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SOME TYPES OF MINIMAL ELEMENT THEOREMS AND EKELAND'S VARIATIONAL PRINCIPLES IN SET OPTIMIZATION

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Abstract. There are two types of criteria of solutions for the set-valued optimization problem, the vectorial criterion and set optimization criterion. The first criterion consists of looking for efficient points of set-valued map and is called set-valued vector optimization problem. On the other hand, Kuroiwa-Tanaka-Ha and Jahn-Ha started developing a new approach to set-valued optimization which is based on comparison among values of the set-valued map. In this paper, we treat the second type criterion and call it a set optimization problem.

As applications of the scalarizing technique, we present several types of minimal element theorems. Next, we derive several types of Ekeland's variational principles, Caristi's fixed point theorems and Takahashi's minimization theorems for set-valued map. Lastly, we prove the equivalences between the above theorems.

1. Introduction

Let Y be a topological vector space ordered by a closed convex cone $C \subset Y$. Let X be a nonempty set and $F : X \to 2^Y$ a set-valued map with domain X ($F(x) \neq \emptyset$ for each $x \in X$). The set-valued optimization problem is formalized as follows:

$$\text{(P)} \begin{cases} \text{Optimize} \quad F(x) \\ \text{Subject to} \quad x \in X \end{cases}$$

There are two types of criteria of solutions for the set-valued optimization problem, the vectorial criterion and set optimization criterion. The first criterion consists of looking for efficient points of the set $F(X) = \cup_{x\in X} F(x)$ and is called set-valued vector optimization problem. On the other hand, Kuroiwa-Tanaka-Ha [15] started developing a new approach to set-valued optimization using the six types of set relations. After that, Jahn-Ha[14] introduced new order relations in set optimization in 2011. The second criterion is based on comparison among values of F, that is, whole images $F(x)$ and seems to be more natural for set-valued optimization problem. In this paper, we treat the second type criterion and call it a set optimization problem.

A set-valued mapping from a set X into a set Y, which is usually regarded as a relation $F \subset X \times Y$, dose not necessarily satisfy the uniqueness property, that is, $(x_1, y_1), (x_2, y_2) \in F$ implies $y_1 = y_2$. In this paper, we consider subsets of $X \times 2^Y$

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and look for minimal elements of them with respect to appropriate order relations on 2^Y.

The aim of this paper is to present new type minimal element theorems with some kind of weak uniqueness property. In contrast to [12], we introduce new type order relations on $X \times 2^Y$ which depend on nonlinear scalarizing functions for sets and equivalent classes on 2^Y.

The organization of this paper is as follows. First, we introduce some types of nonlinear scalarizing technique for sets [2, 3, 4] which are generalization of Tammer-Weidner's scalarizing functions for the vector-valued case [9, 10]. Next, we give some minimal element theorems for set-valued map. There are some previous works ([8, 12, 21, 24] and their references therein), however, we present new type minimal element theorems. Next, we derive several types of Ekeland's variational principles, Caristi's fixed point theorems and Takahashi's minimization theorems for set-valued map which are generalizations of [1]. Lastly, we prove the equivalences between the above theorems. We remark that all existence theorems proved in this paper are special case of Brezis-Browder's theorem.

2. Mathematical preliminaries

2.1. **Mathematical terminology and notation.** Throughout of this paper, let Y be a topological vector space and 0_Y the origin of Y. For a set $A \subset Y$, int$A$ and cl$A$ denote the topological interior and the topological closure of A, respectively. We denote $\mathcal{V}$ by the family of nonempty subsets of Y. The sum of two sets $V_1, V_2 \in \mathcal{V}$ and the product of $\alpha \in \mathbb{R}$ and $V \in \mathcal{V}$ are defined by

$$V_1 + V_2 := \{v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2\} \qquad \alpha V := \{\alpha v \mid v \in V\}.$$

In this paper, we assume that $C \subset Y$ is a closed convex cone, that is, cl$C = C$, $C + C \subset C$ and $t \cdot C \subset C$ for all $t \in [0, \infty)$. For a nonempty set X and $F : X \to \mathcal{V}$, we denote $F(X) := \bigcup_{x \in X} F(x)$. Let P_X and P_Y be projections of $X \times Y$ onto X and Y, respectively, that is,

$$P_X(x, y) = x \qquad P_Y(x, y) = y$$

for every $(x, y) \in X \times Y$.

2.2. **Preliminaries in vector optimization.** A cone C is called pointed if $C \cap (-C) = \{0_Y\}$ and solid if int$C \neq \emptyset$.

Definition 2.1. For $a, b \in Y$ and a solid convex cone $C \subset Y$, we define

$$a \leq_C b \quad \text{by} \quad b - a \in C \qquad a \leq_{\text{int}C} b \quad \text{by} \quad b - a \in \text{int}C.$$

Proposition 2.2. *For $x \in Y$ and $y \in Y$, the following statements hold:*

(i) *$x \leq_C y$ implies that $x + z \leq_C y + z$ for all $z \in Y$,*
(ii) *$x \leq_C y$ implies that $\alpha x \leq_C \alpha y$ for all $\alpha \geq 0$,*
(iii) *$\leq_C$ is reflexive and transitive. Moreover, if C is pointed, $\leq_C$ is antisymmetric and hence a partial order.*

We say that a point $a \in A \subset Y$ is a minimal [resp. weak minimal] point of A if there is no $\hat{a} \in A \setminus \{a\}$ such that $\hat{a} \leq_C a$ [resp. $\hat{a} \leq_{\mathrm{int}C} a$]. The above definition is equivalent to

$$A \cap (a - C) = \{a\} \quad \text{[resp. } A \cap (a - \mathrm{int}C) = \emptyset\text{]}.$$

We denote $\mathrm{Min}(A; C)$[resp. $\mathrm{wMin}(A; \mathrm{int}C)$] by the set of minimal [resp. weak minimal] points of A with respect to C [resp. $\mathrm{int}C$], respectively. We can easily see that

$$\mathrm{Min}(A; C) \subset \mathrm{wMin}(A; \mathrm{int}C) \subset A.$$

2.3. **Preliminaries in set optimization.** We consider several types of binary relationships on $\mathcal{V}$ by using a solid convex cone $C \subset Y$.

Definition 2.3 ([15])**.** For A, $B \in \mathcal{V}$ and a solid closed convex cone $C \subset Y$, we define

$$(\textbf{weak type}) \quad A \leq_C^w B \quad \text{by} \quad B - A \subset C \quad (A \leq_{\mathrm{int}C}^w B \quad \text{by} \quad B - A \subset \mathrm{int}C),$$

$$(\textbf{lower type}) \quad A \leq_C^l B \quad \text{by} \quad B \subset A + C \quad (A \leq_{\mathrm{int}C}^l B \quad \text{by} \quad B \subset A + \mathrm{int}C),$$

$$(\textbf{upper type}) \quad A \leq_C^u B \quad \text{by} \quad A \subset B - C \quad (A \leq_{\mathrm{int}C}^u B \quad \text{by} \quad A \subset B - \mathrm{int}C),$$

$$(\textbf{strong type}) \quad A \leq_C^s B \quad \text{by} \quad 0_Y \in B - A - C \iff 0_Y \in A - B + C$$

$$(A \leq_{\mathrm{int}C}^s B \quad \text{by} \quad 0_Y \in B - A - \mathrm{int}C \iff 0_Y \in A - B + \mathrm{int}C).$$

Remark 1. In [15], they firstly defined the following set relation:

$$(\textbf{type 1}) \quad A \leq_C^{(1)} B \quad \text{by} \quad B - A \subset C.$$

After, Jahn-Ha[14] added reflexivity condition to the above definition. Also in [15], they firstly defined the following type 6 set relation

$$(\textbf{type 6}) \quad A \leq_C^{(6)} B \quad \text{by} \quad A \cap (B - C) \neq \emptyset \iff (A + C) \cap B \neq \emptyset.$$

We can easily show that type 6 set relation is equivalent to strong type set relation

$$0_Y \in B - A - C \iff A \cap (B - C) \neq \emptyset \iff (A + C) \cap B \neq \emptyset,$$

$$0_Y \in B - A - \mathrm{int}C \iff A \cap (B - \mathrm{int}C) \neq \emptyset \iff (A + \mathrm{int}C) \cap B \neq \emptyset.$$

Proposition 2.4 ([2, 16])**.** *For A, $B \in \mathcal{V}$, the following statements hold.*

(i) $A \leq_C^w B$ *implies* $A \leq_C^l B$ *and* $A \leq_C^l B$ *implies* $A \leq_C^s B$.
(ii) $A \leq_C^w B$ *implies* $A \leq_C^u B$ *and* $A \leq_C^u B$ *implies* $A \leq_C^s B$.
(iii) $A \leq_C^l B$ *and* $A \leq_C^u B$ *are not comparable, that is,* $A \leq_C^l B$ *does not imply* $A \leq_C^u B$ *and* $A \leq_C^u B$ *does not imply* $A \leq_C^l B$.

