{2^{n+1}}} B)\right] \tag{5.1}$$
$$\leq A\ell_v B$$
$$\leq \frac{1}{2^n}\left[(A\sharp_v B) \bigtriangledown_{1/2} (A \bigtriangledown_v B) + \sum_{k=1}^{2^n-1}(A\sharp_{\frac{kv}{2^n}} B) \bigtriangledown_v (A\sharp_{v+\frac{k(1-v)}{2^n}} B)\right]$$

Proof. After we divide a in the both sides of the inequalities (3.8) and we put $\frac{b}{a} = t$, we replace t by $A^{-1/2}BA^{-1/2}$ and multiply $A^{1/2}$ from the both sides. Then we obtain the results. □

Theorem 5.2. *For any $v \in (0,1)$ and strictly positive operators A and B, we have*

$$\begin{aligned} &\frac{1}{2^n}\sum_{k=1}^{2^n}\left[\left(A\sharp_{v+\frac{(2k-1)(1-v)}{2^{n+1}}}B\right)\nabla_v\left(A\sharp_{\frac{(2k-1)v}{2^{n+1}}}B\right)\right] \\ \leq\ & A\ell_{1/2}B \\ \leq\ & \frac{1}{2^n}\left[(B\sharp_v A)\nabla_{1/2}(A\nabla_v B)+\sum_{k=1}^{2^n-1}\left(A\sharp_{v+\frac{k(1-v)}{2^n}}B\right)\nabla_v\left(A\sharp_{\frac{kv}{2^n}}B\right)\right] \end{aligned} \tag{5.2}$$

Proof. We obtain the results by the same method of Theorem 5.1. □

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