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MEASURE OF NONCONVEXITY AND FIXED POINTS IN METRIC SPACES

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Abstract. In this paper, the measure of nonconvexity in metric spaces is used for obtaining fixed point results. Fixed point theorems are proved for nonexpansive and γ-condensing maps in strictly convex metric spaces with convex round balls having the so-called property (R), where γ is the measure of noncompactness.

1. Introduction

It is well-known that nonexpansive maps in metric spaces can be viewed as natural extensions of contractions. Nevertheless, fixed point theory for nonexpansive maps differs sharply from that of contractions in the sense that additional assumptions are needed on the structure of the space and/or restrictions on the mapping itself to guarantee the existence of at least one fixed point.

Fixed points of nonexpansive maps in Hilbert spaces were investigated by Browder [4]. Some results in Banach spaces were obtained by Browder [5], Göhde [12], and Kirk [13] using "geometric" properties of the space and/or the domain. Later, a very wide theory has been developed which tries to find more general conditions on the subset K and the space X which still guarantee the existence of a fixed point of a nonexpansive map $T : K \to K$. Two important conditions are the convexity of K and strict convexity of X.

One of the possibilities to define strict convexity of a Banach space $(X, \|\cdot\|)$ is by the following condition [8, 10, 11]:

> For all $x, y \in X$ and $t \in [0, 1]$, there exists a unique $z \in X$ such that $\|x - z\| = t\|x - y\|$ and $\|z - y\| = (1 - t)\|x - y\|$.

In 1999, Bula [6] extended the notion of strict convexity to metric spaces using a similar condition. Namely, a metric space (X, d) is called convex [2, 6] if for each $x, y \in X$ and for each $t \in [0, 1]$, there exists a $z \in X$ such that $d(x, z) = t\,d(x, y)$ and $d(z, y) = (1-t)d(x, y)$. If this point z is unique for all possible combinations of x, y and t, Bula called the space X strictly convex (see also [3]). In strictly convex metric spaces, the intersection of convex sets is convex, however, closed balls in these spaces need not be convex (see [7]). To overcome this difficulty, Bula imposed an additional condition, namely the convex round balls condition: for all $w \in X$, $d(w, z) < \max\{d(w, x), d(w, y)\}$. Among other things, Bula extended Browder-Göhde-Kirk fixed point theorem and De Marr theorem to strictly convex metric

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spaces. Convexity in metric spaces was first introduced by Takahashi in [17]. For detail on the topic, we refere to [1].

Recently, Marrero [14, 15] used the Eisenfeld-Lakshmikantham measure of nonconvexity in reflexive Banach space to prove some fixed point results including the Browder-Göhde-Kirk fixed point theorem without the convexity requirement on the underlying set.

We use in this paper the measure of nonconvexity for obtaining fixed point results in metric spaces. We prove some fixed point theorems for nonexpansive and γ-condensing maps in strictly convex metric spaces with convex round balls having the so-called property (R), where γ is a measure of non-compactness.

2. Preliminaries

In this section, we recall basic definitions and results from the paper [6]. All the way through this section (X, d) denotes a metric space.

Definition 2.1. (1) A set $K \subset X$ is said to be *convex* if for each $x, y \in K$ and for each $t \in [0,1]$, there exists $z \in K$ such that

$$d(x,z) = td(x,y) \text{ and } d(z,y) = (1-t)\, d(x,y).$$

(2) The space X is said to be *strictly convex* if for each $x, y \in X$ and for each $t \in [0,1]$, there exists a unique $z \in X$ such that

$$d(x,z) = td(x,y) \text{ and } d(z,y) = (1-t)\, d(x,y).$$

It is easy to show that the intersection $\bigcap_{\alpha \in I} K_\alpha$ of a family of convex sets in a strictly convex metric space is convex itself. The *convex hull* of $K \subset X$ is the set

$$\text{co}\,(K) = \bigcap \{C \subset X : K \subset C \text{ and } C \text{ is convex}\}.$$

$\overline{\text{co}}\,(K)$ will denote the closure of the convex hull of K. The following properties are easy to prove.

(i) $\text{co}\,(K)$ is convex and $K \subset \text{co}\,(K)$;
(ii) $\text{co}\,(K) = K$ if and only if K is convex;
(iii) $\overline{\text{co}}\,(K) = K$ if and only if K is closed and convex.