Proposition 2.5 ([16])**.** *For A, $B \in \mathcal{V}$ and $y \in Y$, the following statements hold.*

(i) $A \leq_C^w B$ *implies* $(A + y) \leq_C^w (B + y)$.
(ii) $A \leq_C^w B$ *implies* $\alpha A \leq_C^w \alpha B$ *for* $\alpha \geq 0$.
(iii) $\leq_C^w$ *is transitive.*

Remark 2. Since there are some $D \in \mathcal{V}$ such that $D - D \not\subset C$, we have that $A \leq_C^w B$ does not imply $(A + D) \leq_C^w (B + D)$ for all $A, B, D \in \mathcal{V}$.

Proposition 2.6 ([12, 16])**.** *For $A,\ B,\ D \in \mathcal{V}$, the following statements hold.*

(i) *$A \leq^l_C B$ implies $(A+D) \leq^l_C (B+D)$ and $A \leq^u_C B$ implies $(A+D) \leq^u_C (B+D)$.*
(ii) *For $\alpha \geq 0$, $A \leq^l_C B$ implies $\alpha A \leq^l_C \alpha B$ and $A \leq^u_C B$ implies $\alpha A \leq^u_C \alpha B$.*
(iii) *$\leq^l_C$ and $\leq^u_C$ are reflexive and transitive.*

Proposition 2.7 (see also [16])**.** *For $A, B, D \in \mathcal{V}$, the following statements hold.*

(i) *$A \leq^s_C B$ implies $(A+D) \leq^s_C (B+D)$.*
(ii) *$A \leq^s_C B$ implies $\alpha A \leq^s_C \alpha B$ for $\alpha \geq 0$.*
(iii) *$\leq^s_C$ is reflexive.*

Definition 2.8 ([18])**.** It is said that $A \in \mathcal{V}$ is

(i) C-closed $\big[(-C)$-closed$\big]$ if $A+C$ $[A-C]$ is a closed set,
(ii) C-bounded $\big[(-C)$-bounded$\big]$ if for each neighborhood U of zero in Y there is some positive number t such that

$$A \subset tU + C \quad [A \subset tU - C],$$

(iii) C-compact $\big[(-C)$-compact$\big]$ if any cover of A the form $\{U_\alpha + C |\ U_\alpha$ are open$\}$ $[\{U_\alpha - C |\ U_\alpha$ are open$\}]$ admits a finite subcover.

Every C-compact set is C-closed and C-bounded.

Definition 2.9 ([13])**.** It is said that $A \in \mathcal{V}$ is C-proper $\big[(-C)$-proper$\big]$ if

$$A + C \neq Y \qquad [A - C \neq Y].$$

We denote $\mathcal{V}_C$ by the family of C-proper subsets of Y and $\mathcal{V}_{-C}$ the family of $(-C)$-proper subsets of Y, respectively.

Remark 3. It sometimes happens that $\leq^l_C$ is equivalent to $\leq^l_{\mathrm{int}C}$. Thus when we need to distinguish between $\leq^l_C$ and $\leq^l_{\mathrm{int}C}$, we assume C-closedness of $A \in \mathcal{V}$. Similarly, when we need to distinguish between $\leq^u_C$ and $\leq^u_{\mathrm{int}C}$, we assume $(-C)$-closedness of $B \in \mathcal{V}$ (see example [2]).

Introducing the equivalence relations

$$A \sim_l B \Longleftrightarrow A \leq^l_C B \quad \text{and} \quad B \leq^l_C A,$$

$$A \sim_u B \Longleftrightarrow A \leq^u_C B \quad \text{and} \quad B \leq^u_C A,$$

we can generate a partial ordering on the set of equivalence classes which are denoted by $[\cdot]^l$ and $[\cdot]^u$, respectively. We can easily see that

$$A \in [B]^l \Longleftrightarrow A + C = B + C,$$

$$A \in [B]^u \Longleftrightarrow A - C = B - C.$$

Definition 2.10. ($l[u]$-minimal and $l[u]$-weak minimal element [13]) Let $\mathcal{S} \subset \mathcal{V}$. We say that $\bar{A} \in \mathcal{S}$ is a $l[u]$-minimal element if for any $A \in \mathcal{S}$,

$$A \leq^{l[u]}_C \bar{A} \quad \text{implies} \quad \bar{A} \leq^{l[u]}_C A.$$

Moreover, $\bar{A} \in \mathcal{S}$ is a $l[u]$-weak minimal element if for any $A \in \mathcal{S}$,

$$A \leq^{l[u]}_{\text{int}C} \bar{A} \quad \text{implies} \quad \bar{A} \leq^{l[u]}_{\text{int}C} A.$$

We denote the family of $l[u]$-minimal elements of $\mathcal{S}$ by $l[u]$-Min$\mathcal{S}$ and the family of $l[u]$-weak minimal elements of $\mathcal{S}$ by $l[u]$-wMin$\mathcal{S}$.

We can easily see that

$l[u]$-Min$\mathcal{S} \subset l[u]$-wMin$\mathcal{S} \subset \mathcal{S}$.

Remark 4. Since the weak type set relation does not satisfy reflexivity condition, we cannot define the concept of equivalent class and minimal element. Since the strong type set relation does not satisfy transitivity condition, we also cannot define the concept of equivalent class and minimal element (see also [14]).

3. Nonlinear scalarization

In this section, we assume $k^0 \in C \setminus (-C)$. First, we introduce the following scalarizing functions for vector

$$\varphi_{C,k^0} : Y \to (-\infty, \infty] \qquad \text{and} \qquad \psi_{C,k^0} : Y \to [-\infty, \infty),$$

$$\varphi_{C,k^0}(y) = \inf\{t \in \mathbb{R} \,|\, y \leq_C tk^0\} = \inf\{t \in \mathbb{R} \,|\, y \in tk^0 - C\},$$

$$\psi_{C,k^0}(y) = \sup\{t \in \mathbb{R} \,|\, tk^0 \leq_C y\} = \sup\{t \in \mathbb{R} \,|\, y \in tk^0 + C\}.$$