It should be noted that closed balls in strictly convex metric space are not necessarily convex sets. So, one requires the following condition in addition.

Definition 2.2. A strictly convex metric space (X, d) is said to be *strictly convex metric space with convex round balls* if for all $x, y, w \in X$, $x \neq y$ and for all $t \in (0,1)$, there exists $z \in X$ such that

$$d(x,z) = td(x,y) \text{ and } d(z,y) = (1-t)d(x,y),$$
$$d(w,z) < \max\{d(w,x), d(w,y)\}.$$

The above strict inequality shows that if x and y belong to

$$S(w,r) = \{a \in X : d(a,w) = r\}, \quad r > 0,$$

then z does not belong to $S(w,r)$, that is, $S(w,r)$ does not contain straight lines.

Lemma 2.3. *Let (X, d) be a strictly convex metric space with convex round balls. Then the closed ball $B(a, r) = \{y \in X : d(a, y) \leqslant r\}$ is a convex set for every $r > 0$ and every $a \in X$.*

Remark 2.4. The condition:

For all $x, y, w \in X$ $(x \neq y)$ and for all $t \in (0, 1)$, there exists $z \in X$ such that $d(x, z) = td(x, y)$ and $d(z, y) = (1 - t) d(x, y)$ and $d(w, z) \leq \max\{d(w, x), d(w, y)\}$ is equivalent with the condition of convexity of closed balls.

A trivial example of a strictly convex metric space that is not a strictly convex metric space with convex round balls is $X = \{x\}$ with $d(x, x) = 0$. For a nontrivial example see [7, Section 3].

Definition 2.5. A convex set K in X is said to have *normal structure* if for each bounded and convex set $C \subset K$ that contains more than one point, there is some point $y \in C$ such that

$$r_y(C) := \sup\{d(x, y) : x \in C\} < \delta(C) := \sup\{d(x, y) : x, y \in C\}.$$

Lemma 2.6. *Every convex and compact set in a strictly convex metric space X with convex round balls has normal structure.*

3. Main results

Throughout this section (X, d) will be a metric space.

On the lines of Nadler [16], we adopt that:

(1) $\mathcal{B}(X) = \{A : A$ is a non-empty closed and bounded subset of $X\}$,
(2) for $A, B \in \mathcal{B}(X)$ and $x \in X$,

$$d(x, A) = \inf\{d(x, a) : a \in A\}$$

and

$$H(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(b, A) : b \in B\}\}.$$

It is well known that $\mathcal{B}(X)$ is a metric space with the distance H which is known as the Hausdorff-Pompeiu metric on $\mathcal{B}(X)$.

Definition 3.1. [9] The *Eisenfeld-Lakshmikantham measure of nonconvexity* (E-L measure of nonconvexity, for short) of a bounded subset K of X is defined by

$$\mu(K) = \sup_{x \in \overline{\text{co}}(K)} \inf_{y \in K} d(x, y) = H(K, \overline{\text{co}}(K)),$$

where H is the Hausdorff-Pompeiu distance.

The following properties of μ can be derived in a fairly straightforward manner from its definition. Let $K, K_1 \in B(X)$ and $\overline{K}$ denote the closure of K. Then:

(i) $\mu(K) = 0$ if and only if $\overline{K}$ is convex;
(ii) $\mu(\overline{K}) = \mu(K)$;
(iii) $\mu(K) \leq \delta(K)$;
(iv) $|\mu(K) - \mu(K_1)| \leq 2H(K, K_1)$.

Definition 3.2. A convex metric space (X,d) is said to have *property* (R) if every decreasing sequence of nonempty closed bounded convex subsets of X has a nonempty intersection.

Lemma 3.3. *Let X be a strictly convex metric space with convex round balls having property (R). Then for any $K \subset X$,*

$$C(K) = \{x \in K : r_x(K) = r(K)\}$$

is nonempty, closed and convex, where $r(K) = \inf\{r_x(K) : x \in C\}$.

Proof. Let

$$K(n,x) = \left\{y \in K : d(x,y) \leq r(K) + \frac{1}{n}\right\}$$

for $x \in K$ and set $L_n = \bigcap_{x \in K} K(n,x)$. Then $\{L_n\}$ is a decreasing sequence of nonempty (by property (R)) closed convex sets and so

$$\bigcap_{n=1}^{\infty} L_n = C(K)$$

is nonempty, closed and convex. □

Recall that a self map $T : X \to X$ of a metric space (X,d) is called nonexpansive if $d(Tx,Ty) \leq d(x,y)$ holds for all $x, y \in X$.

Theorem 3.4. *Let X be a strictly convex metric space with convex round balls having property (R). Let $K \subset X$ be a nonempty bounded closed convex set having normal structure and $T : K \to K$ a nonexpansive map. Then T has a fixed point.*

Proof. Let

$$\Gamma = \{C \subset K : C \text{ is nonempty closed convex and } T(C) \subseteq C\}.$$

Then from Zorn's Lemma and property (R), Γ has a minimal element K_0. We shall show that K_0 is a singleton. Let $x \in C(K_0)$ (note that $C(K_0)$ is nonempty by Lemma 3.3). Then

$$d(Tx,Ty) \leq d(x,y) \leq r_x(K_0) = r(K_0)$$

for all $y \in K_0$ and so $T(K_0) \subseteq B(Tx, r(K_0))$. Since

$$T(K_0 \cap B(Tx, r(K_0))) \subseteq K_0 \cap B(Tx, r(K_0)),$$

by the minimality of K_0, we have

$$K_0 \subset B(Tx, r(K_0)).$$

So $r_{Tx}(K_0) \leq r(K_0)$. But $r(K_0) \leq r_x(K_0)$ for $x \in K_0$. Therefore, $r_{Tx}(K_0) = r(K_0)$ and so $Tx \in C(K_0)$. This implies that $T(C(K_0)) \subseteq C(K_0)$. By Lemma 3.3, $C(K_0) \in \Gamma$.

If $z, w \in C(K_0)$, then

$$d(z,w) \leq r_z(K_0) = r(K_0).$$

Since K_0 has a normal structure, and there exists $x \in K_0$ such that

$$\delta\left(C\left(K_0\right)\right) \le r_x\left(K_0\right) < \delta\left(K_0\right).$$

This contradicts the minimality of K_0. Thus $\delta\left(K_0\right) = 0$. This implies that K_0 is a singleton. □

Lemma 3.5. *Let* $\{A_n\}_{n=1}^{\infty}$ *be a decreasing sequence of nonempty closed bounded subsets of a convex metric space* X *with* $\lim_{n\to\infty} \mu\left(A_n\right) = 0$ *and let* $A_\infty = \bigcap_{n=1}^{\infty} A_n$. *Then* $A_\infty = \bigcap_{n=1}^{\infty} \overline{co}\left(A_n\right)$.

Proof. Clearly,

$$A_\infty = \bigcap_{n=1}^{\infty} A_n \subseteq \bigcap_{n=1}^{\infty} \overline{\mathrm{co}}\left(A_n\right).$$

On the other hand, let $x \in \bigcap_{n=1}^{\infty} \overline{\mathrm{co}}\left(A_n\right)$. Then $x \in \overline{\mathrm{co}}\left(A_n\right)$ for all n. Since $\lim_{n\to\infty} \mu\left(A_n\right) = 0$, for each $\epsilon > 0$ there is N such that $\mu(A_n) < \epsilon$ for $n > N$. For such n, $\inf_{y\in A_n} d(x, y_n) < \epsilon$, so one can choose $y_n \in A_n$ satisfying $d(x, y_n) < \epsilon$. Hence, $y_n \to x$ as $n \to \infty$. Each A_n is closed and so $x \in A_n$ for each n, i.e., $x \in \bigcap_{n=1}^{\infty} A_n = A_\infty$. As a result, we have

$$A_\infty = \bigcap_{n=1}^{\infty} \overline{\mathrm{co}}\left(A_n\right).$$

□

Recall that a subset K of a metric space (X, d) is called a Chebyshev set if for each $x \in X$ there is a unique $y \in K$ such that $d(x, y) = d(x, K)$.

Theorem 3.6. *Let* (X, d) *be a strictly convex metric space with convex round balls having property* (R) *and let* μ *be the E-L measure of nonconvexity. Let* $K \subset X$ *be a nonempty closed set and* $x_0 \in X \setminus K$. *If* $\lim_{n\to\infty} \mu\left(K\left(n, x_0\right)\right) = 0$, *where*

$$K\left(n, x_0\right) = \left\{x \in K : d\left(x_0, x\right) \le d + \frac{1}{n}\right\}$$
$$\text{and } d = d\left(x_0, K\right),$$

then K *is a Chebyshev set.*

Proof. Since $\{K\left(n, x_0\right)\}$ is a decreasing sequence of nonempty closed bounded sets with $\lim_{n\to\infty} \mu\left(K\left(n, x_0\right)\right) = 0$, property (R) guarantees that

$$K_\infty = \bigcap_{n=1}^{\infty} K\left(n, x_0\right) = P_K\left(x_0\right) = \{x \in K : d\left(x_0, x\right) = d\}$$

is nonempty and convex.