The above scalarization method is based on the sublinearity of φ_{C,k^0} and hence it is called "sublinear scalarization". This approach is found in Rubinov [22] and Pascoletti-Serafini [20], and it was developed and investigated by Tammer [10] and Luc [18]. It is similar to the idea of Minkowski functional, which is a type of gauge function. Moreover, φ_{C,k^0} has the order-monotone property (see for detail [19]). The above scalarizing functions for vector have the following property (see also [19])

$$\varphi_{C,k^0}(y) = -\psi_{C,k^0}(-y).$$

In the last 20 years, the investigations of scalarizing functions for sets which are generalizations of the above scalarizing function for vector have developed widely (see the history of the investigation [2, 3, 4] and their references therein). Agreeing $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$, we define $h^w_{\inf}, h^l_{\inf}, h^u_{\inf}, h^s_{\inf} : \mathcal{V} \to [-\infty, \infty]$

$$h^w_{\inf}(V) = \inf\{t \in \mathbb{R} \,|\, V \leq^w_C \{tk^0\}\} = \inf\{t \in \mathbb{R} \,|\, tk^0 - V \subset C\},$$

$$h^l_{\inf}(V) = \inf\{t \in \mathbb{R} \,|\, V \leq^l_C \{tk^0\}\} = \inf\{t \in \mathbb{R} \,|\, tk^0 \in V + C\},$$

$$h^u_{\inf}(V) = \inf\{t \in \mathbb{R} \,|\, V \leq^u_C \{tk^0\}\} = \inf\{t \in \mathbb{R} \,|\, V \subset tk^0 - C\},$$

$$h^s_{\inf}(V) = \inf\{t \in \mathbb{R} \,|\, V \leq^s_C \{tk^0\}\} = \inf\{t \in \mathbb{R} \,|\, (V + C) \cap \{tk^0\} \neq \emptyset\},$$

and $h^w_{\sup}, h^l_{\sup}, h^u_{\sup}, h^s_{\sup} : \mathcal{V} \to [-\infty, \infty]$

$$h^w_{\sup}(V) = \sup\{t \in \mathbb{R} \,|\, \{tk^0\} \leq^w_C V\} = \sup\{t \in \mathbb{R} \,|\, V - tk^0 \subset C\},$$

$$h^l_{\sup}(V) = \sup\{t \in \mathbb{R} \,|\, \{tk^0\} \leq^l_C V\} = \sup\{t \in \mathbb{R} \,|\, V \subset tk^0 + C\},$$

$$h^u_{\sup}(V) = \sup\{t \in \mathbb{R} \,\big|\, \{tk^0\} \leq^u_C V\} = \sup\{t \in \mathbb{R} \,\big|\, tk^0 \in V - C\},$$

$$h^s_{\sup}(V) = \sup\{t \in \mathbb{R} \,\big|\, \{tk^0\} \leq^s_C V\} = \sup\{t \in \mathbb{R} \,\big|\, \{tk^0\} \cap (V - C) \neq \emptyset\}.$$

The functions $h^w_{\inf}, h^l_{\inf}, h^u_{\inf}, h^s_{\inf}$ and $h^w_{\sup}, h^l_{\sup}, h^u_{\sup}, h^s_{\sup}$ play the role of utility functions. By the definitions of the above scalarizing functions for sets, we obtain the following relationships.

Proposition 3.1 ([2])**.** *The following statements hold:*

(i) $h^l_{\sup}(V) = -h^u_{\inf}(-V)$;
(ii) $h^u_{\sup}(V) = -h^l_{\inf}(-V)$.

Proposition 3.2. *The following statements hold:*

(i) $h^s_{\inf}(V) = h^l_{\inf}(V) \leq h^u_{\inf}(V) = h^w_{\inf}(V)$;
(ii) $h^w_{\inf}(V) = h^l_{\sup}(V) \leq h^u_{\sup}(V) = h^s_{\sup}(V)$;
(iii) $h^w_{\sup}(V) = -h^w_{\inf}(-V)$;
(iv) $h^s_{\sup}(V) = -h^s_{\inf}(-V)$.

Proof. We have for all $t \in \mathbb{R}$

$$\{V \in \mathcal{V} \,\big|\, V \subset tk^0 - C\} \subset \{V \in \mathcal{V} \,\big|\, \{tk^0\} \subset V + C\} \qquad \text{and}$$

$$\{V \in \mathcal{V} \,\big|\, V \subset tk^0 + C\} \subset \{V \in \mathcal{V} \,\big|\, \{tk^0\} \subset V - C\}$$

and hence conclusion (i) and (ii) follows. Conclusion (iii) and (iv) are easily derived by the definitions of scalarizing functions. □

Definition 3.3. We say that the function $f : \mathcal{V} \to [-\infty, \infty]$ is

(i) $\leq^l_C$-increasing if $V_1 \leq^l_C V_2$ implies $f(V_1) \leq f(V_2)$,
(ii) strictly $\leq^l_{\text{int}C}$-increasing if $V_1 \leq^l_{\text{int}C} V_2$ $(V_1 \neq V_2)$ implies $f(V_1) < f(V_2)$.

The definitions of $\leq^u_C$-increasing, $\leq^w_C$-increasing, $\leq^s_C$-increasing, strictly $\leq^u_{\text{int}C}$-increasing, strictly $\leq^w_{\text{int}C}$-increasing and strictly $\leq^s_{\text{int}C}$-increasing are similar to the above ones, respectively.

3.1. l-type.

Lemma 3.4 ([4])**.** *Let $k^0 \in \text{int}C$. The function $h^l_{\inf} : \mathcal{V}_C \to (-\infty, \infty]$ has the following properties:*

(i) $h^l_{\inf}(V) \leq t \iff tk^0 \in \text{cl}(V + C)$;
(ii) $h^l_{\inf}$ *is* $\leq^l_C$*-increasing;*
(iii) $h^l_{\inf}(V + \lambda k^0) = h^l_{\inf}(V) + \lambda$ *for every* $\lambda \in \mathbb{R}$;
(iv) $\hat{V} \in [V]^l \Longrightarrow h^l_{\inf}(\hat{V}) = h^l_{\inf}(V)$;
(v) $h^l_{\inf}$ *is sublinear.*
(vi) $h^l_{\inf}$ *achieves a real value;*
(vii) $h^l_{\inf}(V) < t \iff tk^0 \in V + \text{int}C$;
(viii) $h^l_{\inf}$ *is strictly* $\leq^l_{\text{int}C}$*-increasing.*

Example 1. We remark that the monotonicity of $\leq_C^s$ is not guaranteed. We set

$$Y = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0\}, \qquad k^0 = (1, 1),$$

$$V_1 = [0, 1] \times [3, 4] \qquad V_2 = [2, 3] \times [2, 3].$$

We can check that $V_1 \leq_C^s V_2$ and $(3 =)h^s_{\inf}(V_1) > h^s_{\inf}(V_2)(= 2)$.

Example 2 ([4])**.** Assumption of closedness in conclusion (i) of Lemma 3.4 is needed. We set

$$Y = \mathbb{R}^2, \quad C = \mathbb{R}^2_+, \quad k^0 = (1, 1), \quad V = \{(x, y) \mid y \leq -\frac{1}{x},\ x > 0\}.$$

We can check that

$$h^l_{\inf}(V) \leq 0 \iff 0_Y \in V + C$$

is false since $V + C = \{(x, y) \mid x > 0\}$ is an open set which does not contain 0_Y.

Example 3 ([4])**.** Assumption of C-properness and $k^0 \in \mathrm{int}C$ are needed to show that $h^l_{\inf} > -\infty$. We set

$$Y = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+, \qquad k^0 = (1, 1), \qquad V = \{(x, y) \mid y \leq \frac{1}{x},\ x < 0\}.$$

We can check that $V + C = Y$ and $h^l_{\inf}(V) = -\infty$.

Example 4 ([4])**.** Assumption $k^0 \in \mathrm{int}C$ is needed to show that $h^l_{\inf}$ is a real-valued function and $h^l_{\inf}$ is a strictly $\leq^l_{\mathrm{int}C}$-increasing function. We set

$$Y = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+, \qquad k^0 = (1, 0), \qquad V_1 = (1, 2)\times(1, 2), \qquad V_2 = (1, 3)\times(1, 3).$$

We can check that $V_1 \leq^l_{\mathrm{int}C} V_2$, however, $h^l_{\inf}(V_1) = h^l_{\inf}(V_2) = \infty$.

3.2. u**-type.**

Lemma 3.5 ([4])**.** *Let $k^0 \in \mathrm{int}C$. The function $h^u_{\inf} : \mathcal{V} \to (-\infty, \infty]$ has the following properties:*

(i) $h^u_{\inf}(V) \leq t \iff V \subset tk^0 - C$;
(ii) $h^u_{\inf}$ *is* $\leq^u_C$*-increasing;*
(iii) $h^u_{\inf}(V + \lambda k^0) = h^u_{\inf}(V) + \lambda$ *for every* $\lambda \in \mathbb{R}$;
(iv) $\hat{V} \in [V]^u \Longrightarrow h^u_{\inf}(\hat{V}) = h^u_{\inf}(V)$;
(v) $h^u_{\inf}$ *is sublinear.*
(vi) $h^u_{\inf}(V) < t \Longrightarrow V \subset tk^0 - \mathrm{int}C$.

Moreover, if $k^0 \in \mathrm{int}C$ and V is $(-C)$-bounded then $h^u_{\inf}$ has the following property:

(vii) $h^u_{\inf}$ *achieves a real value.*

Furthermore, if $k^0 \in \mathrm{int}C$ and V is $(-C)$-compact then $h^u_{\inf}$ has the following properties:

(viii) $V \subset tk^0 - \mathrm{int}C \Longrightarrow h^u_{\inf}(V) < t$;
(ix) $h^u_{\inf}$ *is strictly* $\leq^u_{\mathrm{int}C}$*-increasing.*

Example 5 ([4])**.** Assumption of $(-C)$-boundedness is needed to show that $h^u_{\inf} < \infty$. We set

$$Y = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+, \qquad k^0 = (1,1), \qquad V = \{(x,y) \mid x = 1\}.$$

We can check that V is not $(-C)$-bounded and $h^u_{\inf}(V) = \infty$.

Example 6 ([4])**.** Assumption $k^0 \in \mathrm{int}C$ is needed to show that $h^u_{\inf}$ is a real-valued function and $h^u_{\inf}$ is a strictly $\leq^u_{\mathrm{int}C}$-increasing function. We set

$$Y = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+, \qquad k^0 = (1,0),$$

$$V_1 = \{(x,y) \mid x \leq 0,\ 0 \leq y \leq 1\}, \qquad V_2 = \{(x,y) \mid x \leq 1,\ 1 \leq y \leq 2\}.$$

We can check that even if $V_1, V_2 \in \mathcal{V}$ is $(-C)$-compact and $V_1 \leq^u_{\mathrm{int}C} V_2$, however, $h^u_{\inf}(V_1) = h^u_{\inf}(V_2) = \infty$.

Example 7 ([4])**.** Assumption of $(-C)$-compactness is needed to show that $h^u_{\inf}$ is a strictly $\leq^u_{\mathrm{int}C}$-increasing function. We set

$$Y = \mathbb{R}^2, \qquad C = \mathbb{R}^2_+, \qquad k^0 = (1,1),$$

$$V_1 = \{(x,y) \mid 0 \leq y \leq 1\}, \quad V_2 = \{(x,y) \mid 2 \leq y \leq 3\}.$$

We can check that $V_1 \leq^u_{\mathrm{int}C} V_2$, however, $h^u_{\inf}(V_1) = h^u_{\inf}(V_2) = \infty$.

3.3. **Nonconvex separation type theorems for sets.**

Corollary 3.6 (l**-type**, revised version of [2])**.** *Let Y be a topological vector space, $C \subset Y$ a solid closed convex cone, $k^0 \in \mathrm{int}C$ and $V \in \mathcal{V}_C$ a C-closed set. Then we have*

$$0_Y \notin V + C \iff h^l_{\inf}(V) > 0.$$

Moreover, if $k^0 \in \mathrm{int}C$, then we have

$$0_Y \notin V + \mathrm{int}C \iff h^l_{\inf}(V) \geq 0.$$

Proof. The proof of this Corollary is a consequence of (i) and (vii) of Lemma 3.4. □

Corollary 3.7 (u**-type**, revised version of [2])**.** *Let Y be a topological vector space, $C \subset Y$ a solid closed convex cone, $k^0 \in \mathrm{int}C$ and $V \in \mathcal{V}$ is a $(-C)$-bounded set. Then we have*

$$V \not\subset -C \iff h^u_{\inf}(V) > 0.$$

Moreover, if $k^0 \in \mathrm{int}C$ and $V \in \mathcal{V}$ is a $(-C)$-compact set, then we have

$$V \not\subset -\mathrm{int}C \iff h^u_{\inf}(V) \geq 0.$$

Proof. The proof of this Corollary is a consequence of (i), (vi) and (viii) of Lemma 3.5. □

4. Main results

Brezis and Browder generalized Ekeland's variational principle, which is a minimal point theorem on a quasi-ordered set.

Theorem 4.1 (Brezis-Browder[5]). *Let $(W, \preceq)$ be a quasi-ordered set (that is, $\preceq$ is a reflexive and transitive relation on W) and let $\phi : W \to \mathbb{R}$ be a function satisfying*

(A1) *ϕ is bounded below,*
(A2) *$w_1 \preceq w_2$ implies $\phi(w_1) \leq \phi(w_2)$,*
(A3) *for every $\preceq$-decreasing sequence $\{w_n\}_{n\in\mathbb{N}} \subseteq W$ there exists some $w \in W$ such that $w \preceq w_n$ for all $n \in \mathbb{N}$.*

Then for every $w_0 \in W$ there exists some $\bar{w} \in W$ such that

(i) *$\bar{w} \preceq w_0$,*
(ii) *$\hat{w} \preceq \bar{w}$ implies $\phi(\hat{w}) = \phi(\bar{w})$.*

The above theorem plays an important role in this paper. We define the following ordering relation on $X \times \mathcal{V}_C$ (l-type and strong type) and $X \times \mathcal{V}$ (u-type and weak type), respectively, where X is a metric space.

$$(x_1, V_1) \preceq^w_{k^0} (x_2, V_2) \Longleftrightarrow V_1 + d(x_1, x_2)k^0 \leq^w_C V_2$$
$$(x_1, V_1) \preceq^l_{k^0} (x_2, V_2) \Longleftrightarrow V_1 + d(x_1, x_2)k^0 \leq^l_C V_2$$
$$(x_1, V_1) \preceq^u_{k^0} (x_2, V_2) \Longleftrightarrow V_1 + d(x_1, x_2)k^0 \leq^u_C V_2$$
$$(x_1, V_1) \preceq^s_{k^0} (x_2, V_2) \Longleftrightarrow V_1 + d(x_1, x_2)k^0 \leq^s_C V_2$$

Proposition 4.2. *We have the following properties:*

(i) *$\preceq^l_{k^0}$ and $\preceq^u_{k^0}$ are reflexive and transitive on $X \times \mathcal{V}_C$ and $X \times \mathcal{V}$, respectively;*
(ii) *$\preceq^w_{k^0}$ is transitive on $X \times \mathcal{V}$;*
(iii) *$\preceq^s_{k^0}$ is reflexive on $X \times \mathcal{V}_C$;*
(iv) *$(\preceq^w_{k^0}) \Longrightarrow (\preceq^l_{k^0}) \Longrightarrow (\preceq^s_{k^0})$ and $(\preceq^w_{k^0}) \Longrightarrow (\preceq^u_{k^0}) \Longrightarrow (\preceq^s_{k^0})$.*

4.1. **Minimal element theorems for set-valued map.** The aim of this subsection is to present minimal element theorems by using Brezis-Browder's principle, sublinear scalarizing functions for sets and nonconvex separation type theorems. In [12], they defined order relations $\preceq^l_{k^0}, \preceq^u_{k^0}$ and presented firstly minimal element theorems with respect to $\preceq^l_{k^0}$ and $\preceq^u_{k^0}$.