For uniqueness, let $x, y \in P_K\left(x_0\right)$ with $x \neq y$. Then $d\left(x_0, x\right) = d$ and $d\left(x_0, y\right) = d$. It follows that there exists $z \in K$ such that

$$d\left(x, z\right) = td\left(x, y\right) \text{ and } d\left(z, y\right) = \left(1 - t\right) d\left(x, y\right).$$

From the condition of convex round balls, we have

$$d(x_0, z) < \max\{d(x_0, x), d(x_0, y)\} = d = d(x_0, K).$$

This is a contradiction. Hence K is Chebyshev. □

Definition 3.7. Let K be a nonempty closed bounded subset of a strictly convex metric space (X, d). A map $T : K \to K$ is said to have *property* (C) if $\lim_{n\to\infty} \mu(A_n) = 0$, where μ is the E-L measure of nonconvexity in X and $\{A_n\}_{n=1}^{\infty}$ is the decreasing sequence of nonempty, closed and bounded subset of X defined by

$$A_1 = \overline{T(K)},\ A_{n+1} = \overline{T(A_n)}\ \ (n \in \mathbb{N}).$$

Definition 3.8. Let K be a nonempty bounded subset of a strictly convex metric space (X, d) with E-L measure of nonconvexity μ. A map $T : K \to K$ is called a *μ-contraction* (with constant α) if

$$\mu(T(A)) \le \alpha\mu(A)$$

for some $\alpha \in (0, 1)$ and every $A \subset K$.

Remark 3.9. When K is closed, every μ-contraction $T : K \to K$ has property (C). Indeed, if $\{A_n\}_{n=1}^{\infty}$ is as in Definition 3.7, then $\mu(A_n) \le \alpha^n\mu(K)$ $(n \in \mathbb{N})$ implies that $\lim_{n\to\infty} \mu(A_n) = 0$.

Proposition 3.10. *Let X be a strictly convex metric space with convex round balls having property (R). Let $K \subset X$ be a nonempty closed bounded set and let $T : K \to K$ have property (C). Then K contains a nonempty closed convex set A such that $T(A) \subset A$.*

Proof. Let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of nonempty closed bounded subset of X defined by

$$A_1 = \overline{T(K)},\ A_{n+1} = \overline{T(A_n)}\ \ (n \in \mathbb{N}).$$

The set $A_\infty = \bigcap_{n=1}^{\infty} A_n$ is closed with $T(A_\infty) \subset A_\infty$. Since T has property (C), we have $\lim_{n\to\infty} \mu(A_n) = 0$. By property (R) and Lemma 3.5, A_∞ is nonempty and convex. □

Theorem 3.11. *Let (X, d) be a strictly convex metric space with convex round balls having property (R). Let $K \subset X$ be a nonempty closed bounded set having normal structure and $T : K \to K$ a nonexpansive map having property (C). Then T has a fixed point.*

Proof. By Proposition 3.10, there exists a nonempty closed convex subset A of K with $T(A) \subset A$. Since T is nonexpansive, by Theorem 3.4, T has a fixed point. □

In what follows, we will use the (Kuratowski) *measure of noncompactness* of a bounded subset K of a metric space (X, d) defined by

$$\gamma(K) = \inf\{\epsilon > 0 : K \text{ can be covered by a finite number of sets of diameter } \le \epsilon\}.$$

It is well known that the function γ in a complete metric space (X, d) have the following properties:

(i) $0 \leq \gamma(K) \leq \delta(K)$;
(ii) $\gamma(K) = 0$ if and only if K is precompact (that is, $\overline{K}$ is compact);
(iii) $\gamma(K) = \gamma(\overline{K})$;
(iv) $\gamma(K_1 \cup K_2) = \max\{\gamma(K_1), \gamma(K_2)\}$ for $K_1, K_2 \subset X$.