We define the following new order relations on $X \times \mathcal{V}_C$ and $X \times \mathcal{V}$, respectively, where X is a metric space. The idea of these relations depends on [8] and chapter 2 of [11].

$$(x_1, V_1) \preceq^l_{k^0, h^l_{\inf}} (x_2, V_2) \Longleftrightarrow \begin{cases} (x_1, V_1) \preceq^l_{k^0} (x_2, V_2) \\ h^l_{\inf}(V_1) < h^l_{\inf}(V_2) \end{cases} \text{or} \quad \begin{cases} x_1 = x_2 \\ V_2 \in [V_1]^l \end{cases}$$

$$(x_1, V_1) \preceq^u_{k^0, h^u_{\inf}} (x_2, V_2) \Longleftrightarrow \begin{cases} (x_1, V_1) \preceq^u_{k^0} (x_2, V_2) \\ h^u_{\inf}(V_1) < h^u_{\inf}(V_2) \end{cases} \text{or} \quad \begin{cases} x_1 = x_2 \\ V_2 \in [V_1]^u \end{cases}$$

We also see that $\preceq^l_{k^0, h^l_{\inf}}$ and $\preceq^u_{k^0, h^u_{\inf}}$ are reflexive and transitive on $X \times \mathcal{V}_C$ and $X \times \mathcal{V}$, respectively.

Theorem 4.3 (**l-type**). *Let X be a complete metric space and Y a topological vector space, $C \subset Y$ a solid closed convex cone, $\mathcal{V}_C$ a family of C-closed subsets of Y, $k^0 \in \mathrm{int}C$ and $\mathcal{A} \subset X \times \mathcal{V}_C$ a nonempty set. We assume the following conditions:*

(i) *$\mathcal{A}$ is bounded below (there exists $\tilde{V} \in \mathcal{V}_C$ such that $0_Y \notin P_{\mathcal{V}_C}(\mathcal{A}) - \tilde{V} + C$);*

(ii) *For all $\preceq^l_{k^0}$-decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with $x_n \to x \in X$, there exists $(x, V) \in \mathcal{A}$ such that $(x, V) \preceq^l_{k^0} (x_n, V_n)$ for all $n \in \mathbb{N}$.*

Then for every $(x_0, V_0) \in \mathcal{A}$ there exists $(\bar{x}, \bar{V}) \in \mathcal{A}$ such that

(a) *$(\bar{x}, \bar{V}) \preceq^l_{k^0} (x_0, V_0)$, and*

(b) *If $(\hat{x}, \hat{V}) \in \mathcal{A}$ such that $(\hat{x}, \hat{V}) \preceq^l_{k^0} (\bar{x}, \bar{V})$ then $\hat{x} = \bar{x}$.*

Moreover, if we replace $\preceq^l_{k^0}$ with $\preceq^l_{k^0,h^l_{\inf}}$, conclusion (b) can be replaced to

(b') *If $(\hat{x}, \hat{V}) \in \mathcal{A}$ such that $(\hat{x}, \hat{V}) \preceq^l_{k^0,h^l_{\inf}} (\bar{x}, \bar{V})$ then $\hat{x} = \bar{x}$ and $\hat{V} \in [\bar{V}]^l$.*

Proof. Let

$$\mathcal{A}_0 := \{(x, V) \in \mathcal{A} \mid (x, V) \preceq^l_{k^0} (x_0, V_0)\}.$$

We apply the Brezis-Browder principle to the quasi-ordered set $(\mathcal{A}_0, \preceq^l_{k^0})$ and the following functional

$$\phi : \mathcal{A}_0 \to \mathbb{R}, \qquad \phi(x, V) := h^l_{\inf}(V).$$

We show that ϕ satisfies the assumptions of Theorem 4.1. By Corollary 3.6 and (v) of Lemma 3.4, we have for $\tilde{V} \in \mathcal{V}_C$

$$0 < h^l_{\inf}(P_{\mathcal{V}_C}(\mathcal{A}) - \tilde{V}) \leq h^l_{\inf}(P_{\mathcal{V}_C}(\mathcal{A})) + h^l_{\inf}(-\tilde{V})$$

for all $x \in X$ and hence

$$-\infty < -h^l_{\inf}(-\tilde{V}) \leq h^l_{\inf}(P_{\mathcal{V}_C}(\mathcal{A})).$$

Then, we have that $h^l_{\inf}(P_{\mathcal{V}_C}(\mathcal{A}))$ is bounded from below on X, that is, (A1) holds. By condition (ii) and (iii) of Lemma 3.4, we have that

$$(x_1, V_1) \preceq^l_{k^0} (x_2, V_2) \quad \left(\iff V_1 + d(x_1, x_2)k^0 \leq^l_C V_2\right)$$

implies

$$h^l_{\inf}(V_1) + d(x_1, x_2) \leq h^l_{\inf}(V_2)$$

and hence

$$h^l_{\inf}(V_1) \leq h^l_{\inf}(V_2),$$

that is, (A2) holds. We easily see that condition (ii) implies (A3). Therefore, by Theorem 4.1, for every $(x_0, V_0) \in \mathcal{A}_0$ there exists $(\bar{x}, \bar{V}) \in \mathcal{A}_0$ such that

(1) $(\bar{x}, \bar{V}) \preceq^l_{k^0} (x_0, V_0)$,

(2) $(\hat{x}, \hat{V}) \preceq^l_{k^0} (\bar{x}, \bar{V})$ implies $\phi(\hat{x}, \hat{V}) = \phi(\bar{x}, \bar{V})$.

Condition (1) implies conclusion (a). Since $(\hat{x}, \hat{V}) \in \mathcal{A}_0$, by condition (ii) and (iii) of Lemma 3.4, we have that

$$h^l_{\inf}(\hat{V}) + d(\hat{x}, \bar{x}) \leq h^l_{\inf}(\bar{V}).$$

Now we have $h^l_{\inf}(\hat{V}) = \phi(\hat{x}, \hat{V}) = \phi(\bar{x}, \bar{V}) = h^l_{\inf}(\bar{V})$, we obtain $d(\hat{x}, \bar{x}) = 0$ and hence $\hat{x} = \bar{x}$, that is, conclusion (b) holds.

To prove (b'), let

$$\mathcal{B}_0 := \{(x, V) \in \mathcal{A} \mid (x, V) \preceq^l_{k^0, h^l_{\inf}} (x_0, V_0)\},$$

$$\phi : \mathcal{B}_0 \to \mathbb{R}, \qquad \phi(x, V) := h^l_{\inf}(V).$$

Similarly, we also show that ϕ satisfies the assumptions of Theorem 4.1 and we obtain conclusion (b'). □

In a similar way as Theorem 4.3, we obtain u-type minimal element theorem.

Theorem 4.4 (**u-type**). *Let X be a complete metric space and Y a topological vector space, $C \subset Y$ a solid closed convex cone, $\mathcal{V}$ a family of $(-C)$-bounded subsets of Y, $k^0 \in \text{int}C$ and $\mathcal{A} \subset X \times \mathcal{V}$ a nonempty set. We assume the following conditions:*

(i) *$\mathcal{A}$ is bounded below*
(there exists $\tilde{V} \in \mathcal{V}$ and $\tilde{V} \neq Y$ such that $P_{\mathcal{V}}(\mathcal{A}) - \tilde{V} \not\subset -C$);

(ii) *For all $\preceq^u_{k^0}$-decreasing sequence $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subset \mathcal{A}$ with $x_n \to x \in X$, there exists $(x, V) \in \mathcal{A}$ such that $(x, V) \preceq^u_{k^0} (x_n, V_n)$ for all $n \in \mathbb{N}$.*

Then for every $(x_0, V_0) \in \mathcal{A}$ there exists $(\bar{x}, \bar{V}) \in \mathcal{A}$ such that

(a) *$(\bar{x}, \bar{V}) \preceq^u_{k^0} (x_0, V_0)$, and*

(b) *If $(\hat{x}, \hat{V}) \in \mathcal{A}$ such that $(\hat{x}, \hat{V}) \preceq^u_{k^0} (\bar{x}, \bar{V})$ then $\hat{x} = \bar{x}$.*

Moreover, if we replace $\preceq^u_{k^0}$ with $\preceq^u_{k^0, h^u_{\inf}}$, conclusion (b) can be replaced to

(b') *If $(\hat{x}, \hat{V}) \in \mathcal{A}$ such that $(\hat{x}, \hat{V}) \preceq^u_{k^0, h^u_{\inf}} (\bar{x}, \bar{V})$ then $\hat{x} = \bar{x}$ and $\hat{V} \in [\bar{V}]^u$.*

Remark 5. In [12], they introduced the following condition (Theorem 5.1 and 5.2):

(Hamel-Löhne): Let Y a topological vector space, $C \subset Y$ be a proper closed convex cone and $K \subset Y$ a convex cone. If, in addition, $k^0 \in K \backslash \{0_Y\} \subset \text{int}C$ and if for each $(x, V) \in \mathcal{A}_0$, V is compact, then (ii) can be strengthened to (ii') If $(\hat{x}, \hat{V}) \in \mathcal{A}$ such that $(\hat{x}, \hat{V}) \preceq^{l[u]} (\bar{x}, \bar{V})$ then $\hat{x} = \bar{x}$ and $\hat{V} \cap \bar{V} \neq \emptyset$.

We compare the above condition (ii') to (b') of Theorem 4.3. Then we see that condition "$\hat{V} \cap \bar{V} \neq \emptyset$" is clearly different from "$\hat{V} \in [\bar{V}]^l$ ($\iff$ $\hat{V} + C = \bar{V} + C$)". (Theorem 4.4 is similar). Additionally in Theorem 4.3 and 4.4, we do not assume compactness condition on $V \in \mathcal{V}$.

Moreover, if we consider some scalarizing function which satisfies monotonicity condition, that is, for $A, B \in \mathcal{V}$ and scalarizing function $f : \mathcal{V} \to \mathbb{R}$ we have that $A \leq^l_C B$ implies $f(A) \leq f(B)$, then

$$A \in [B]^l \quad \Longrightarrow \quad f(A) = f(B).$$

Therefore, we conclude that Theorem 4.3 and 4.4 are new type minimal element theorems in set optimization problem. Especially, we remark that the conclusion (b') in Theorem 4.3 and 4.4 are some kind of weak "uniqueness" condition.

By using Proposition 4.2, we obtain the following theorem.

Theorem 4.5 (**strong type**). *We assume either hypothesis of Theorem 4.3 or hypothesis of Theorem 4.4. Then for every* $(x_0, V_0) \in \mathcal{A}$ *there exists* $(\bar{x}, \bar{V}) \in \mathcal{A}$ *such that*

(a) $(\bar{x}, \bar{V}) \preceq^s_{k^0} (x_0, V_0)$, *and*

(b) *If* $(\hat{x}, \hat{V}) \in \mathcal{A}$ *such that* $(\hat{x}, \hat{V}) \preceq^s_{k^0} (\bar{x}, \bar{V})$ *then* $\hat{x} = \bar{x}$.

Remark 6. Since $\preceq^s_{k^0}$ satisfies only reflexive condition, we conclude that Theorem 4.5 is like a pseudo minimal element theorem.

4.2. **Ekeland's variational principles for set-valued map.** In 1972, Ekeland[7] presented the following variational principle, which provides powerful tools in modern variational analysis.

Theorem 4.6 (Ekeland[7]). *Let* (X, d) *be a complete metric space and* $f : X \to (-\infty, \infty]$ *a l.s.c. function,* $\not\equiv +\infty$*, bounded from below. Let* $\varepsilon > 0$ *and* $u \in X$ *satisfy*

$$f(u) \leq \inf_{x \in X} f(x) + \varepsilon.$$

Then there exists $v \in X$ *such that*

(i) $f(v) \leq f(u)$,

(ii) $d(u, v) \leq 1$, *and*

(iii) *for each* $w \neq v$, $\quad f(v) - \varepsilon d(v, w) < f(w)$.

In this subsection, by using scalarizing functions $h^l_{\inf}$ and $h^u_{\inf}$, we obtain three types of Ekeland's variational principles for set-valued map. We consider the following conditions:

(C-$\leq^l_C$): Let X be a complete metric space and Y a topological vector space, $C \subset Y$ a solid closed convex cone, $k^0 \in \text{int}C$ and $F : X \to \mathcal{V}_C$ a C-closed valued function.

(i) F is bounded below (there exists $\tilde{V} \in \mathcal{V}_C$ such that $0_Y \notin F(X) - \tilde{V} + C$);

(ii) $\{\hat{x} \in X | (\hat{x}, F(\hat{x})) \preceq^l_{k^0} (x, F(x))\}$ is closed for all $x \in X$.

(C-$\leq^u_C$): Let X be a complete metric space and Y a topological vector space, $C \subset Y$ a solid closed convex cone, $k^0 \in \text{int}C$ and $F : X \to \mathcal{V}$ a $(-C)$-bounded valued function.

(i) F is bounded below
(there exists $\tilde{V} \in \mathcal{V}$ and $\tilde{V} \neq Y$ such that $F(X) - \tilde{V} \not\subset -C$);

(ii) $\{\hat{x} \in X | (\hat{x}, F(\hat{x})) \preceq^u_{k^0} (x, F(x))\}$ is closed for all $x \in X$.

Theorem 4.7 (*l*-**type**). *We assume* **(C-$\leq^l_C$)**. *Moreover, we assume*

(Ekeland-$\leq^l_C$): *for* $k^0 \in \text{int}C$ *and* $x_0 \in X$ *with* $F(x_0) \not\subset F(X) + k^0 + \text{int}C$.

Then there exists $\bar{x} \in X$ *such that*

(i) $F(\bar{x}) \leq^l_C F(x_0)$,

(ii) $d(\bar{x}, x_0) \leq 1$ *and*

(iii) $F(x) + d(\bar{x}, x)k^0 \not\leq^l_C F(\bar{x})$ *for all* $x \neq \bar{x}$.

Proof. Let $\mathcal{A} = \text{gr}F := \{(x, F(x)) | x \in X\} \subset X \times \mathcal{V}_C$. Of course, $P_{\mathcal{V}_C}(\mathcal{A}) = F(X)$. Let us show that condition (ii) of Theorem 4.3 holds. Let $\{(x_n, V_n)\}_{n \in \mathbb{N}} \subset \mathcal{A}$ be a $\preceq^l_{k^0}$-decreasing sequence with $x_n \to x \in X$. Of course, $V_n = F(x_n)$. For all $n, p \in \mathbb{N}$, we have that

$$x_{n+p} \in \mathcal{A}_n := \{x \in X | F(x) + d(x, x_n)k^0 \leq^l_C F(x_n)\}$$

By condition (ii) of **(C-$\leq^l_C$)**, $\mathcal{A}_n$ contains a limit x of the sequence $(x_{n+p})_{p\in\mathbb{N}}$. Therefore, $(x, F(x)) \preceq^l_{k^0} (x_n, F(x_n))$ for every n. Applying Theorem 4.3, for every $x_0 \in X$ we obtain $\bar{x} \in X$ such that $(\bar{x}, F(\bar{x})) \in \mathrm{gr}F$ satisfies

(a) $(\bar{x}, F(\bar{x})) \preceq^l_{k^0} (x_0, F(x_0))$,
(b) $(x, F(x)) \not\preceq^l_{k^0} (\bar{x}, F(\bar{x}))$ for all $x \neq \bar{x}$.

Condition (b) is condition (iii) of Theorem 4.7. To prove condition (ii), we suppose that $d(\bar{x}, x_0) > 1$. Then we have that

$$(d(\bar{x}, x_0) - 1)k^0 + C \subset \mathrm{int}C.$$

By condition (a), we have that

$$F(x_0) \subset F(\bar{x}) + d(\bar{x}, x_0)k^0 + C$$

and hence

$$F(x_0) - k^0 \subset F(\bar{x}) + d(\bar{x}, x_0)k^0 - k^0 + C \subset F(\bar{x}) + \mathrm{int}C.$$

Therefore, we obtain $F(x_0) \subset F(\bar{x}) + k^0 + \mathrm{int}C$, which contradicts **(Ekeland-$\leq^l_C$)**. □

In a similar way as Theorem 4.7, we obtain u-type and strong type Ekeland's variational principles.