Definition 3.12. Let K be a nonempty and bounded subset of a metric space (X, d), and let γ stands for the measure of noncompactness in X. A map $T : K \to K$ is called γ-*condensing* provided that

$$\gamma(T(A)) < \gamma(A)$$

for every $A \subset K$ with $T(A) \subset A$ and $\gamma(A) > 0$.

Theorem 3.13. *Let (X, d) be a strictly convex complete metric space with convex round balls. Let $K \subset X$ be a nonempty closed bounded convex set and suppose $T : K \to K$ is γ-condensing and nonexpansive. Then T has a fixed point.*

Proof. Fix $x \in K$ and let Γ denote the family of all closed convex subsets C of K for which $x \in C$ and $T : C \to C$. Now set

$$B = \bigcap_{C \in \Gamma} C, \quad D = \overline{\text{co}}\,(T(B) \cup \{x\}).$$

Since $x \in B$ and $T : B \to B$, it must be the case that $D \subseteq B$. This implies that $T(D) \subseteq T(B) \subseteq D$. Since $x \in D$, it follows that $D \in \Gamma$. Therefore $B \subseteq D$, from which we conclude that $B = D$.

We now have $T(D) = T(B) \subseteq D$ and

$$\gamma(D) = \gamma\big(\overline{\text{co}}\,(T(B) \cup \{x\})\big) = \gamma(T(B) \cup \{x\}) = \gamma(T(B)) = \gamma(T(D)).$$

Since T is γ-condensing, this can only happen if $\gamma(D) = 0$, that is, if D is compact. Therefore T is a nonexpansive map of the compact convex set D into itself, so by Theorem 3.4, T must have a fixed point. □

Theorem 3.14. *Let (X, d) be a strictly convex complete metric space with convex round balls having property (R). Let γ be the measure of noncompactness in X and let $K \subset X$ be a nonempty closed bounded set. Assume that the map $T : K \to K$ is nonexpansive, γ-condensing and has property (C). Then T has a fixed point.*

Proof. By Proposition 3.10 and Theorem 3.13, T has a fixed point. □

References

[1] T. S. Alahmadi, Z. Kadelburg, S. Radenovic and N. Shahzad, *Strict convexity and De Marr's theorem in metric spaces,* Linear Nonlinear Anal. **1** (2015), 79–87.

[2] N. Aronszajn and P. Panitchpakdi, *Extension of uniformly continuous transformations and hyperconvex metric spaces*, Pacific J. Math. **6** (1956), 405–439.

[3] M. Borkowski, D. Bugajewski, Dariusz and D. Phulara, *On some properties of hyperconvex spaces*, Fixed Point Theory Appl. **2010**, Art. ID 213812, 19 pp.

[4] F. E. Browder, *Fixed-point theorems for noncompact mappings in Hilbert space*, Proc. Nat. Acad. Sci. U.S.A. **53** (1965), 1272–1276.

[5] F. E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. U.S.A. **54** (1965), 1041–1044.

[6] I. Bula, *Strictly convex metric spaces and fixed points*, Math. Moravica **3** (1999), 5–16.
[7] I. Bula and J. Viksna, *Example of strictly convex metric spaces with not convex balls*, Int. J. Pure Appl. Math. **25** (2005), 87–93.
[8] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers Group, Dordrecht, 1990.
[9] J. Eisenfeld and V. Lakshmikantham, *On a measure of nonconvexity and applications*, Yokohama Math. J. **24** (1976), 133–140.
[10] I. Galina, *On strict convexity*, Mathematics (Latvian), 193–198, Latv.Univ. Zinat. Raksti, 576, Latv. Univ., Riga, 1992
[11] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge, 1990.
[12] D. Göhde, *Zum Prinzip der kontraktiven Abbildung*, (in German) Math. Nachr. **30** (1965), 251–258.
[13] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006.
[14] I. Marrero, *A note on reflexivity and nonconvexity*, Nonlinear Anal. **74** (2011), 6890–6894.
[15] I. Marrero, *Weak compactness and the Eisenfeld-Lakshmikantham measure of nonconvexity*, Fixed Point Theory Appl. **5** (2012), 1–7.
[16] S. B. Nadler Jr., *Multivalued contraction mappings*, Pacific J. Math. **20** (2) (1969), 457–488.
[17] W. Takahashi, *A convexity in metric space and nonexpansive mappings,* I, Kodai Math. Semin. Rep. **22** (1970), 142–149.

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