Theorem 4.8 (**u-type**). *We assume* **(C-$\leq^u_C$)**. *Moreover, we assume*

(Ekeland-$\leq^u_C$): *for $k^0 \in \mathrm{int}C$ and $x_0 \in X$ with $F(X) + k^0 \not\subset F(x_0) - \mathrm{int}C$.*

Then there exists $\bar{x} \in X$ such that

(i) $F(\bar{x}) \leq^u_C F(x_0)$,
(ii) $d(\bar{x}, x_0) \leq 1$ *and*
(iii) $F(x) + d(\bar{x}, x)k^0 \not\leq^u_C F(\bar{x})$ *for all* $x \neq \bar{x}$.

Theorem 4.9 (**strong type**). *We assume either* **(C-$\leq^l_C$)** *or* **(C-$\leq^u_C$)**. *Moreover, we assume*

(Ekeland-$\leq^s_C$): *for $k^0 \in \mathrm{int}C$ and $x_0 \in X$ with $\big(F(X) + k^0 + \mathrm{int}C\big) \cap F(x_0) = \emptyset$.*

Then there exists $\bar{x} \in X$ such that

(i) $F(\bar{x}) \leq^s_C F(x_0)$,
(ii) $d(\bar{x}, x_0) \leq 1$ *and*
(iii) $F(x) + d(\bar{x}, x)k^0 \not\leq^s_C F(\bar{x})$ *for all* $x \neq \bar{x}$.

Proof. Applying Theorem 4.3 or Theorem 4.4 and Proposition 4.2, for every $x_0 \in X$ we obtain $\bar{x} \in X$ such that $(\bar{x}, F(\bar{x})) \in \mathrm{gr}F$ satisfies

(a) $(\bar{x}, F(\bar{x})) \preceq^s_{k^0} (x_0, F(x_0))$,
(b) $(x, F(x)) \not\preceq^s_{k^0} (\bar{x}, F(\bar{x}))$ for all $x \neq \bar{x}$.

Condition (b) is condition (iii) of Theorem 4.9. To prove condition (ii), we suppose that $d(\bar{x}, x_0) > 1$. Then we have that $(d(\bar{x}, x_0) - 1)k^0 + C \subset \text{int}C$ and hence

$$F(\bar{x}) + d(\bar{x}, x_0)k^0 + C \subset F(\bar{x}) + k^0 + \text{int}C.$$

By condition (a), we have that

$$\big(F(\bar{x}) + d(\bar{x}, x_0)k^0 + C\big) \cap F(x_0) \neq \emptyset$$

and hence

$$\big(F(\bar{x}) + k^0 + \text{int}C\big) \cap F(x_0) \neq \emptyset,$$

which contradicts **(Ekeland-$\le_C^s$)**. By condition (a), we have also

$$\big(F(\bar{x}) + C\big) \cap F(x_0) \neq \emptyset,$$

which implies condition (i). □

4.3. Caristi's fixed point theorems for set-valued map.

Theorem 4.10 (Caristi[6]). *Let (X, d) be a complete metric space and $f : X \to (-\infty, \infty]$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Assume that $T : X \to X$ satisfies*

$$d(x, Tx) \le f(x) - f(Tx)$$

for each $x \in X$, then T has a fixed point in X, that is, there exists $x_0 \in X$ with $Tx_0 = x_0$.

In this subsection, by using scalarizing functions $h^l_{\inf}$ and $h^u_{\inf}$, we obtain three types of Caristi's fixed point theorems for set-valued map.

Theorem 4.11 (*l*-**type**). *We assume* **(C-$\le_C^l$)**. *Moreover, we assume*

(Caristi-$\le_C^l$): *if $T : X \to 2^X$ is a multivalued mapping such that for every $x \in X$ there exists $y \in Tx$ such that $F(y) + d(x, y)k^0 \le_C^l F(x)$,*

then T has a fixed point in X, that is, there exists $\bar{x} \in X$ with $\bar{x} \in T\bar{x}$. Furthermore, for every $x \in X$ we have $Tx \neq \emptyset$ and for every $y \in Tx$, f satisfies the above inequality, then T has a critical point in X, that is, there exists $\bar{x} \in X$ such that $T\bar{x} = \{\bar{x}\}$.

Proof. By Theorem 4.7, there exists $\bar{x} \in X$ such that

$$F(y) + d(\bar{x}, y)k^0 \not\le_C^l F(\bar{x}) \quad \text{for all} \quad y \in X \setminus \{\bar{x}\}. \tag{4.1}$$

On the other hand by condition **(Caristi-$\le_C^l$)**, there exists $y \in X$ such that $y \in T\bar{x}$ and

$$F(y) + d(\bar{x}, y)k^0 \le_C^l F(\bar{x}).$$

Because of (4.1), we have $\bar{x} = y$. Therefore, T has at least one fixed point. Moreover, all the $y \in T\bar{x}$ being equal to $\bar{x}$, we have that T has a critical point. □

In a similar way as Theorem 4.11, we obtain u-type and strong type Caristi's fixed point theorems.

Theorem 4.12 (*u*-**type**). *We assume* **(C-$\le_C^u$)**. *Moreover, we assume*

(Caristi-$\leq_C^u$): *if $T : X \to 2^X$ is a multivalued mapping such that for every $x \in X$ there exists $y \in Tx$ such that $F(y) + d(x,y)k^0 \leq_C^u F(x)$,*

then T has a fixed point in X. Furthermore, for every $x \in X$ we have $Tx \neq \emptyset$ and for every $y \in Tx$, F satisfies the above inequality, then T has a critical point in X.

Theorem 4.13 (**strong type**)**.** *We assume either* **(C-$\leq_C^l$)** *or* **(C-$\leq_C^u$)**. *Moreover, we assume*

(Caristi-$\leq_C^s$): *if $T : X \to 2^X$ is a multivalued mapping such that for every $x \in X$ there exists $y \in Tx$ such that $F(y) + d(x,y)k^0 \leq_C^s F(x)$,*

then T has a fixed point in X. Furthermore, for every $x \in X$ we have $Tx \neq \emptyset$ and for every $y \in Tx$, F satisfies the above inequality, then T has a critical point in X.

4.4. **Takahashi's minimization theorems for set-valued map.** Takahashi presents the following theorem, which is useful in optimization theory.

Theorem 4.14 (Takahashi[23])**.** *Let (X, d) be a complete metric space and $f : X \to (-\infty, \infty]$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Suppose that for each $u \in X$ with $\inf_{x\in X} f(x) < f(u)$, there exists $v \in X$ such that $v \neq u$ and $f(v) + d(u,v) \leq f(u)$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x\in X} f(x)$.*

In this subsection, by using scalarizing functions $h^l_{\inf}$ and $h^u_{\inf}$, we obtain two types of Takahashi's minimization theorems for set-valued map.

Theorem 4.15 (**l-type**)**.** *We assume* **(C-$\leq_C^l$)**. *Moreover, we assume*

(Takahashi-$\leq_C^l$): *for each $y \in X$ with $F(y) \notin l$-wMin$F(X)$, there exists $z \in X \setminus \{y\}$ such that $F(z) + d(y,z)k^0 \leq_C^l F(y)$.*

Then there exists $\bar{x} \in X$ such that $F(\bar{x}) \in l$-wMin$F(X)$.

Proof. By Theorem 4.7, there exists $u \in X$ such that

$$F(v) + d(u,v)k^0 \not\leq_C^l F(u)$$

for all $v \in X\setminus\{u\}$. If for all $u \in X$ such that $F(u) \notin l$-wMin$F(X)$, by **(Takahashi-$\leq_C^l$)**, there exists $v \in X \setminus \{u\}$ with $F(v) + d(u,v)k^0 \leq_C^l F(u)$, which is a contradiction. □

In a similar way as Theorem 4.15, we obtain u-type Takahashi's minimization theorem.

Theorem 4.16 (**u-type**)**.** *We assume* **(C-$\leq_C^u$)**. *Moreover, we assume*

(Takahashi-$\leq_C^u$): *for each $y \in X$ with $F(y) \notin u$-wMin$F(X)$, there exists $z \in X \setminus \{y\}$ such that $F(z) + d(y,z)k^0 \leq_C^u F(y)$.*

Then there exists $\bar{x} \in X$ such that $F(\bar{x}) \in u$-wMin$F(X)$.

Remark 7. Since we cannot define the concept of minimal element of the weak and strong type set relations, we cannot obtain Takahashi's minimization theorem for set-valued map with respect to $\leq_C^w$ and $\leq_C^s$.

4.5. **Equivalences.** In this subsection, we prove the equivalences between the above existence theorems. The proof is similar as [1].

Theorem 4.17 (*l***-type).** *Theorem 4.7, Theorem 4.11 and Theorem 4.15 are equivalent to each other.*

Theorem 4.18 (*u***-type).** *Theorem 4.8, Theorem 4.12 and Theorem 4.16 are equivalent to each other.*

Theorem 4.19 (**strong type).** *Theorem 4.9 and Theorem 4.13 are equivalent.*

Proof. (Theorem 4.9⇒Theorem 4.13) The proof is similar to Theorem 4.11.
(Theorem 4.13⇒Theorem 4.9) Let $k^0 \in \mathrm{int}C$ and $x_0 \in X$ with

$$\big(F(X) + k^0 + \mathrm{int}C\big) \cap F(x_0) = \emptyset.$$

We define

$$X_0 := \{x \in X | F(x) + d(x, x_0)k^0 \leq^l_C F(x_0)\} \cup \{x \in X | F(x) + d(x, x_0)k^0 \leq^u_C F(x_0)\}.$$

Since $x_0 \in X_0$, we have that X_0 is nonempty. Moreover by **(C-**$\leq^l_C$**)** or **(C-**$\leq^u_C$**)**, X_0 is closed and hence complete. We also define

$$Sx := \{y \in X | x \neq y, F(y) + d(x, y)k^0 \leq^s_C F(x)\}$$

$$Tx := \begin{cases} x & if\ Sx = \emptyset \\ Sx & if\ Sx \neq \emptyset. \end{cases}$$

By the definition of Sx and Tx, we have that $x \notin Sx$, $Tx \neq \emptyset$ for all $x \in X$ and $T : X_0 \to 2^{X_0}$. Also we have that for every $x \in X$ there exists $y \in Tx$ such that $F(y) + d(x, y)k^0 \leq^s_C F(x)$. By theorem 4.13, there exists $\bar{x} \in X_0$ such that $\bar{x} \in T\bar{x}$. By the definition of T and $\bar{x} \in X_0$, we have for each $x \neq \bar{x}$

$$F(\bar{x}) + d(\bar{x}, x_0)k^0 \leq^s_C F(x_0)$$

$$F(x) + d(x, \bar{x})k^0 \not\leq^s_C F(\bar{x}).$$

In a similar way as Theorem 4.9, we obtain $F(\bar{x}) \leq^s_C F(x_0)$ and $d(\bar{x}, x_0) \leq 1$. □

5. Conclusions

In this paper, we present some new types of minimal element theorems on $X \times \mathcal{V}$, where X is a complete metric space and $\mathcal{V}$ a family of nonempty subsets of topological vector space. First, we introduce new type order relation $\preceq^s_{k^0}$ and obtained minimal element theorem with respect to $\preceq^s_{k^0}$. Second, we introduce new type order relations $\preceq^l_{k^0,h^l_{\inf}}$ and $\preceq^u_{k^0,h^u_{\inf}}$ which depend on nonlinear scalarizing functions for sets $h^l_{\inf}$, $h^u_{\inf}$ and equivalent classes $[\cdot]^l$, $[\cdot]^u$, respectively. By the definitions of $\preceq^l_{k^0,h^l_{\inf}}$ and $\preceq^u_{k^0,h^u_{\inf}}$, we obtain (b') of Theorem 4.3, Theorem 4.4, respectively. These results are some kind of characteristic minimality conclusions in set optimization (see also remark 5). Lastly, we present three types of set-valued Ekeland's variational principle, Caristi's fixed point theorem and two types of set-valued Takahashi's minimization theorem, which are generalizations of [1].

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References

[1] Y. Araya, *Ekeland's variational principle and its equivalent theorems in vector optimization*, J. Math. Anal. Appl. **346** (2008), 9–16.

[2] Y. Araya, *Four types of nonlinear scalarizations and some applications in set optimization*, Nonlinear Anal. **75** (2012), 3821–3835.

[3] Y. Araya, *New types of nonlinear scalarizations in set optimization*, Nonlinear Analysis and Optimization, S. Akashi, W. Takahashi and T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2014, pp. 7–21.

[4] Y. Araya, *Conjugate duality in set optimization via nonlinear scalarization*, submitted.

[5] H. Brezis and F. E. Browder, *A general principle on ordered sets in nonlinear functional analysis*, Advances in Math. **21** (1976), 355–364.

[6] J. Caristi, *Fixed point theorems for mapping satisfying inwardness conditions*, Trans. Amer. Math. Soc. **215** (1976), 241–251.

[7] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl. **47** (1974), 324–354.

[8] F. Flores-Bazan, C. Gutierrez and V. Novo, *A Brezis-Browder principle on partially ordered spaces and related ordering theorems*, J. Math. Anal. Appl. **375** (2011), 245–260.

[9] C. Gerth and P. Weidner, *Nonconvex separation theorems and some applications in vector optimization*, J. Optim. Theory Appl. **67** (1990), 297–320.

[10] A. Göpfert, H. Riahi, C. Tammer, and C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, Springer-Verlag, New York, 2003.

[11] A. Göpfert, C. Tammer, and C. Zălinescu, *On the vectorial Ekeland's variational principle and minimal points in product spaces*, Nonlinear Anal. **39** (2000), 909–922.

[12] A. Hamel and A. Löhne, *Minimal element theorems and Ekeland's principle with set relations*, J. Nonlinear and Convex Anal. **7** (2006), 19–37.

[13] E. Hernández and L. Rodríguez-Marín, *Nonconvex scalarization in set-optimization with set-valued maps*, J. Math. Anal. Appl. **325** (2007), 1–18.

[14] J. Jahn and T. X. D. Ha, *New order relations in set optimization*, J. Optim. Theory Appl. **148** (2011), 209–236.

[15] D. Kuroiwa, T. Tanaka and T. X. D. Ha, *On cone convexity of set-valued maps*, Nonlinear Anal. **30** (1997), 1487–1496.

[16] I. Kuwano, T. Tanaka and S. Yamada, *Characterization of nonlinear scalarizing functions for set-valued maps*, Nonlinear Analysis and Optimization, S. Akashi, W. Takahashi and T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2009, pp. 193–204.

[17] I. Kuwano, T. Tanaka and S. Yamada, *Inherited properties of nonlinear scalarizing functions for set-valued maps*, Nonlinear Analysis and Convex Analysis, S. Akashi, W. Takahashi and T. Tanaka (eds.), Yokohama Publishers, Yokohama, 2010, pp. 161–177.

[18] D. T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, 319, Springer-Verlag, Berlin, 1989.

[19] Y. Ogata, Y. Saito, T. Tanaka and S. Yamada, *Sublinear scalarization methods for sets with respect to set-relations*, Linear Nonlinear Anal. **3** (2017), 121–132.

[20] A. Pascoletti and P. Serafini *Scalarizing vector optimization problems*, J. Optim. Theory Appl. **42** (1984), 499–524.

[21] J. H. Qiu, *A pre-order principle and set-valued Ekeland variational principle*, J. Math. Anal. Appl. **419** (2014), 904–937.

[22] A. M. Rubinov, *Sublinear operators and their applications (Russian)*, Uspehi Mat. Nauk **32** (1977), 113–174.

[23] W. Takahashi, *Existence theorems generalizing fixed point theorems for multivalued mappings*, Fixed Point Theory and Applications (Marseille, 1989), Pitman Res. Notes Math. Ser. 252, Longman Sci. Tech., Harlow, 1991, pp. 397–406.

[24] C. Tammer and C. Zălinescu, *Vector variational principles for set-valued functions*, Optimization **60** (2011), 839–857.

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